Mathematics

By a group of supervisors

The Main Book

SECOND TERM

1 SEC.

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Unit 1

MATRICES.

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Learning outcomes

By the end of this unit, the student should be able to :

- Recognize the concept of the matrix and its order.
- Model some real life problems using the matrices.
- Recognize some special matrices.
- Recognize the equality of two matrices.
- Find the transpose of a matrix.
- Multiply a real number by a matrix.
- Recognize the concept of symmetric matrix and skew symmetric matrix.
- Carry out the operations of addition , subtraction and multiplication on matrices.
- Recognize the properties of addition and multiplication of the matrices.
- Use matrices in other domains.
- Recognize the determinant of a matrix of order 2×2 and 3×3
- Find the value of the determinant of order 2×2 and 3×3
- Find the surface area of the triangle using the determinants.
- Solve a system of linear equations using Cramar's rule.
- Find the inverse of the square matrix of order 2×2
- Solve two simultaneous equations using the inverse of a matrix.

Brief History

- The British scientist J.J. Sylvester was the first to use the expression "matrix".
- The British scientist Cayley was the first to use the matrices, and he is a mathematical scientist and has a lot of searches especially in algebra which included the theorem of matrix.
- Matrix is commonly used in modern times, and it includes many branches of science and knowledge, we use it in statistics and economics, sociology, psychology and so on. Fruther more, matrices have an important role in mathematics especially in the branch of linear algebra.

J.J. Sylvester
(1814 - 1897)

Arthur Cayley
(1821 - 1895)

Lesson One — Organizing data in matrices

Illustrated example

- A pizzeria sells four kinds of pizzas : (Vegetarian - Chicken - Beef - Cheese) and serves for each kind three different sizes : (small - medium - large)
- To remember these data and compare between them easily , the shop owner arranged the average of the number of sold pieces daily in the following table :

		Size		
		Small	**Medium**	**Large**
Kind	**Vegetarian**	15	13	9
	Chicken	16	18	12
	Beef	13	10	8
	Cheese	18	20	17

- Each number in this table has a certain meaning , for example , the number 10 refers to the number of sold pieces of beef with medium size and the number 12 refers to the number of sold pieces of chicken with large size and so on.
- For we know that the numbers of the first row refer to the average of sold pieces of vegetarian daily with the sizes : (small - medium - large) respectively , similarly , the numbers of the second row for chicken , the numbers of the third row for beef and the numbers of the fourth row for cheese respectively , then we don't need the previous table , and we are satisfied to write the data in a simple form by writting only the contained

numbers in the table in the same order inside two large parentheses as $\begin{pmatrix} & & \end{pmatrix}$

Then we write the daily averages for the sales of the shop = $\begin{pmatrix} 15 & 13 & 9 \\ 16 & 18 & 12 \\ 13 & 10 & 8 \\ 18 & 20 & 17 \end{pmatrix}$

- This form is called a **number matrix** and the numbers contained inside the two parentheses are called **the elements of the matrix**.
- This matrix is formed from four rows and three columns as in the opposite figure , so we say that it is a matrix of order 4×3 or simply "a 4×3 matrix" and we notice that we mention the number of rows firstly , then the number of columns.

$$\begin{array}{cccc} 1^{st}\text{ column} & 2^{nd}\text{ column} & 3^{rd}\text{ column} & \\ \downarrow & \downarrow & \downarrow & \end{array}$$

$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 15 & 13 & 9 \\ 16 & 18 & 12 \\ 13 & 10 & 8 \\ 18 & 20 & 17 \end{pmatrix}\begin{array}{l} \leftarrow 1^{st}\text{ row} \\ \leftarrow 2^{nd}\text{ row} \\ \leftarrow 3^{rd}\text{ row} \\ \leftarrow 4^{th}\text{ row} \end{array}$$

Remark

The owner of the shop can organize his previous data in another table as the following table :

		Kind			
		Vegetarian	**Chicken**	**Beef**	**Cheese**
Size	**Small**	15	16	13	18
	Medium	13	18	10	20
	Large	9	12	8	17

Similarly , we don't need the previous table and we are satisfied with writting the numbers inside a matrix , then the daily averages for the sales of the

shop = $\begin{pmatrix} 15 & 16 & 13 & 18 \\ 13 & 18 & 10 & 20 \\ 9 & 12 & 8 & 17 \end{pmatrix}$ and it is a matrix of order 3×4

Unit 1

From the previous , we can define the matrix as follows :

Definition of the matrix

- The matrix is an arrangement of a number of elements (variables or numbers) in rows and columns enclosed by two parentheses as ( ) , such that the position of each element in the matrix has a meaning.
- If the number of rows is m and the number of columns is n , then the form of the matrix is $m \times n$ or of order $m \times n$ or of type $m \times n$ (it is read as m by n) where m and n are positive integers.
- The number of the elements of the matrix = number of rows × number of columns $= m \times n$

Expressing an element inside a matrix

- Capital letters are used to name the matrix or to symbolize it as : A , B , C , X , Y , ... , but small letters are used to name the elements of the matrix as : a , b , c , x , y , ...
- If we want to express an element inside the matrix A that lies in the i^{th} row and the j^{th} column , then we can express it by the form : a_{ij}

For example :

The element a_{23} lies in the 2^{nd} row and the 3^{rd} column (it is read as : a two three)

The element a_{32} lies in the 3^{rd} row and the 2^{nd} column (it is read as : a three two)

Example 1

If $A = \begin{pmatrix} -2 & 0 & \frac{1}{2} \\ 1 & 6 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ -1 & 8 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & -3 & \sqrt{5} \\ -1 & 0 & -2 \\ 2 & \frac{1}{4} & 9 \end{pmatrix}$

1. Write the order of each of A , B and C
2. Write the following elements : a_{22} , b_{21} , a_{23} , b_{12} , c_{32} , c_{23}

Solution

1. A is a matrix of order 2×3 ,

   B is a matrix of order 2×2 and

   C is a matrix of order 3×3
2. $a_{22} = 6$, $b_{21} = -1$, $a_{23} = -4$, $b_{12} = 5$, $c_{32} = \frac{1}{4}$, $c_{23} = -2$

TRY TO SOLVE

If the matrix $X = \begin{pmatrix} 5 & -2 \\ 4 & 1 \\ -7 & 0 \end{pmatrix}$

1 Write the order of the matrix X.

2 Write the following elements x_{32} , x_{21} , x_{12}

Remark

If A is a matrix of order $m \times n$, then we can express it by the form :

$A = (a_{ij})$ where :

$i = 1 , 2 , 3 , \ldots , m$ and $j = 1 , 2 , 3 , \ldots , n$

In our study , the cases in which $m \leq 3$ and $n \leq 3$ will be sufficient.

Example 2

Write all the elements of the following matrices, showing the order of each matrix:

1 The matrix $A = (a_{ij})$ where $i = 1 , 2 , 3$ and $j = 1 , 2$

2 The matrix $B = (b_{ij})$ where $i = 1$ and $j = 1 , 2 , 3$

Solution

1 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, a matrix of order 3×2

2 $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \end{pmatrix}$, a matrix of order 1×3

Example 3

Write the matrix (a_{ij}) of order 3×2 where $a_{ij} = 2i - j$

Solution

$a_{11} = 2 \times 1 - 1 = 1$, $a_{12} = 2 \times 1 - 2 = 0$, $a_{21} = 2 \times 2 - 1 = 3$,

$a_{22} = 2 \times 2 - 2 = 2$, $a_{31} = 2 \times 3 - 1 = 5$, $a_{32} = 2 \times 3 - 2 = 4$

$\therefore (a_{ij}) = \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{pmatrix}$

TRY TO SOLVE

Write all the elements of the matrix $C = (c_{ij})$ where $i = 1 , 2$ and $j = 1 , 2 , 3$

Unit 1

Some special matrices

1 The row matrix

It is a matrix that consists of one row and any number of columns.

For example :

$A = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$ is a row matrix of order 1×3

2 The column matrix

It is a matrix that consists of one column and any number of rows.

For example :

$X = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a column matrix of order 2×1

3 The square matrix

It is a matrix in which the number of rows = the number of columns.

For example :

$A = \begin{pmatrix} \sqrt{3} & 5 \\ -2 & 6 \end{pmatrix}$ is a square matrix of order 2×2

4 The zero matrix

It is a matrix whose all elements are zeroes. We denote it by O_{mn} , it may be a square matrix or not.

For example :

$O_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero matrix of order 2×2

, $O_{3 \times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a zero matrix of order 3×1

5 The diagonal matrix

It is a square matrix in which all elements are zeroes except the elements of its main diagonal , then at least one of them is not equal to zero. (Where the main diagonal that contains the elements a_{11} , a_{22} , a_{33} , ...)

For example :

$Y = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a diagonal matrix of order 3×3

, $R = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix of order 2×2

6 The unit matrix

It is a diagonal matrix in which each element on the main diagonal is the number 1 , and it is denoted by I

For example : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unit matrix of order 2×2

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix of order 3×3

The equality of two matrices

The two matrices A and B are said to be equal if and only if , they have the same order and their corresponding elements are equal.

i.e. $a_{ij} = b_{ij}$ for each i and j

For example : $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ \frac{4}{2} & 3 & -1 \end{pmatrix}$

while $\begin{pmatrix} 2 & 8 \\ -3 & 5 \end{pmatrix} \neq \begin{pmatrix} 2 & 5 \\ -3 & 8 \end{pmatrix}$ because the corresponding elements are different.

and so $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ because they don't have the same order.

Example 4

Find the value of each of x , y and z if $\begin{pmatrix} z & 0 & 2 \\ 4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & x+5 \\ 4 & 2y-3 & 5 \end{pmatrix}$

Solution

$\because$ The two matrices are equal. $\therefore z = -1$

, $x + 5 = 2$ $\therefore x = -3$, $2y - 3 = 7$ $\therefore y = 5$

TRY TO SOLVE

Find the value of each of x and y if $\begin{pmatrix} x^3 & -2 \\ 3 & 2y-9 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 3 & -y \end{pmatrix}$

Multiplying a real number by a matrix

If A is a matrix of order $m \times n$, then the product of any real number k by the matrix A is the matrix $C = kA$ of the same order $m \times n$, and each element of the elements of the matrix C equals the corresponding element to it in the matrix A multiplied by the real number k

i.e. $c_{ij} = k\, a_{ij}$, where $i = 1 , 2 , \ldots , m$ and $j = 1 , 2 , \ldots , n$

i.e. Multiplying a real number by a matrix means multiplying each element of the elements of the matrix by that real number.

For example :

If $A = \begin{pmatrix} 6 & -2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$, then :

- $2A = \begin{pmatrix} 6\times 2 & -2\times 2 & 3\times 2 \\ 2\times 2 & 4\times 2 & 0\times 2 \end{pmatrix} = \begin{pmatrix} 12 & -4 & 6 \\ 4 & 8 & 0 \end{pmatrix}$
- $-3A = \begin{pmatrix} -18 & 6 & -9 \\ -6 & -12 & 0 \end{pmatrix}$

Remark

From the previous , we deduce that it is possible to take a common factor among all the elements of the matrix.

For example : $\begin{pmatrix} 4 & 8 & 6 \\ 2 & 0 & -14 \end{pmatrix} = 2\begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & -7 \end{pmatrix}$

Example 5

If $\begin{pmatrix} 10 & -20 \\ -8 & 16 \end{pmatrix} = -2 \begin{pmatrix} -5 & 5x \\ 4 & -2y \end{pmatrix}$, then find the value of : $\sqrt[3]{xy}$

Solution

$\because \begin{pmatrix} 10 & -20 \\ -8 & 16 \end{pmatrix} = \begin{pmatrix} 10 & -10x \\ -8 & 4y \end{pmatrix}$ $\quad \therefore -20 = -10x \quad \therefore x = 2$

, $16 = 4y$ $\quad \therefore y = 4 \quad \therefore \sqrt[3]{xy} = \sqrt[3]{8} = 2$

TRY TO SOLVE

1 If $A = \begin{pmatrix} 3 & -4 \\ -1 & 0 \end{pmatrix}$, then find : $3A$, $-A$, $-5A$

2 If $\begin{pmatrix} 24 & -8 \\ 32 & 0 \\ 12 & -4 \end{pmatrix} = 4 \begin{pmatrix} 6 & 2y \\ -2x & 0 \\ 3 & -1 \end{pmatrix}$, then find : xy

Matrix transpose

In any matrix A of order $m \times n$, if we replace the rows by columns or the columns by rows in the same order , then we will get a matrix of order $n \times m$ that is called the transpose of the matrix A and we denote it by A^t

i.e. If $A = (a_{ij})$, then $A^t = (a_{ji})$

For example :

- If $A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}$ is a matrix of order 2×3

, then $A^t = \begin{pmatrix} 2 & -1 \\ 3 & 5 \\ 0 & 6 \end{pmatrix}$ is a matrix of order 3×2 , $(A^t)^t = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}$

Notice that

$(A^t)^t = A$

- If $B = \begin{pmatrix} 9 \\ -2 \\ 4 \end{pmatrix}$ is a matrix of order 3×1 (column matrix)

, then $B^t = \begin{pmatrix} 9 & -2 & 4 \end{pmatrix}$ is a matrix of order 1×3 (row matrix) , $(B^t)^t = \begin{pmatrix} 9 \\ -2 \\ 4 \end{pmatrix} = B$

Example 6

If $A=\begin{pmatrix} \cot 30^\circ & \sec 30^\circ \\ \csc 30^\circ & \sin 30^\circ \end{pmatrix}$, $B=\begin{pmatrix} \sqrt{3} & \frac{1}{2}y \\ \sqrt{3}\,x & \frac{1}{2} \end{pmatrix}$, and $A=B^t$

, then find the value of each of : x , y

Solution

$\because A=\begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 2 & \frac{1}{2} \end{pmatrix}$, $B^t=\begin{pmatrix} \sqrt{3} & \sqrt{3}\,x \\ \frac{1}{2}y & \frac{1}{2} \end{pmatrix}$, $\because A=B^t$

$\therefore \sqrt{3}\,x=\frac{2}{\sqrt{3}}$ $\therefore x=\frac{2}{3}$, $\frac{1}{2}y=2$ $\therefore y=4$

TRY TO SOLVE

If $\begin{pmatrix} 2 & x+2 & -5 \\ 0 & 6 & 3 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ -4 & 6 \\ -5 & y \end{pmatrix}^t$, then find : $\frac{x}{y}$

Symmetric and skew symmetric matrices

If A is a square matrix , then :

- A is called a symmetric matrix if and only if $A=A^t$
- A is called a skew symmetric matrix if and only if $A=-A^t$

For example :

- If $A=\begin{pmatrix} 2 & -1 & -3 \\ -1 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$, then $A^t=\begin{pmatrix} 2 & -1 & -3 \\ -1 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$

i.e. A is a symmetric matrix because $A=A^t$

- If $B=\begin{pmatrix} 0 & -\frac{1}{2} & -4 \\ \frac{1}{2} & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}$

, then $B^t=\begin{pmatrix} 0 & \frac{1}{2} & 4 \\ -\frac{1}{2} & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix}=-\begin{pmatrix} 0 & -\frac{1}{2} & -4 \\ \frac{1}{2} & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}$

i.e. B is a skew symmetric matrix because $B=-B^t$

Remarks

- If A is a symmetric matrix , we notice that its elements are symmetric about the main diagonal , then $a_{ij} = a_{ji}$ as in the opposite figure , where $a_{21} = a_{12} = d$, $a_{31} = a_{13} = e$, $a_{32} = a_{23} = f$

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

The main diagonal

- The elements of the main diagonal in the skew symmetric matrix have the numeral zero , and its elements satisfy the relation $a_{ij} = -a_{ji}$

and we notice that in the matrix $A = \begin{pmatrix} 0 & -\frac{1}{2} & -4 \\ \frac{1}{2} & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}$

where $a_{21} = -a_{12} = \frac{1}{2}$, $a_{31} = -a_{13} = 4$, $a_{32} = -a_{23} = -2$

- Any diagonal matrix is a symmetric matrix.

Example 7

1 **If** $A = \begin{pmatrix} 5 & 2x & 8 \\ -4 & -3 & 6 \\ x + 2y & 6 & 4 \end{pmatrix}$ **is a symmetric matrix , then find the values of : x , y**

2 **If** $B = \begin{pmatrix} 0 & 3x & 7 \\ z + 3 & 0 & -2z \\ 3y - x & 6 & 0 \end{pmatrix}$ **is a skew symmetric matrix**

, then find the values of : x , y , z

Solution

1 $\because$ A is a symmetric matrix

$\therefore 2x = -4$ $\qquad\qquad$ $\therefore x = -2$

, $x + 2y = 8$ $\qquad\qquad$ $\therefore -2 + 2y = 8$

$\therefore y = 5$

2 $\because$ B is a skew symmetric matrix

$\therefore z + 3 = -3x$ (1) , $3y - x = -7$ (2)

, $-2z = -6$ $\therefore z = 3$

Substituting in (1) : $\therefore 3 + 3 = -3x$ $\therefore x = -2$

Substituting in (2) : $\therefore 3y + 2 = -7$ $\therefore y = -3$

TRY TO SOLVE

Complete the following :

1 If $A = \begin{pmatrix} 5 & 8 \\ -2x & 6 \end{pmatrix}$ is a symmetric matrix , find the value of : x

2 If $B = \begin{pmatrix} 0 & -8 & 5 \\ \frac{1}{2}x & 0 & 12 \\ -5 & y - x & 0 \end{pmatrix}$ is a skew symmetric matrix , find the values of : x , y

Lesson Two Adding and subtracting matrices

First Adding matrices

If A and B are two matrices of the same order , then the addition operation is possible and the result of addition is a matrix of the same order and each of its elements is the sum of the two corresponding elements in A and B

For example :

If $A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$, then $A + B = \begin{pmatrix} 2+5 & 1+2 \\ 3+2 & -2+3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 5 & 1 \end{pmatrix}$

Example 1

If $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -2 \\ 5 & 1 \\ 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}$

Find each of the following if it is possible : **1** $2A + C^t$ **2** $B + C$

Solution

1 $2A + C^t = 2\begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}^t$

$$= \begin{pmatrix} 2 & -4 \\ 6 & 10 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & 4 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 6 & 14 \\ 7 & 10 \end{pmatrix}$$

2 It is impossible to add B and C , because they don't have the same order , since B is of order 3×2 and C is of order 2×3

Example 2

If $A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix}$, check that : $(A + B)^t = A^t + B^t$

Solution

$\because A + B = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 2 \\ 10 & -1 \end{pmatrix}$

$\therefore (A + B)^t = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix}$ (1)

$\because A^t = \begin{pmatrix} 2 & -1 & 6 \\ 3 & 5 & 7 \end{pmatrix}$, $B^t = \begin{pmatrix} 0 & 2 & 4 \\ 4 & -3 & -8 \end{pmatrix}$

$\therefore A^t + B^t = \begin{pmatrix} 2 & -1 & 6 \\ 3 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 4 & -3 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix}$ (2)

From (1) and (2) , we deduce that : $(A + B)^t = A^t + B^t$

TRY TO SOLVE

If $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ -1 & 8 \end{pmatrix}$, then find : $\frac{1}{3}(A + B^t)$

Example 3

Find the values of a , b and c that satisfy the equation :

$3\begin{pmatrix} a & b \\ c & 3 \end{pmatrix} = 2\begin{pmatrix} a & 6 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & b + 4 \\ c + 3 & 3 \end{pmatrix}$

Solution

$\begin{pmatrix} 3a & 3b \\ 3c & 9 \end{pmatrix} = \begin{pmatrix} 2a & 12 \\ -2 & 6 \end{pmatrix} + \begin{pmatrix} 4 & b + 4 \\ c + 3 & 3 \end{pmatrix}$ (multiplying a real number by a matrix)

$\therefore \begin{pmatrix} 3a & 3b \\ 3c & 9 \end{pmatrix} = \begin{pmatrix} 2a + 4 & b + 16 \\ c + 1 & 9 \end{pmatrix}$

and from the property of equality of two matrices :

$\therefore 3a = 2a + 4$ $\therefore a = 4$

$, 3b = b + 16$ $\therefore b = 8$

$, 3c = c + 1$ $\therefore c = \frac{1}{2}$

TRY TO SOLVE

If $3\begin{pmatrix} a & b \\ c & -2 \end{pmatrix} = 2\begin{pmatrix} a+1 & 3 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4+b \\ c+3 & -10 \end{pmatrix}$

, then find the value of each of : a , b and c

Properties of adding matrices

Let A , B and C be three matrices of the order m × n and **O** is a zero matrix of the same order , then the following properties will be satisfied :

1 Closure property : A + B is a matrix of the same order m × n

2 Commutative property : A + B = B + A

For example :

$$\begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -3 & 2 \end{pmatrix}$$

3 Associative property : (A + B) + C = A + (B + C)

For example :

If $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}$

, then $(A+B)+C = \left[\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix}\right] + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}$

$= \begin{pmatrix} -1 & -1 & 7 \\ 11 & 13 & -15 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 17 & 20 & 1 \end{pmatrix}$

, $A+(B+C) = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \left[\begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}\right]$

$= \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 4 & -1 \\ 13 & 15 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 17 & 20 & 1 \end{pmatrix}$

i.e. (A + B) + C = A + (B + C)

4 The existence of the additive identity :

Zero matrix O is the additive identity "neutral"

i.e. $A + \mathbf{O} = \mathbf{O} + A = A$

For example :

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

5 The existence of the additive inverse :

$A + (-A) = (-A) + A = \mathbf{O}$ where $(-A)$ is the additive inverse of the matrix A

For example :

If $A = \begin{pmatrix} 4 & 1 & 0 \\ -3 & 2 & 5 \end{pmatrix}$, then the additive inverse of A is $-A = \begin{pmatrix} -4 & -1 & 0 \\ 3 & -2 & -5 \end{pmatrix}$

where

$$\begin{pmatrix} 4 & 1 & 0 \\ -3 & 2 & 5 \end{pmatrix} + \begin{pmatrix} -4 & -1 & 0 \\ 3 & -2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}_{2 \times 3}$$

Second Subtracting matrices

If A and B are two matrices of the same order $m \times n$, then the remainder of subtracting $(A - B)$ is the matrix C of the order $m \times n$ that is defined as follows :

$C = A - B = A + (-B)$ where $(-B)$ is the additive inverse of the matrix B

For example :

If $A = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 5 & 0 \end{pmatrix}$

, then $A - B = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} -4 & -3 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & -4 \end{pmatrix}$

Remark

We can carry out the subtraction operation directly by subtracting the corresponding elements of the two matrices.

For example :

$$\begin{pmatrix} 3 & 4 & 5 \\ -2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} -3 & -2 & 6 \\ 0 & -1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -1 \\ -2 & 2 & -8 \end{pmatrix}$$

Example 4

If $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix}$

, **find the value of :** $4A - 2B + \frac{1}{2}C$

Solution

$$4A - 2B + \frac{1}{2}C = 4\begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix} - 2\begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 4 \\ -4 & 12 \\ 0 & 16 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 2 & -10 \\ -6 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & -3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -1 \\ -2 & 21 \end{pmatrix}$$

TRY TO SOLVE

If $A = \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ **and** $C = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

, then find the value of : $2A + 3C - 2B$

Remark

Subtracting matrices operation is not commutative and not associative.

Example 5

If $A = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}$

, **find the matrix X such that :** $3X + 2B = A$

Solution

$\because 3X + 2B = A$ (Add the additive inverse of the matrix (2 B) to both sides)

$\therefore 3X + 2B + (-2B) = A + (-2B)$

$\therefore 3X + \mathbf{O}_{3 \times 3} = A - 2B$

$\therefore 3X = A - 2B$ $\left(\text{Multiply the two sides by } \frac{1}{3}\right)$

$\therefore X = \frac{1}{3}(A - 2B)$

$$\therefore X = \frac{1}{3}\left[\begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}\right]$$

$$\therefore X = \frac{1}{3}\begin{pmatrix} -3 & -3 & -2 \\ 6 & -3 & 0 \\ 3 & 9 & 8 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -\frac{2}{3} \\ 2 & -1 & 0 \\ 1 & 3 & \frac{8}{3} \end{pmatrix}$$

Example 6

If $A = \begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix}$

, find the matrix X that satisfies : $2\left[X^t - A\right] = 3\,B$

Solution

$\because 2\left[X^t - A\right] = 3\,B$

$\therefore 2\,X^t - 2\,A = 3\,B$ (Add the matrix 2 A to both sides)

$\therefore 2\,X^t - 2\,A + 2\,A = 3\,B + 2\,A$

$\therefore 2\,X^t + \mathbf{O}_{2\times 3} = 3\,B + 2\,A$

$$\therefore 2\,X^t = 3\,B + 2\,A = 3\begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix} + 2\begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 13 & 14 & -2 \end{pmatrix}$$

$$\therefore X^t = \frac{1}{2}\begin{pmatrix} 4 & 3 & 5 \\ 13 & 14 & -2 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & 7 & -1 \end{pmatrix}$$ (Take the transpose of both sides)

$$\therefore (X^t)^t = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & 7 & -1 \end{pmatrix}^t \qquad \therefore X = \begin{pmatrix} 2 & \frac{13}{2} \\ \frac{3}{2} & 7 \\ \frac{5}{2} & -1 \end{pmatrix}$$

Example 7

If $X + 2\,X^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}$ **, then find the matrix X**

Solution

$$\because X + 2\,X^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}$$ (1) (Take the transpose of both sides)

$$\therefore (X + 2\,X^t)^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}^t$$

Notice that

- $(A + B)^t = A^t + B^t$
- $(A^t)^t = A$

$\therefore X^t + 2X = \begin{pmatrix} 9 & 13 \\ 14 & 6 \end{pmatrix}$ (2)

Multiply (2) by – 2 :

$\therefore -2X^t - 4X = \begin{pmatrix} -18 & -26 \\ -28 & -12 \end{pmatrix}$ (3)

Add (1) and (3) :

$\therefore -3X = \begin{pmatrix} -9 & -12 \\ -15 & -6 \end{pmatrix}$

$\therefore X = \frac{-1}{3} \begin{pmatrix} -9 & -12 \\ -15 & -6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$

TRY TO SOLVE

If $A = \begin{pmatrix} 2 & 4 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & -2 \\ 4 & \frac{1}{2} \end{pmatrix}$, then find the matrix X where :

$3A - 2B = 2X - 3I$ where I is of the order 2×2

Remark

You can use the scientific calculator to add and subtract the matrices , and we will show that at the end of the unit.

Lesson Three — Multiplying matrices

Introductory example

If the matrix A expresses the results of 20 matches for Al-Ahly team and Zamalek team in general league of football where :

$$A = \begin{array}{c} \begin{matrix} \text{Win} & \text{Drawn} & \text{Loss} \end{matrix} \\ \begin{pmatrix} 12 & 6 & 2 \\ 11 & 4 & 5 \end{pmatrix} \end{array} \begin{matrix} \longrightarrow \text{Al-Ahly} \\ \longrightarrow \text{Zamalek} \end{matrix}$$

and the matrix B expresses the number of points that the team gains in the cases of win , drawn and loss where :

$$B = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \begin{matrix} \longrightarrow \text{Win} \\ \longrightarrow \text{Drawn} \\ \longrightarrow \text{Loss} \end{matrix}$$

, then the sum of points that Al-Ahly gained $= 12 \times 3 + 6 \times 1 + 2 \times 0 = 42$ points

, the sum of points that Zamalek gained $= 11 \times 3 + 4 \times 1 + 5 \times 0 = 37$ points and we can express the sum of points that each team gained by the matrix $C = \begin{pmatrix} 42 \\ 37 \end{pmatrix}$

We notice that : 42 is the sum of products of the elements of first row in the matrix A by the elements of the column in the matrix B , 37 is the sum of products of the elements of second row in the matrix A by the elements of the column in the matrix B

- The matrix C is the product of multiplying the matrix A × the matrix B

i.e. $C = A\,B = \begin{pmatrix} 12 & 6 & 2 \\ 11 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \times 3 + 6 \times 1 + 2 \times 0 \\ 11 \times 3 + 4 \times 1 + 5 \times 0 \end{pmatrix} = \begin{pmatrix} 42 \\ 37 \end{pmatrix}$

Multiplying matrices

If A is a matrix of the order $m \times \ell$, B is a matrix of the order $r \times n$, then :

- Their product $C = AB$ will be defined if and only if $\ell = r$

i.e. the number of columns of the matrix A = the number of rows of the matrix B

- The matrix $C = AB$ will be of the order $m \times n$

$$\underset{m \times \boxed{\ell \quad \text{Equal} \quad r} \times n}{A \qquad\qquad\qquad B} \quad = \quad \underset{m \times n}{(AB)}$$

- Each element c_{ij} in the matrix $C = AB$ equals the sum of products of elements of i^{th} row in the matrix A by the elements of j^{th} column in the matrix B , one by one corresponding to it.

To explain the concept of multiplying matrices :

For example :

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then A is a matrix of order 3×2 and B is a matrix of order 2×2 Since , the number of columns of matrix A = the number of rows of matrix B = 2 , then :

$$\underset{3 \times \boxed{2 \quad \text{Equal} \quad 2} \times 2}{A \qquad\qquad\qquad B} \quad = \quad \underset{3 \times 2}{(AB)}$$

i.e. The multiplying operation of the matrix A by the matrix B is defined and produces a matrix AB of order 3×2 and can be obtained as follow :

- Multiply each element from the first row in the matrix A by the corresponding element in the first column in the matrix B and adding up their products to get the element in (the first row and first column) in the matrix (AB) as follow :

$$\begin{pmatrix} a_{11} & a_{12} \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} b_{11} & \cdots \\ b_{21} & \cdots \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

- Then multiply each element from the first row in the matrix A by the corresponding element in the second column in the matrix B and adding up their products to get the element in (the first row and the second column) in the matrix (AB) as follow :

$$\begin{pmatrix} a_{11} & a_{12} \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots & b_{12} \\ \cdots & b_{22} \end{pmatrix} = \begin{pmatrix} \cdots & a_{11}b_{12} + a_{12}b_{22} \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

- and so on till we get all elements of the matrix (AB) as follow :

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \\ a_{31}\, b_{11} + a_{32}\, b_{21} & a_{31}\, b_{12} + a_{32}\, b_{22} \end{pmatrix}$$

Notice that

The multiplying operation of matrix B by matrix A is not defined.

i.e. BA is not defined because the number of columns of matrix B ≠ the number of rows of matrix A

Example 1

Find AB if possible in each of the following :

1 $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$

2 $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 & 3 \\ -1 & 2 & -4 \end{pmatrix}$

3 $A = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

Solution

1 ∵ A is a matrix of order 3×2 and B is a matrix of order 2×2

∴ The number of columns of matrix A = the number of rows of matrix B

∴ AB is defined of order 3×2

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} (2)(-1) + (1)(0) & (2)(2) + (1)(-3) \\ (3)(-1) + (-1)(0) & (3)(2) + (-1)(-3) \\ (0)(-1) + (4)(0) & (0)(2) + (4)(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ -3 & 9 \\ 0 & -12 \end{pmatrix}$$

2 $\because$ A is a matrix of order 2×3 and B is a matrix of order 2×3

$\therefore$ The number of columns of matrix A $\neq$ the number of rows of matrix B

$\therefore$ AB is not defined.

3 $\because$ A is a matrix of order 1×3 and B is a matrix of order 3×1

$\therefore$ The number of columns of matrix A = the number of rows of matrix B

$\therefore$ AB is defined of order 1×1

$$\therefore AB = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} (2)(-2) + (-1)(1) + (3)(4) \end{pmatrix} = \begin{pmatrix} 7 \end{pmatrix}$$

Example 2

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 & 3 \\ -1 & 2 & -4 \end{pmatrix}$

, **find** : AB^t if possible.

Solution

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$ of order 2×3 , $B^t = \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ 3 & -4 \end{pmatrix}$ of order 3×2

$\because$ The number of columns of matrix A = the number of rows of matrix B^t

$\therefore AB^t$ is defined of order 2×2

$$\therefore AB^t = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 20 & -9 \\ -5 & 7 \end{pmatrix}$$

TRY TO SOLVE

If $A = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 0 \\ 4 & -1 \\ 2 & 1 \end{pmatrix}$, then find if possible : AB , AB^t , $A^t B$

Unit 1

Properties of multiplying matrices

If A , B and C are three matrices , I is the identity matrix , then the following properties are satisfied.

1 Associative property :

$\boxed{(AB)C = A(BC)}$ where multiplying operations are defined.

For example :

If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

, then $AB = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -23 \\ 5 & -1 & 24 \end{pmatrix}$

$\therefore (AB)C = \begin{pmatrix} 0 & 2 & -23 \\ 5 & -1 & 24 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ -17 \end{pmatrix}$

, $BC = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$\therefore A(BC) = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 19 \\ -17 \end{pmatrix}$

$\therefore (AB)C = A(BC)$

2 The existence of multiplicative neutral (identity) property :

The identity matrix I is the multiplicative neutral matrix.

i.e. $\boxed{AI = IA = A}$ where A is a square matrix of the same order of I

For example :

$$\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$$

3 Distributing multiplication of matrices on addition property :

$$\boxed{\begin{array}{l} A(B + C) = AB + AC \\ , (A + B)C = AC + BC \end{array}}$$ where multiplying and adding operations are defined.

For example :

If $A=\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ and $C=\begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}$

, then $B+C=\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}+\begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\therefore A(B+C)=\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix}$ (1)

, $\because AB+AC=\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}+\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}$

$=\begin{pmatrix} 5 & -9 \\ -9 & 15 \end{pmatrix}+\begin{pmatrix} -4 & 8 \\ 6 & -18 \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix}$ (2)

$\therefore$ From (1) and (2) , we deduce that : $A(B+C)=AB+AC$

Remark

If A and B are two matrices whose multiplying operation is possible in any form i.e. AB is defined and BA is defined too , then it is not necessary that : $AB = BA$ that means that : the multiplying operation is not commutative.

For example :

If $A=\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, $B=\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}$ and $C=\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

, then :

1 $AB=\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}=\begin{pmatrix} -23 & -14 \\ -9 & -4 \end{pmatrix}$

, $BA=\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} -6 & 5 \\ 32 & -21 \end{pmatrix}$

i.e. $AB \neq BA$

Notice that

It is possible to multiply any two square matrices of the same order.

2 $AC=\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}=\begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$

, $CA=\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$

i.e. $AC = CA$

Example 3

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, find : A^2 , A^3

Solution

$$A^2 = A \times A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 8 \\ -4 & -1 \end{pmatrix}$$

$$, A^3 = A^2 \times A = \begin{pmatrix} 7 & 8 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 22 \\ -11 & -9 \end{pmatrix}$$

Notice that

If A is not a square matrix , then A^2 is not defined.

Example 4

If $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$, then prove that : $A^2 - 2A - 3I = O$

Solution

$$A^2 - 2A - 3I = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} - 2\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

TRY TO SOLVE

If $A = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$, then prove that : $A^2 - 5A + 2I = O$

Critical thinking

If A and B are two matrices , AB = O

Does it mean that A = O or B = O always ?

Answer : No

Explanation of the answer :

Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$, then

$$AB = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

, $\because A \neq O$, $B \neq O$

i.e. If $AB = O$, it is not always that : $A = O$ or $B = O$

Transpose of the product of two matrices

If A and B are two matrices and AB is defined , then $(AB)^t = B^t A^t$

Generally ,

$(ABC \text{ .... } E)^t = E^t \text{ .... } C^t B^t A^t$ where multiplying operations are defined.

Example 5

If $A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ **and** $B = \begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix}$, **check that :** $(AB)^t = B^t A^t$

Solution

$$\because AB = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix} = \begin{pmatrix} 23 & -39 \\ 17 & -8 \end{pmatrix}$$

$$\therefore (AB)^t = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix} \qquad (1)$$

$$, \because B^t = \begin{pmatrix} 6 & -3 & 2 \\ 1 & 9 & -8 \end{pmatrix} , \quad A^t = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 4 & 7 \end{pmatrix}$$

$$\therefore B^t A^t = \begin{pmatrix} 6 & -3 & 2 \\ 1 & 9 & -8 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix} \qquad (2)$$

From (1) and (2) :

$\therefore (AB)^t = B^t A^t$

Example 6

If $A = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 & 6 \\ 5 & -7 & 4 \end{pmatrix}$ **and** $C = \begin{pmatrix} 1 & -3 \\ -5 & 2 \\ 3 & 4 \end{pmatrix}$,

find the matrix X that satisfies the relation : $15\, X^t = A^2 + (BC)^t$

Solution

$$\because A^2 = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & -7 \\ -21 & 28 \end{pmatrix}$$

$$, BC = \begin{pmatrix} -2 & 3 & 6 \\ 5 & -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -5 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 36 \\ 52 & -13 \end{pmatrix}$$

$$\therefore (BC)^t = \begin{pmatrix} 1 & 52 \\ 36 & -13 \end{pmatrix}$$

$$\therefore 15\,X^t = \begin{pmatrix} 7 & -7 \\ -21 & 28 \end{pmatrix} + \begin{pmatrix} 1 & 52 \\ 36 & -13 \end{pmatrix} = \begin{pmatrix} 8 & 45 \\ 15 & 15 \end{pmatrix}$$

$$\therefore X^t = \begin{pmatrix} \frac{8}{15} & 3 \\ 1 & 1 \end{pmatrix} \qquad \therefore X = \begin{pmatrix} \frac{8}{15} & 1 \\ 3 & 1 \end{pmatrix}$$

Example 7

Find the values of a , b and c if :

$$\begin{pmatrix} 1 & a & 2 \\ 0 & 2 & 4 \\ 5 & -1 & b \end{pmatrix}\begin{pmatrix} -1 & 0 & 2 \\ 7 & 6 & 4 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} -19 & -6 & 4 \\ 6 & c & 28 \\ -24 & 12 & 36 \end{pmatrix}$$

Solution

We can find a , b and c without carrying out a complete multiplication operation, but we will just :

- multiply the elements of the first row of the first matrix by the elements of the first column of the second matrix

$\therefore 1 \times -1 + a \times 7 + 2 \times -2 = -19 \qquad \therefore a = -2$

- multiply the elements of the third row of the first matrix by the elements of the first column of the second matrix.

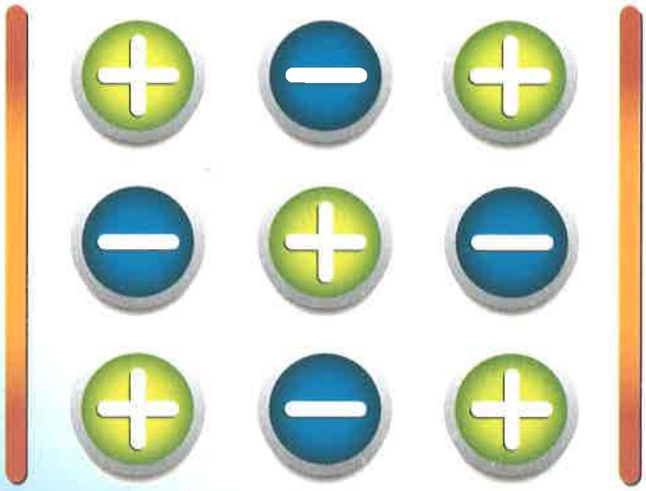
$\therefore 5 \times -1 - 1 \times 7 + b \times -2 = -24 \qquad \therefore b = 6$

- multiply the elements of the second row of the first matrix by the elements of the second column of the second matrix.

$\therefore 0 \times 0 + 2 \times 6 + 4 \times 3 = c \qquad \therefore c = 24$

Remark

We can use the scientific calculator for multiplying matrices and we will represent it at the end of the unit.



Lesson Four — Determinants

The second order determinant

If A is a square matrix of order 2×2 where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by the symbol $|A|$ and is called determinant of the second order and it is the number defined as follows :

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

i.e. The value of the determinant of the second order equals the product of the two elements of the principal diagonal minus the product of the two elements of the other diagonal.

Example 1

Find the value of each of the following determinants :

1 $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}$ **2** $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix}$

3 $\begin{vmatrix} 6 & 3 \\ -8 & -4 \end{vmatrix}$ **4** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

Solution

1 $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2 \times 8 - 3 \times 5 = 16 - 15 = 1$

2 $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix} = 4 \times 6 - 2 \times (-7) = 24 + 14 = 38$

3 $\begin{vmatrix} 6 & 3 \\ -8 & -4 \end{vmatrix} = 6 \times (-4) - (-8) \times 3 = -24 + 24 = 0$

4 $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = \sin\theta \times \sin\theta - (-\cos\theta) \times \cos\theta = \sin^2\theta + \cos^2\theta = 1$

TRY TO SOLVE

Find the value of each of the following determinants :

1 $\begin{vmatrix} 3 & 7 \\ 0 & -4 \end{vmatrix}$ **2** $\begin{vmatrix} -1 & 3 \\ -5 & -2 \end{vmatrix}$

Example 2

Find the value of x which satisfies each of the following equations :

1 $\begin{vmatrix} x^2-4 & 1 \\ 0 & 1 \end{vmatrix} = 0$ **2** $\begin{vmatrix} x+2 & 3 \\ -2 & x-2 \end{vmatrix} = 1$

Solution

1 $\because \begin{vmatrix} x^2-4 & 1 \\ 0 & 1 \end{vmatrix} = (x^2-4) \times 1 - 0 \times 1 = x^2 - 4$

$\therefore x^2 - 4 = 0$ $\therefore x^2 = 4$

$\therefore x = \pm\sqrt{4} = \pm 2$

2 $\because \begin{vmatrix} x+2 & 3 \\ -2 & x-2 \end{vmatrix} = (x+2)(x-2) - (-2) \times 3 = x^2 - 4 + 6 = x^2 + 2$

$\therefore x^2 + 2 = 1$ $\therefore x^2 = -1$

$\therefore x = \pm\sqrt{-1}$

$\therefore x = i$ or $x = -i$ (where $i^2 = -1$)

TRY TO SOLVE

Find the value of x which satisfies the equation :

$\begin{vmatrix} 2x & -2 \\ 4 & 1 \end{vmatrix} = 6$

The third order determinant

If A is a square matrix of order 3 × 3 where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

, then the determinant of the matrix A is denoted by the symbol $|A|$ and is called

determinant of the third order where $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

And before knowing how to expand the third order determinant , we will recognize the "minor determinant" corresponding to any element of the matrix A , and how to determine its sign.

For every element of the matrix A there exists a minor determinant which we can get by elemenating the row and the column intersected at this element.

For example : We can get the minor determinant corresponding to each element of the first row as follows :

To get the minor determinant corresponding to a_{11} which is denoted by $\lvert a_{11} \rvert$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
To get the minor determinant corresponding to a_{12} which is denoted by $\lvert a_{12} \rvert$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
To get the minor determinant corresponding to a_{13} which is denoted by $\lvert a_{13} \rvert$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- To determine the sign of the minor determinant of any element of a matrix , we add the two orders of the row and the column intersected at this element.

If the sum of the two orders is :
- **even :** then the sign is positive.
- **odd :** then the sign is negative.

For example :

- The sign of $|a_{11}|$ is positive because $1 + 1 = 2$ (even)
- The sign of $|a_{12}|$ is negative because $1 + 2 = 3$ (odd)
- The sign of $|a_{13}|$ is positive because $1 + 3 = 4$ (even)

- Hence , we can write the rule of signs of the minor determinant as in the opposite figure :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Note that :

The sign of the minor determinant corresponding to the element a_{ij} is determined by the rule : $(-1)^{i+j}$

Expanding the third order determinant

It is possible to expand the third order determinant in terms of the elements of any row or column and its minor determinants , taking into account the rule of signs.

For example : If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then :

* $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ (Using the elements of the first row)

* $|A| = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ (Using the elements of the second column)

Example 3

Find the value of the determinant : $\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix}$

Solution

Using the elements of the first row , we find that :

$$\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 2(0 \times (-3) - 1 \times 4) - 3(-2 \times (-3) - 1 \times 4) - (-2 \times 1 - 1 \times 0)$$

$$= 2(0 - 4) - 3(6 - 4) - (-2 - 0)$$

$$= 2 \times (-4) - 3 \times 2 + 2 = -12$$

Remark

It is possible to expand the determinant using any row or column as mentioned before , and we will expand it again using the elements of the second column taking into account the rule of signs.

$$\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix} = -3\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ -2 & 4 \end{vmatrix}$$

$$= -3\left(-2 \times (-3) - 1 \times 4\right) + 0 - \left(2 \times 4 - (-2) \times (-1)\right)$$

$$= -3\,(6 - 4) - (8 - 2) = -3 \times 2 - 6 = -12$$

Which is the same result we get before (try to use any other row or column)

Example 4

Find the value of the determinant : $\begin{vmatrix} 4 & -1 & 3 \\ 0 & 5 & -2 \\ 0 & -3 & -1 \end{vmatrix}$

Solution

It is preferable to expand this determinant in terms of the elements of the first column because of the existence of the greatest number of zeroes

$\therefore$ The value of the determinant $= 4\begin{vmatrix} 5 & -2 \\ -3 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ -3 & -1 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix}$

$$= 4\left(5 \times (-1) - (-3) \times (-2)\right) - 0 + 0$$

$$= 4\,(-5 - 6) = 4 \times (-11) = -44$$

TRY TO SOLVE

Find the value of the determinant : $\begin{vmatrix} 3 & 0 & -5 \\ -2 & 4 & 1 \\ 7 & -3 & 6 \end{vmatrix}$

The determinant of the triangular matrix

The triangular matrix

It is a matrix in which all its elements above or below the principal diagonal are zeros as :

$$\begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix} , \begin{pmatrix} -1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{pmatrix} , \begin{pmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 4 & 2 & 7 \end{pmatrix}$$

The value of the determinant of the triangular matrix equals the product of the elements of its principal diagonal.

i.e. $\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}\, a_{22}$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\, a_{22}\, a_{33}$

Proof : $\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}\, a_{22} - 0 \times a_{12} = a_{11}\, a_{22} - 0 = a_{11}\, a_{22}$

, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - 0 \begin{vmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{vmatrix} + 0 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$ (Using the elements of the first column)

$= a_{11}\,(a_{22}\, a_{33} - a_{23} \times 0) = a_{11}\, a_{22}\, a_{33}$

, then $\begin{vmatrix} 5 & 0 \\ -3 & 2 \end{vmatrix} = 10$, $\begin{vmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 4 & 2 & 7 \end{vmatrix} = 2 \times (-3) \times 7 = -42$

TRY TO SOLVE

Find the value of the determinant : $\begin{vmatrix} -1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{vmatrix}$

Example 5

Solve the equation : $\begin{vmatrix} x & 0 & 1 \\ 8 & 1-x & -x \\ x & -1 & 1+x \end{vmatrix} = 0$

Solution

Expanding the determinant :

$\therefore x \begin{vmatrix} 1-x & -x \\ -1 & 1+x \end{vmatrix} - 0 \begin{vmatrix} 8 & -x \\ x & 1+x \end{vmatrix} + 1 \begin{vmatrix} 8 & 1-x \\ x & -1 \end{vmatrix} = 0$

$\therefore x\Big((1-x)(1+x) - (-x) \times (-1)\Big) - 0 + 1\Big(8 \times (-1) - (1-x) \times x\Big) = 0$

$\therefore x\,(1 - x^2 - x) + 1\,(-8 - x + x^2) = 0$ $\quad\therefore x - x^3 - x^2 - 8 - x + x^2 = 0$

$\therefore -x^3 - 8 = 0$ $\quad\therefore x^3 = -8$

$\therefore\ x = -2$

TRY TO SOLVE

Solve the equation : $\begin{vmatrix} x & 2 & -2 \\ 2 & x & -2 \\ -2 & 2 & x \end{vmatrix} = 0$

Example 6

If A is a matrix of order 2 × 2 and | A | = 7 , find | 3 A |

Solution

Let $A = \begin{pmatrix} x & y \\ z & \ell \end{pmatrix}$ $\therefore |A| = x\ell - yz = 7$ (1)

, $3A = 3\begin{pmatrix} x & y \\ z & \ell \end{pmatrix} = \begin{pmatrix} 3x & 3y \\ 3z & 3\ell \end{pmatrix}$

$\therefore |3A| = \begin{vmatrix} 3x & 3y \\ 3z & 3\ell \end{vmatrix} = 9x\ell - 9yz = 9(x\ell - yz)$ (2)

from (1) , (2) : $\therefore |3A| = 9 \times 7 = 63$

from the previous example we can conclude that :

Remarks

1. If A is a matrix of order $n \times n$, $K \in \mathbb{R}$, then $|KA| = K^n |A|$

For example :

- If A is a matrix of order 2×2 and $|A| = 3$, then $|5A| = 5^2 \times |A| = 25 \times 3 = 75$
- If A is a matrix of order 3×3 and $|A| = 5$, then $|2A| = 2^3 \times |A| = 8 \times 5 = 40$

2. If A is a square matrix then $|A| = |A^t|$
3. If A and B are two square matrices such that AB exists then , $|AB| = |A| \times |B|$

Finding the area of a triangle by using determinants

We can use determinants to find the area of a triangle in terms of the coordinates of its vertices as follows :

If XYZ is a triangle where X (a , b) , Y (c , d) , Z (e , f)

, then the area of Δ XYZ is | A |

Where $A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$

x – coordinates of the triangle vertices

y – coordinates of the triangle vertices

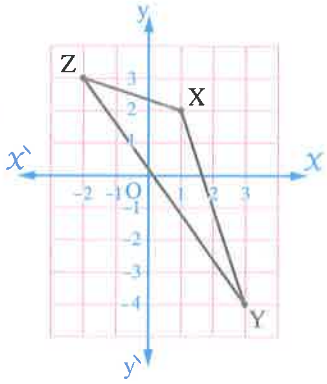
Remember that

| A | means the absolute value of A (i.e. only its positive value)

And we will represent the proof of the previous law at the end of this lesson as an enrich activity.

Example 7

Find using determinants the area of the opposite triangle whose vertices are X (1 , 2) , Y (3 , – 4) , Z (– 2 , 3)



Solution

$$\because A = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 1 \\ -2 & 3 & 1 \end{vmatrix}$$

By using the elements of the third column

$$\therefore A = \frac{1}{2}\left[\begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left((9-8)-(3+4)+(-4-6)\right)$$

$$= \frac{1}{2}(1-7-10) = -8$$

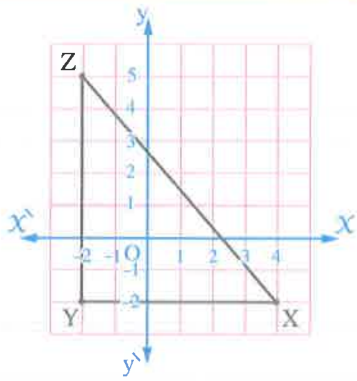
$\therefore$ The area of Δ XYZ = | A | = | – 8 | = 8 square units.

Note that we used the elements of the third row to expand the determinant because it is easily for performing mathematical operations because of the existence of ones.

TRY TO SOLVE

In the opposite figure :

XYZ is a triangle where X = (4 , –2) , Y = (–2 , –2) , Z = (–2 , 5) find using determinants the area of Δ XYZ , and check your answer using the rule of calculating the triangle area.



Remark

To prove that the three points X (a , b) , Y (c , d) , Z (e , f) are collinear by using determinants , then we prove that : $\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} = 0$

Example 8

Prove using determinants that the points (–2 , 4) , (3 , 0) , (8 , –4) are collinear.

Solution

$$\because \begin{vmatrix} -2 & 4 & 1 \\ 3 & 0 & 1 \\ 8 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 8 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 4 \\ 8 & -4 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix}$$

$$= (-12 - 0) - (8 - 32) + (0 - 12)$$

$$= -12 + 24 - 12 = 0$$

$\therefore$ The points (– 2 , 4) , (3 , 0) , (8 , – 4) are collinear.

TRY TO SOLVE

Prove using determinants that the points (4 , 4) , (2 , 1) , (–2 , – 5) are collinear.

Solving a system of linear equations by Cramer's rule

First Solving a system of linear equations in two variables

- Solving a system of linear equations in two variables means to find the values of the two variables satisfying the two equations together.
- If we have a system of linear equations in two variables as follows : $a x + b y = m$, $c x + d y = n$, then to solve this system we do the following :

1 Find the values of three determinants , after putting the two equations in the previous form , and these determinants are :

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	• Is called the determinant of the matrix of coefficients and denoted by the symbol Δ (is read as delta) • We get it by putting the two coefficients of x in the two equations in the first column , and the two coefficients of y in the two equations in the second column.
$\begin{vmatrix} m & b \\ n & d \end{vmatrix}$	• Is called the determinant of the variable x and denoted by the symbol Δ_x (is read as delta x) • We get it from the determinant Δ by changing the elements of the first column (coefficients of x) by the constants m and n
$\begin{vmatrix} a & m \\ c & n \end{vmatrix}$	• Is called the determinant of the variable y and is denoted by the symbol Δ_y (is read as delta y) • We get it from the determinant Δ by changing the elements of the second column (coefficients of y) by the constants m and n

2 Find the values of x and y as follows (where $\Delta \neq 0$) :

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{md - nb}{ad - cb}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{an - cm}{ad - cb}$$

* **Note that :** If $\Delta \neq 0$, then the system has a unique solution while if $\Delta = 0$, then the system either has an infinite number of solutions or has no solution.

The following example shows the previous steps.

Example 9

Solve the system of the following equations using Cramer's rule :

$6x - 5y = -23$, $3x + 3y = 16$

Solution

$$\Delta = \begin{vmatrix} 6 & -5 \\ 3 & 3 \end{vmatrix} = 6 \times 3 - 3 \times (-5) = 18 + 15 = 33$$

$$, \Delta_x = \begin{vmatrix} -23 & -5 \\ 16 & 3 \end{vmatrix} = -23 \times 3 - 16 \times (-5) = -69 + 80 = 11$$

$$, \Delta_y = \begin{vmatrix} 6 & -23 \\ 3 & 16 \end{vmatrix} = 6 \times 16 - 3 \times (-23) = 96 + 69 = 165$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{11}{33} = \frac{1}{3}$$

$$, y = \frac{\Delta_y}{\Delta} = \frac{165}{33} = 5$$

and the S.S. $= \left\{\left(\frac{1}{3}, 5\right)\right\}$

Remark

You can check your answer by substituting the values of x and y in the two equations.

TRY TO SOLVE

Solve the following two equations using Cramer's rule :

$4x + 3y = -4$, $3x - y = -3$

Second Solving a system of linear equations in three variables

If we have a system of linear equations in three variables as follows :

1 $a_1 x + b_1 y + c_1 z = m$ **2** $a_2 x + b_2 y + c_2 z = n$ **3** $a_3 x + b_3 y + c_3 z = k$

, then similarly as we did in case of system of linear equations in two variables :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{determinant of the coefficients}$$

$$\Delta_x = \begin{vmatrix} m & b_1 & c_1 \\ n & b_2 & c_2 \\ k & b_3 & c_3 \end{vmatrix} = \text{determinant of the variable } x$$

and we get it by changing the elements of the first column (coefficients of x) by the constants m , n , k

$$\Delta_y = \begin{vmatrix} a_1 & m & c_1 \\ a_2 & n & c_2 \\ a_3 & k & c_3 \end{vmatrix} = \text{determinant of the variable } y$$

and we get it by changing the elements of the second column (coefficients of y) by the constants m , n , k

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & m \\ a_2 & b_2 & n \\ a_3 & b_3 & k \end{vmatrix} = \text{determinant of the variable } z$$

and we get it by changing the elements of the third column (coefficients of z) by the constants m , n , k

Let $\Delta \neq 0$, then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$

The following example shows the previous steps.

Example 10

Solve the system of the following equations using Cramer's rule :

$3y + 2x = z + 1$, $3x + 2z = 8 - 5y$, $3z - 1 = x - 2y$

Solution

1 Put the equations system in the form $a x + b y + cz = m$ as follows :

$2x + 3y - z = 1$, $3x + 5y + 2z = 8$, $x - 2y - 3z = -1$

2 Find Δ , Δ_x , Δ_y , Δ_z as follows :

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = 2(-15+4) - 3(-9-2) + (-1)(-6-5)$$

$$= -22 + 33 + 11 = 22$$

$$, \Delta_x = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = 1(-15+4) - 3(-24+2) + (-1)(-16+5)$$

$$= -11 + 66 + 11 = 66$$

$$, \Delta_y = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix} = 2(-24+2) - 1(-9-2) + (-1)(-3-8)$$

$$= -44 + 11 + 11 = -22$$

$$, \Delta_z = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix} = 2(-5+16) - 3(-3-8) + 1(-6-5)$$

$$= 22 + 33 - 11 = 44$$

3 Find the variables x , y , z as follows :

$$x = \frac{\Delta_x}{\Delta} = \frac{66}{22} = 3 \quad , y = \frac{\Delta_y}{\Delta} = \frac{-22}{22} = -1 \quad , z = \frac{\Delta_z}{\Delta} = \frac{44}{22} = 2$$

$\therefore$ The S.S. = $\{(3 , -1 , 2)\}$

Remarks

- You can check your answer by substituting the three variables in each equation.
- (3 , –1 , 2) is called ordered triple.

TRY TO SOLVE

Solve the system of the following equations using Cramer's rule :

$2x + y - z = 3$, $x + y = 1 - z$, $x = 2y + 3z + 4$

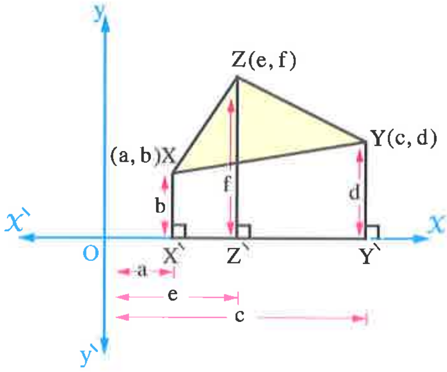
Remark

You can use the scientific calculator to find the value of the determinant and we will represent it at the end of the unit.

Remark

Scientific calculator can be used to calculate the value of the determinant and we will show that at the end of the unit.

Activity A method to prove the law of finding the area of the triangle using the determinants



Let XYZ be a triangle where

X (a , b) , Y (c , d) , Z (e , f) , then

The area of Δ XYZ = the area of the trapezium $XX'Z'Z$

+ the area of the trapezium $ZZ'Y'Y$

– the area of the trapezium $XX'Y'Y$

$$= \frac{b+f}{2}(e-a) + \frac{f+d}{2}(c-e) - \frac{b+d}{2}(c-a)$$

$$= \frac{1}{2}[(b+f)(e-a) + (f+d)(c-e) - (b+d)(c-a)]$$

$$= \frac{1}{2}[be - \cancel{ba} + \cancel{fe} - fa + fc - \cancel{fe} + \cancel{dc} - de - bc + \cancel{ba} - \cancel{dc} + da]$$

$$= \frac{1}{2}[be - fa + fc - de - bc + da] \qquad (1)$$

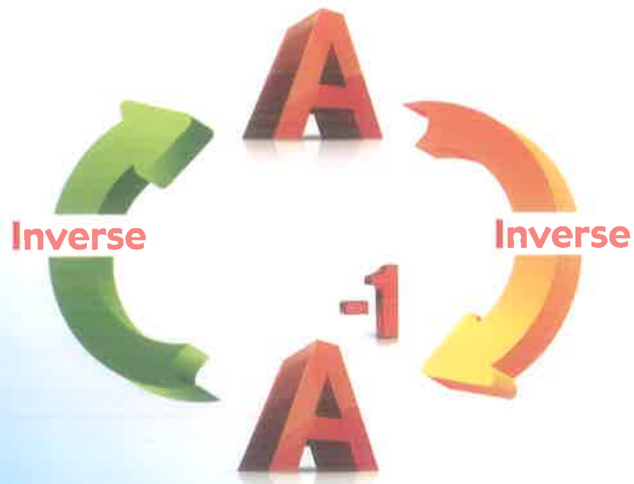
and by expanding the determinant : $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ using the elements of the third column , we find that :

$$\text{The determinant} = \frac{1}{2}\left[\begin{vmatrix} c & d \\ e & f \end{vmatrix} - \begin{vmatrix} a & b \\ e & f \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix}\right]$$

$$= \frac{1}{2}[cf - ed - af + eb + ad - bc] \qquad (2)$$

by comparing the result which we get in (1) and the result which we get in (2) , we find that :

The area of Δ XYZ = $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ (in condition of taking the absolute value of the result)



Lesson Five — Multiplicative inverse of a matrix

If A and B are two square matrices and each of them is of order 2×2 and $AB = BA = I$ where I is the unit matrix of order 2×2, then each of the two matrices A and B is the multiplicative inverse of the other.

For example :

If $A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix}$

, then $AB = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

, $BA = \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

i.e. $AB = BA = I$

$\therefore$ Each of the two matrices A and B is the multiplicative inverse of the other.

Remark

If the matrix $A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix}$ and the matrix $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$

then A is not the multiplicative inverse of B although $AB = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, that is because the matrices A and B are not square matrices.

How to find the multiplicative inverse of a 2 × 2 matrix ?

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the multiplicative inverse of the matrix A which is denoted by the symbol A^{-1} is defined (existed) when the determinant of $A = \Delta \neq 0$, then

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{where} \quad AA^{-1} = A^{-1}A = I$$

Example 1

Find the multiplicative inverse if it existed of each of the following matrices :

1 $A = \begin{pmatrix} -2 & 2 \\ 3 & -4 \end{pmatrix}$

2 $B = \begin{pmatrix} \frac{1}{2} & 2 \\ 3 & 12 \end{pmatrix}$

Solution

1 $\because \Delta = \text{determinant of } A = \begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} = (-2)(-4) - (3)(2) = 2$

$\therefore \Delta \neq 0$

$\therefore$ The matrix A has a multiplicative inverse

$\because A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & -2 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ \frac{-3}{2} & -1 \end{pmatrix}$

2 $\because \Delta = \text{determinant of } B = \begin{vmatrix} \frac{1}{2} & 2 \\ 3 & 12 \end{vmatrix} = \left(\frac{1}{2}\right)(12) - (2)(3) = 0$

$\therefore B^{-1}$ is not defined (not existed)

TRY TO SOLVE

Find the multiplicative inverse if it possible of the matrix : $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

Example 2

Find the real values of x which make the matrix A has a multiplicative inverse in each of the following :

1 $A = \begin{pmatrix} x & 3 \\ 12 & x \end{pmatrix}$

2 $A = \begin{pmatrix} x-1 & 4 \\ 3 & x-2 \end{pmatrix}$

Solution

1 The matrix A has no multiplicative inverse when $|A| = 0$

i.e. $\begin{vmatrix} x & 3 \\ 12 & x \end{vmatrix} = 0$ $\qquad \therefore x^2 - 36 = 0$ $\qquad \therefore x = \pm 6$

$\therefore$ The matrix A has no multiplicative inverse when $x = \pm 6$

$\therefore$ The matrix A has a multiplicative inverse when $x \in \mathbb{R} - \{-6, 6\}$

2 The matrix A has no multiplicative inverse when $|A| = 0$

i.e. $\begin{vmatrix} x-1 & 4 \\ 3 & x-2 \end{vmatrix} = 0$ $\qquad \therefore (x-1)(x-2) - 12 = 0$

$\therefore x^2 - 3x + 2 - 12 = 0$ $\qquad \therefore x^2 - 3x - 10 = 0$

$\therefore (x-5)(x+2) = 0$ $\qquad \therefore x = 5$ or $x = -2$

$\therefore$ The matrix A has no multiplicative inverse when $x = 5$ or $x = -2$

$\therefore$ The matrix A has a multiplicative inverse when $x \in \mathbb{R} - \{5, -2\}$

TRY TO SOLVE

Find the real values of x which make the matrix : $A = \begin{pmatrix} x-1 & 4 \\ 2 & x+1 \end{pmatrix}$ has a multiplicative inverse.

Example 3

If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ **,** $B = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$ **, then prove that :**

1 $(A^{-1})^{-1} = A$

2 $(AB)^{-1} = B^{-1} A^{-1}$

Solution

1 $\because |A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (1)(4) - (3)(2) = -2$ $\qquad \therefore \Delta \neq 0$

$\therefore A^{-1}$ is defined (existed)

$, \because A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

$, \because |A^{-1}| = \begin{vmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{vmatrix} = (-2)\left(\frac{-1}{2}\right) - \left(\frac{3}{2}\right)(1) = \frac{-1}{2} \neq 0$

$\therefore (A^{-1})^{-1}$ is defined (existed)

$$\therefore (A^{-1})^{-1} = \frac{1}{\frac{-1}{2}} \begin{pmatrix} \frac{-1}{2} & \frac{-3}{2} \\ -1 & -2 \end{pmatrix} = -2 \begin{pmatrix} \frac{-1}{2} & \frac{-3}{2} \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = A$$

2 $\because AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ -8 & 2 \end{pmatrix}$

$\therefore |AB| = \begin{vmatrix} -7 & 2 \\ -8 & 2 \end{vmatrix} = (-7)(2) - (2)(-8) = 2 \neq 0$

$\therefore (AB)^{-1}$ is existed. $\therefore (AB)^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -2 \\ 8 & -7 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & \frac{-7}{2} \end{pmatrix}$ (1)

$, |B| = \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = (2)(1) - (-1)(-3) = -1$ $\therefore B^{-1}$ is existed.

$\therefore B^{-1} = \frac{-1}{1} \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$ $, \because A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

$\therefore B^{-1} A^{-1} = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & \frac{-7}{2} \end{pmatrix}$ (2)

From (1) and (2), we get that : $(AB)^{-1} = B^{-1} A^{-1}$

TRY TO SOLVE

Using the two matrices A and B in the previous example, prove that : $(A^{-1} B)^{-1} = B^{-1} A$

Remark

If A is a square matrix of order 2×2 where $|A| \neq 0$, C is another matrix and :

1. $AX = C$ then $X = A^{-1} C$

   By multiplying the two sides of the equation by A^{-1}

   $\therefore A^{-1} A X = A^{-1} C$ $\therefore IX = A^{-1} C$ $\therefore X = A^{-1} C$

2. $XA = C$ then $X = C A^{-1}$

Notice that :

$A^{-1} A = I, I X = X$

Example 4

Find the matrix X which satisfies that : $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \times X = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$, $C = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\therefore$ The equation is $AX = C$ $\qquad\qquad$ $\therefore X = A^{-1} C$

, $\because |A| = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = (2)(0) - (-1)(3) = 3 \neq 0$

$$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 0 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

TRY TO SOLVE

Find the matrix X which satisfies that : $X \times \begin{pmatrix} 3 & 7 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 0 & 3 \end{pmatrix}$

Solving two simultaneous equations by using the multiplicative inverse of a matrix

To solve two linear simultaneous equations in the form : $a_1 x + b_1 y = c_1$, $a_2 x + b_2 y = c_2$ by using the multiplicative inverse of a matrix , follow the following :

1 Write the two equations in the form of a matrix equation :

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ **i.e.** in the form $AX = C$ where

$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is called the matrix of coefficients

, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is called the matrix of variables and $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is called the matrix of constants.

2 Find the solution of the matrix equation :

$AX = C$, then $X = A^{-1} C$ and from that we deduce the values of the variables x and y

Example 5

Solve each system of the following linear equations using the matrices :

1 $2x + 3y = 7 \quad , \quad x - y = 1$ **2** $x = 2y - 1 \quad , \quad 3y = 2x$

Solution

1 The matrix equation is AX = C , where $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $C = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

$\because \Delta = |A| = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (3)(1) - 5 \neq 0$

$\therefore$ For the matrix A , there is a multiplicative inverse A^{-1}

$\therefore A^{-1} = \frac{1}{-5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{pmatrix}$, $\because X = A^{-1}C$

$\therefore X = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore x = 2$, $y = 1$, and the solution = $\{(2 , 1)\}$

2 $x - 2y = -1 \quad , \quad 2x - 3y = 0$

The matrix equation is AX = C , where $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\because \Delta = |A| = \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = (1)(-3) - (-2)(2) = 1 \neq 0$

$\therefore A^{-1} = \frac{1}{1}\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$

, $\because X = A^{-1}C$ $\therefore X = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\therefore x = 3$, $y = 2$, and the solution set = $\{(3 , 2)\}$

TRY TO SOLVE

Solve the system of the following equations using the matrices :

$x + 2y = 4 \quad , \quad y = 2x + 7$

Example 6

If the curve of the function $f : f(X) = aX^2 + b$ passes through the two points $(2, 0)$ and $(-1, -3)$ **use the matrices to find the value of the two constants :** a and b

Solution

$\because$ The curve of the function f passes through the point $(2, 0)$

$\therefore f(2) = 0$ $\qquad\qquad$ $\therefore a \times (2)^2 + b = 0$

$\therefore 4a + b = 0$ $\qquad\qquad$ (1)

, $\because$ the curve of the function f passes through the point $(-1, -3)$

$\therefore f(-1) = -3$ $\qquad\qquad$ $\therefore a \times (-1)^2 + b = -3$

$\therefore a + b = -3$ $\qquad\qquad$ (2)

To solve the two equations (1) and (2) , we write the matrix equation AX = C ,

where $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} a \\ b \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

$\therefore X = A^{-1} C$

, $\because \Delta = |A| = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} (4)(1) - (1)(1) = 3$

$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{4}{3} \end{pmatrix}$

$\therefore X = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore a = 1$, $b = -4$

Remark

We can use the scientific calculator to find the multiplicative inverse of a matrix and we will represent it at the end of the unit.

Technological Activity on Unit One

Using scientific calculator in matrices

We can use scientific calculator which supports matrices for many operations which related to matrices like :

- Finding the transpose of the matrix.
- Performance of adding and subtracting operations on the matrices.
- Finding the value of the determinant.
- Finding the multiplicative inverse of the matrix.

and what we let here , will be by using the calculator of the kind (CASIO fx-991ES PLUS)

First **Entering of the matrix** $A = \begin{pmatrix} -7 & 0 \\ 4 & 7 \end{pmatrix}$:

- Press successively the following buttons from left to right :

→ MODE 6 1 5

and this for choosing a matrix of order 2×2 , then enter the elements of the matrix A by pressing successively the following buttons :

Entering of the elements of the first row → (−) 7 = 0 =

Entering of the elements of the second row → 4 = 7 =

Second **Entering of the matrix** $B = \begin{pmatrix} -8 & 4 \\ 0 & 7 \end{pmatrix}$:

- Press successively the following buttons from left to right :

→ SHIFT [MATRIX] 4 2 2 5

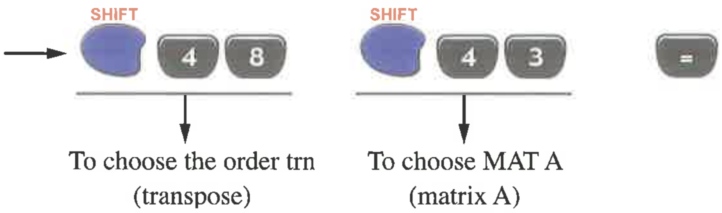
for choosing another matrix of order 2×2 , then enter the elements of the matrix B by pressing successively the following buttons :

Entering of the elements of the first row → (−) 8 = 4 =

Entering of the elements of the second row → 0 = 7 = AC

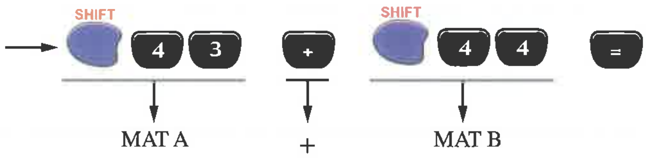
Now , we entered the two matrices A and B , and we can do some of the operations on them as the following :

1 To find A^t , press successively the buttons from left to right :



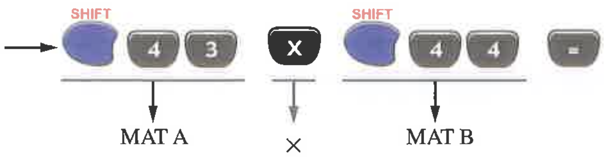
The matrix $\begin{pmatrix} -7 & 4 \\ 0 & 7 \end{pmatrix}$ will appear on the screen which represents A^t

2 To find A + B , press successively the buttons from left to right :



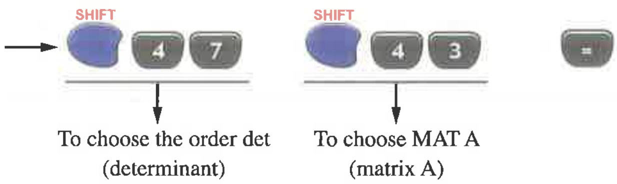
The matrix $\begin{pmatrix} -15 & 4 \\ 4 & 14 \end{pmatrix}$ will appear on the screen which represents A + B

3 To find AB , press successively the buttons from left to right :



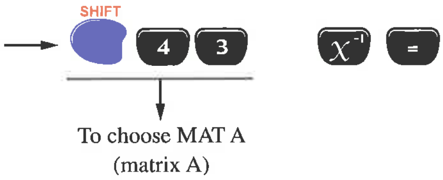
The matrix $\begin{pmatrix} 56 & -28 \\ -32 & 65 \end{pmatrix}$ will appear on the screen which represents AB

4 To find the value of the determinant of the matrix A , press successively the buttons from left to right :



– 49 will appear on the screen which represents the value of the determinant of the matrix A

5 To find the multiplicative inverse of the matrix A , press successively the buttons from left to right :



The matrix $\begin{pmatrix} -\frac{1}{7} & 0 \\ \frac{4}{49} & \frac{1}{7} \end{pmatrix}$ will appear on the screen which represents the multiplicative inverse of matrix A

6 To find $A^t + B$, press successively the buttons from left to right :

The matrix $\begin{pmatrix} -15 & 8 \\ 0 & 14 \end{pmatrix}$ will appear on the screen which represents $A^t + B$

TRY TO SOLVE

Use the calculator to find each of the following :

B^t , $A - B$, BA , the determinant B , the multiplicative inverse of B , $A + B^t$, $A^t B$ and BA^t

Unit 2

LINEAR PROGRAMMING

Unit Lessons

Lesson One	:	**Linear inequalities - Solving systems of linear inequalities graphically.**
Lesson Two	:	**Linear programming and optimization.**

Learning outcomes

By the end of this unit, the student should be able to :

- Solve first degree inequalities in one variable and represent the solution graphically.
- Solve first degree inequalities in two variables and determine the region of solution graphically.
- Solve a system of linear inequalities graphically.
- Solve life problems on systems of linear inequalities.
- Use linear programming to solve life mathematical problems.
- Record the data of a mathematical life problem in a suitable table , and transfer these data in the form of linear inequalities , then determine the region of solution graphically.
- Determine the objective function in terms of the coordinates and determine the points which belong to the solution set , giving the optimum solution to the objective function.

Lesson One

Linear inequalities - Solving systems of linear inequalities graphically

- Remember the properties of the inequality relation in $\mathbb{R}$:

> Assuming that a , b and c are three real numbers , then :
> - If $a \leq b$, then $a + c \leq b + c$ whether c is positive or negative.
> - If $a \leq b$, then $ac \leq bc$ if c is positive.
> - If $a \leq b$, then $ac \geq bc$ if c is negative.

- You can deduce the previous properties in cases of the other inequality relation signs «$\geq$, $>$, $<$»

Solving the first degree inequality in one variable graphically

- Each of the inequalities :

  $3x < 5$, $4 - x \geq 2x$, $3 \leq x < 6$

  is called an inequality of the first degree in one variable.

- Solving the inequality means finding all the elements of the substitution set which satisfy the inequality.

- The substitution set may be $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$
  and the following illustrative example shows how to solve the first degree inequality in the two cases.

Illustrative example

Show graphically the S.S. of the inequality : $3x + 10 > 1$

1 If the substitution set is $\mathbb{R}$

2 If the substitution set is $\mathbb{R} \times \mathbb{R}$

$$3x + 10 > 1$$

$$\therefore 3x > -9$$

$$\therefore x > -3$$

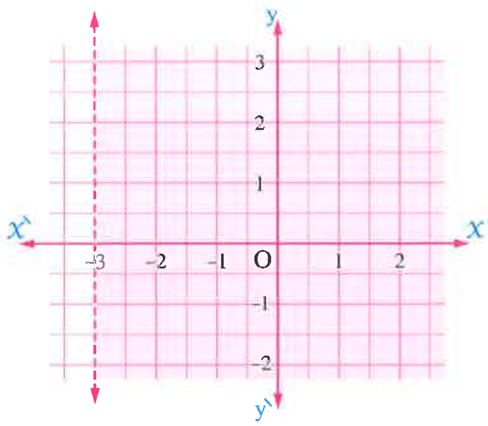
Case (1)

If the substitution set is $\mathbb{R}$, then the S.S. is represented on the number line.

- The S.S. is all the real numbers greater than – 3
- The S.S. is the part of the number line on the right of – 3
- The unclosed circle at – 3 means – 3 does not belong to the S.S.

Case (2)

If the substitution set is $\mathbb{R} \times \mathbb{R}$, then the S.S. is represented on a lattice.

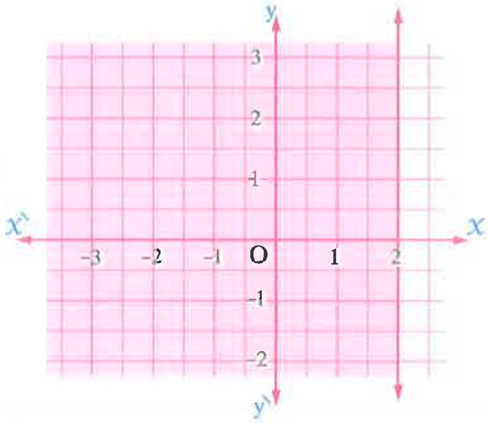


- The S.S. is all the ordered pairs whose x-projection is greater than – 3
- The S.S. is the region on the right of the straight line $x = -3$ (is called half plane).
- The straight line $x = -3$ is drawn dashed because its points don't belong to the S.S.

Unit 2

Example 1

Show graphically the S.S. of the inequality :
$5x - 7 \le 2x - 1$ in $\mathbb{R} \times \mathbb{R}$



Solution

$\because 5x - 7 \le 2x - 1$

$\therefore 5x - 2x \le -1 + 7$

$\therefore 3x \le 6$ $\qquad \therefore x \le 2$

Notice that

1. The shaded region is on the left of the straight line $x = 2$ because the inequality relation is "smaller than".
2. The straight line $x = 2$ is drawn solid because the inequality contains the symbol of equality **i.e.** $\le$

Example 2

Show graphically the S.S. of the inequality : $x - 1 \le 4x + 5 < x + 17$ where $x \in \mathbb{R}$

Solution

$\because x - 1 \le 4x + 5 < x + 17$ $\qquad \therefore -1 \le 3x + 5 < 17$

$\therefore -6 \le 3x < 12$ $\qquad \therefore -2 \le x < 4$ $\qquad \therefore$ The S.S. $= [-2 , 4[$

Example 3

Find graphically the S.S. of the inequality : $2x - 2 \le 3x - 1 < x + 5$ where $x \in \mathbb{R}$

Solution

Parting the inequality into two inequalities as the following :

$2x - 2 \le 3x - 1$	$3x - 1 < x + 5$
$\therefore 2x - 3x \le -1 + 2$	$\therefore 3x - x < 5 + 1$
$\therefore -x \le 1$ $\qquad \therefore x \ge -1$	$\therefore 2x < 6$ $\qquad \therefore x < 3$
$\therefore$ The S.S. $= [-1 , \infty[$	$\therefore$ The S.S. $=]-\infty , 3[$

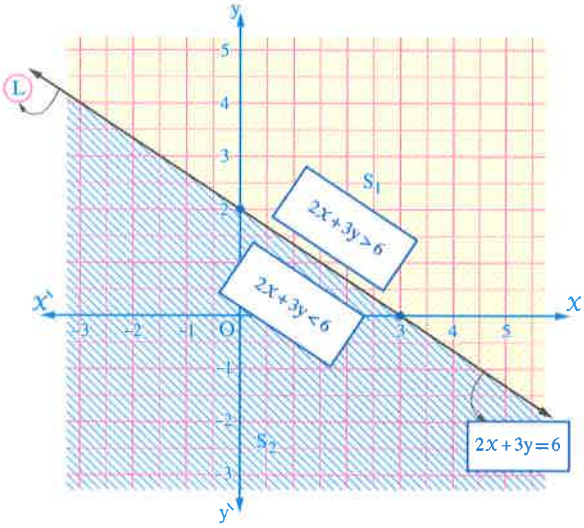
$\therefore$ The S.S. of the original inequality $= [-1 , \infty[\cap]-\infty , 3[= [-1 , 3[$

Solving the first degree inequality in two variables graphically

- We know that we can represent the linear equation : $2x + 3y = 6$ graphically by a straight line as follows :

x	0	3
y	2	0

«We should get a third ordered pair to check the graph»



- From the graph , we notice that this straight line divides the Cartesian plane into three sets of points :

1. The set of points of the straight line L (is called a boundary line) and each of these points satisfies that $2x + 3y = 6$
2. The set of points of the plane that lies on one side of the straight line L (and it is called a half plane) and is denoted by S_1 and each of them satisfies that $2x + 3y > 6$
3. The set of points of the plane that lies on the other side of the straight line L (and it is called a half plane also) and is denoted by S_2 and each of them satisfies that $2x + 3y < 6$

From the previous , we deduce that

- The half plane S_1 is the region representing the S.S. of the inequality : $2x + 3y > 6$
- The union of the points of the half plane S_1 and the straight line L represents the S.S. of the inequality : $2x + 3y \geq 6$
- The half plane S_2 is the region representing the S.S. of the inequality : $2x + 3y < 6$
- The union of the points of the half plane S_2 and the straight line L represents the S.S. of the inequality : $2x + 3y \leq 6$

Unit 2

Steps of solving the first degree inequality in two variables graphically

1. Represent the straight line equation related to the inequality by a solid line in case of $\geq$ or $\leq$, and by a dashed line in case of $>$ or $<$

2. Determine the half plane in which the feasible "or solution" region lies by choosing any point (x_1, y_1) belonging to one half plane as a test point and substitute it in the inequality.
   - If the chosen point satisfied the inequality, then the half plane containing this point is the feasible region of the inequality.
   - If the chosen point did not satisfy the inequality, then the other half plane is the feasible region of the inequality.

Remark

To make it easier, choose the origin point $(0, 0)$ if the boundary line does not pass through it.

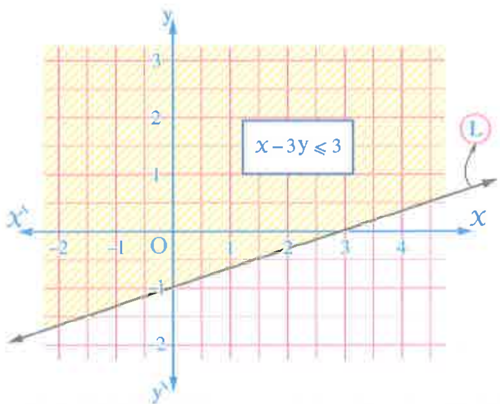
Example 4

Represent graphically the S.S. of the inequality : $x - 3y \leq 3$ in $\mathbb{R} \times \mathbb{R}$

Solution

1. Draw the boundary line L whose equation is :
   $x - 3y = 3$ as a solid straight line
   because the inequality relation is $\leq$
   using the following table :

x	0	3
y	-1	0



2. Choose the origin point as a test point.
   $\because$ The point $(0, 0)$ satisfies the inequality (because $0 < 3$)
   $\therefore$ The S.S. of the inequality is the boundary line L $\cup$ the half plane that contains the point $(0, 0)$ and that is represented by the shaded region in the previous graph.

Notice that

You can draw the boundary line L without the previous table by using the slope of the straight line and the intercepted part of the y-axis as you studied before.

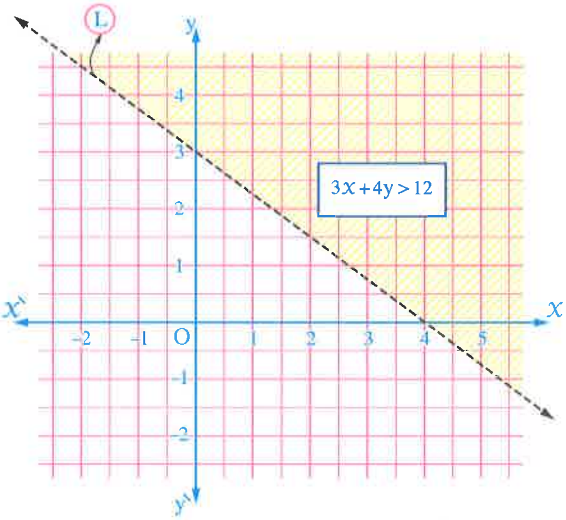
Example 5

Represent graphically the S.S. of the inequality : $3x + 4y > 12$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 Draw the boundary line L whose equation is : $3x + 4y = 12$ as a dashed line because the inequality relation is > using the following table :

x	0	4
y	3	0



2 Choose the origin point as a test point.

, ∵ (0 , 0) does not satisfy the inequality (because 0 < 12)

∴ The S.S. of the inequality is the half plane that does not contain the point (0 , 0) and is represented by the shaded region in the graph.

Remarks

- The equation : y = 0 is represented by the x-axis.
- The equation : $x = 0$ is represented by the y-axis.
- The equation : y = a is represented by a straight line parallel to the x-axis and passing through the point (0 , a)
- The equation : $x = a$ is represented by a straight line parallel to the y-axis and passing through the point (a , 0)
- The straight line whose equation is in the form : $\frac{x}{a} + \frac{y}{b} = 1$ passes through the two points (a , 0) and (0 , b)

TRY TO SOLVE

Represent graphically the S.S. of the inequality : $2x - 5y \leq 10$ in $\mathbb{R} \times \mathbb{R}$

Solving systems of linear inequalities graphically

To find the graphical solution of two inequalities , we do as the following :

1 We shade the region S_1 that represents the S.S. of the 1^{st} inequality.

2 We shade the region S_2 that represents the S.S. of the 2^{nd} inequality.

- The common region S of the two shaded regions S_1 and S_2 represents the S.S. of the two inequalities where $S = S_1 \cap S_2$

Example 6

Represent graphically the S.S. of the two inequalities :

$x + 3y \le 3$, $2x + y \le 4$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 Draw the boundary line $L_1 : x + 3y = 3$ as a solid line using the following table :

x	0	3
y	1	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 3$)

$\therefore$ The region S_1 is the S.S. of the inequality : $x + 3y \le 3$

and it is represented by $L_1 \cup$ the half plane in which the origin point lies [***Fig. (1)***]

2 Draw the boundary line $L_2 : 2x + y = 4$ as a solid line using the following table :

x	0	2
y	4	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 4$)

$\therefore$ The region S_2 is the S.S. of the inequality : $2x + y \le 4$

and it is represented by $L_2 \cup$ the half plane in which the origin point lies [***Fig. (2)***]

3 The S.S. of the two inequalities simultaneously is $S = S_1 \cap S_2$ and it is represented by the common region in the two shaded parts [***Fig. (3)***]

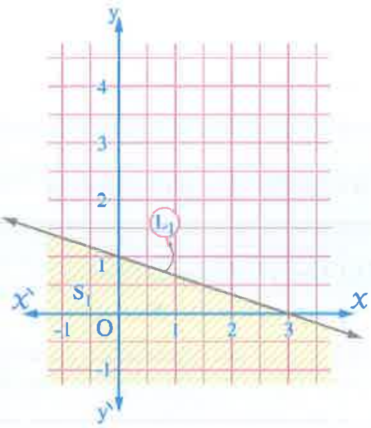


Fig. (1)

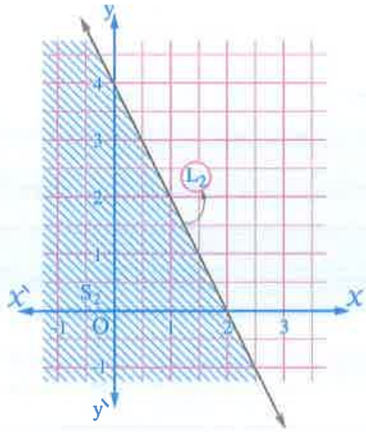


Fig. (2)

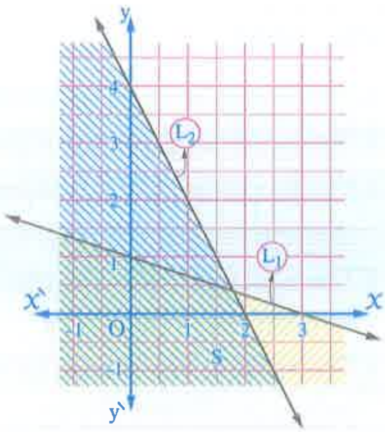


Fig. (3)

Remark

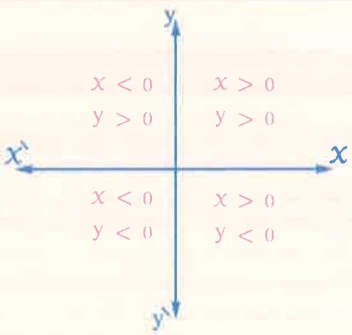
The two coordinate axes divide the Cartesian plane into four quadrants :

1^{st} quadrant : where $x > 0$, $y > 0$

, 2^{nd} quadrant : where $x < 0$, $y > 0$

, 3^{rd} quadrant : where $x < 0$, $y < 0$

and 4^{th} quadrant : where $x > 0$, $y < 0$



Example 7

Represent graphically the S.S. of the inequalities :

$x \geq 0$, $y \geq 0$, $y + 3x \leq 9$ and $y - x < 1$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 The S.S. of the two inequalities : $x \geq 0$ and $y \geq 0$

is represented by $\overrightarrow{OX} \cup \overrightarrow{Oy} \cup$ the 1st quadrant of the Cartesian plane.

2 Draw the boundary line $L_1 : y + 3x = 9$ as a solid line.

x	2	3
y	3	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 9$)

$\therefore$ The region S_1 is the S.S. of the inequality : $y + 3x \leq 9$ and it is represented by $L_1 \cup$ the half plane in which the point (0 , 0) lies [*Fig. (1)*]

3 Draw the boundary line $L_2 : y - x = 1$ as a dashed line.

x	0	−1
y	1	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 1$)

$\therefore$ The region S_2 is the S.S. of the inequality : $y - x < 1$ and it is represented by the half plane in which the point (0 , 0) lies [*Fig. (2)*]

4 S is the S.S. of the four inequalities which is represented by the region in the 1st quadrant that has the common shade [*Fig. (3)*]

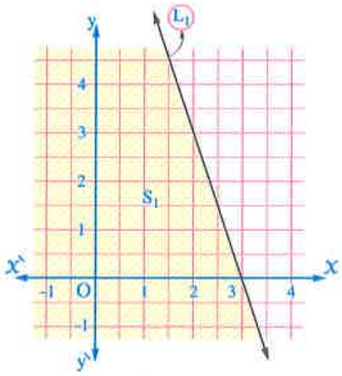


Fig. (1)

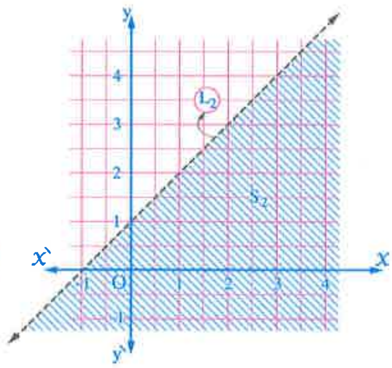


Fig. (2)

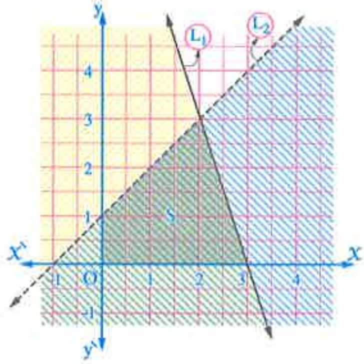


Fig. (3)

Remark

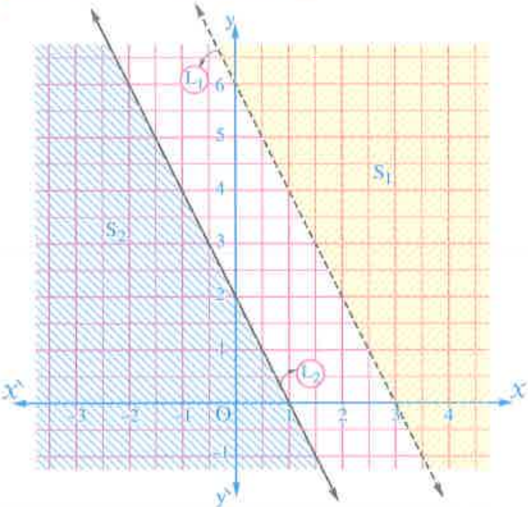
In the previous two examples , we draw a separate figure to show the feasible region of each inequality. After that we deduced the last figure which shows the feasible region of all the inequalities simultaneously. You (after some practice) won't be in need of drawing all these figures, but you will satisfy the last figure only.

Example 8

Represent graphically the S.S. of the inequalities :

$2x + y > 6$, $4x + 2y \leq 4$ in $\mathbb{R} \times \mathbb{R}$

Solution



1 Draw the boundary line $L_1 : 2x + y = 6$ as a dashed line that passes through the two points (0 , 6) and (3 , 0)

, ∵ the point (0 , 0) does not satisfy the inequality.

∴ The region S_1 is the S.S. of the inequality : $2x + y > 6$ and it is represented by the half plane which does not contain the origin point.

2 Draw the boundary line $L_2 : 4x + 2y = 4$ as a solid line that passes through the two points (0 , 2) and (1 , 0)

, ∵ the point (0 , 0) satisfies the inequality.

∴ The region S_2 is the S.S of the inequality : $4x + 2y \leq 4$ and it is represented by $L_2 \cup$ the half plane which contains the origin point.

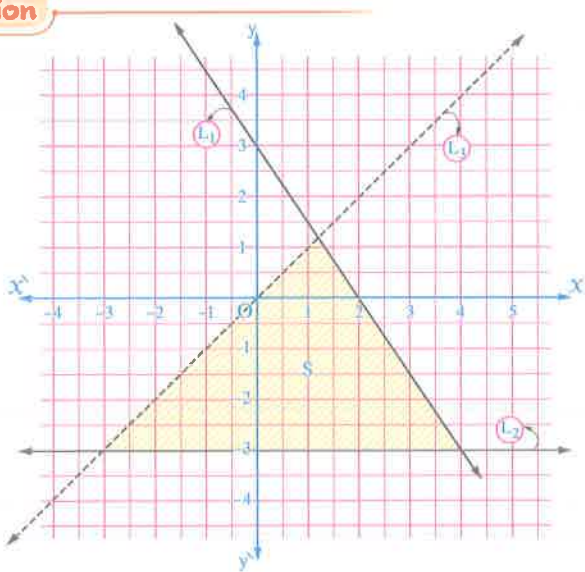
3 The S.S. of the two inequalities simultaneously is $S = S_1 \cap S_2 = \varnothing$

Example 9

Represent graphically the S.S. of the following inequalities :

$3x + 2y \leq 6$, $y + 3 \geq 0$ and $x - y > 0$ in $\mathbb{R} \times \mathbb{R}$

Solution



1 Draw the boundary line $L_1 : 3x + 2y = 6$ as a solid line that passes through the two points (2 , 0) and (0 , 3)

, ∵ the point (0 , 0) satisfies the inequality (because $0 < 6$)

∴ The S.S. (S_1) is represented by $L_1 \cup$ the half plane in which the origin point lies.

2 Draw the boundary line $L_2 : y = -3$ as a solid line [A straight line is parallel to the x-axis and passes through the point $(0 , -3)$]

, $\because$ the point $(0 , 0)$ satisfies the inequality (because $0 > -3$)

$\therefore$ The S.S. (S_2) is represented by $L_2 \cup$ the half plane in which the origin point lies.

3 Draw the boundary line $L_3 : x - y = 0$

as a dashed line that passes through the two points $(0 , 0)$ and $(1 , 1)$

, $\because$ the point $(0 , 2)$ does not satisfy the inequality (because $-2 \ngtr 0$)

$\therefore$ The S.S. (S_3) is represented by the half plane in which the point $(0 , 2)$ does not lie.

4 The S.S. of the three inequalities simultaneously is $S = S_1 \cap S_2 \cap S_3$

and it is represented by the shaded region in the shown Cartesian plane.

Example 10

A factory for children toys produces cars and planes. It produces 250 toys daily at most. If the cost of one car is L.E. 15 and of one plane is L.E. 10 and the total cost of the daily production is not more than L.E. 3000 , write a system of linear inequalities representing the previous, then represent graphically the solution region of this system.

Solution

Let the number of cars = x , the number of planes = y

- The system of inequalities is :

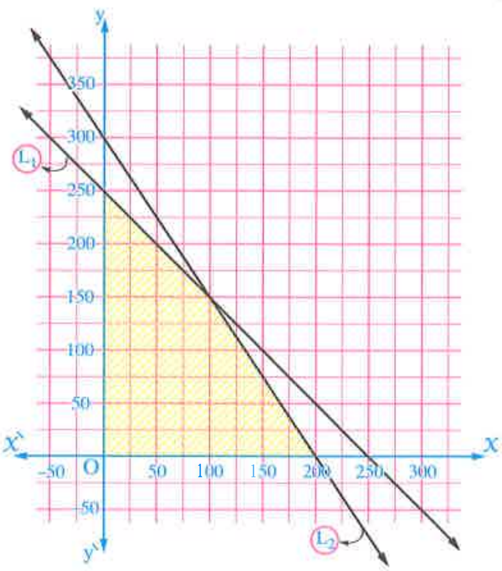
1 $x \geq 0$ **2** $y \geq 0$

3 $x + y \leq 250$

4 $15x + 10y \leq 3000$ **i.e.** $3x + 2y \leq 600$

- Determining the region which represents the S.S. of the inequalities as follows :

1 The inequalities $x \geq 0$, $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the 1st quadrant.



2 Draw the boundary line L_1 : $x + y = 250$ as a solid line that passes through the two points (0 , 250) and (250 , 0)

, $\because$ the point (0 , 0) satisfies the inequality (because $0 < 250$)

$\therefore$ The S.S. of this inequality is represented by $L_1 \cup$ the half plane in which the origin point lies.

3 Draw the boundary line L_2 : $3x + 2y = 600$ as a solid line that passes through the two points (0 , 300) and (200 , 0)

, $\because$ the point (0 , 0) satisfies the inequality (because $0 < 600$)

$\therefore$ The S.S. of this inequality is represented by $L_2 \cup$ the half plane in which the origin point lies.

4 The ordered pairs that its x-coordinates and y-coordinates are integers in the shaded region is the S.S. of the required system of linear inequalities.

Lesson TWO — Linear programming and optimization

Linear programming

It is one of the scientific methods that is used to give the best decision of solving a problem or it is the optimal solution that satisfies a certain object in view of some restrictions and available abilities or materials where the object can be put in the form of a linear function called "the objective function" and the stipulations and available abilities are put in the form of linear inequalities.

The method of linear programming depends on :

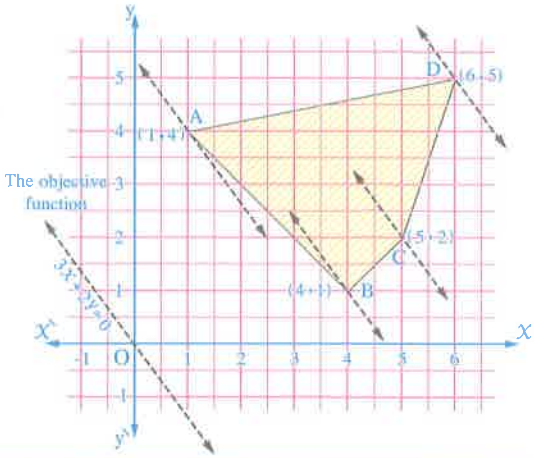
1. Representing the system of inequalities that expresses the stipulations such that we obtain a ribbed region representing the S.S. of the inequalities.

   Often the restrictions contain the two inequalities : $x \geq 0$, $y \geq 0$, that means the S.S. (the region representing the S.S.) lies in the first quadrant.

2. Determining the objective function in the form $P = \ell x + m y$ where ℓ and m are constants we represent the equation $\ell x + m y = 0$ by a straight line that passes through the origin point , then we let this straight line move parallel to itself upwards till it passes through the vertices of the polygon that determines the region of the S.S.
   Since all these parallel straight lines have the same slope and differ only in the value of "P" and each point (x , y) belonging to the S.S. and to the same straight line gives a value to the number "P"

   So , we can determine the greatest value or the smallest value of the objective function.

For example :



If the S.S. representing the set of inequalities that represents the restrictions is the shaded region in the opposite graph and the required is finding the greatest and smallest value of the expression $P = 3x + 2y$, then we substitute by the coordinates of the points : A , B , C and D "the vertices of the polygon" in the objective function.

Notice that

The value of the objective function at any point that lies on a side of the shaded region sides is included between its values at the two vertices of the polygon for the side that joins them.

$\therefore [P]_A = 3 \times 1 + 2 \times 4 = 11$, $[P]_B = 3 \times 4 + 2 \times 1 = 14$

, $[P]_C = 3 \times 5 + 2 \times 2 = 19$, $[P]_D = 3 \times 6 + 2 \times 5 = 28$

So , we find that the greatest value is 28 at the vertex D (6 , 5) and the smallest value is 11 at the vertex A (1 , 4)

Example 1

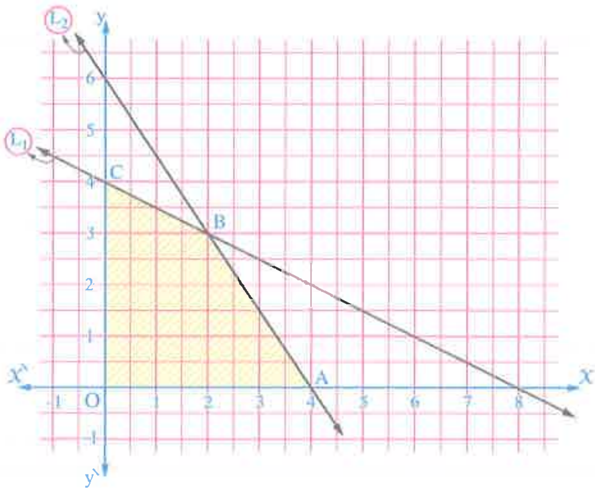
Determine the S.S. of the following inequalities simultaneously :

$x \geq 0$, $y \geq 0$, $x + 2y \leq 8$ and $3x + 2y \leq 12$

Then find from the S.S. (x , y) that makes "P" maximum where $P = 50x + 75y$

Solution

First Determine the region that represents the S.S. of the inequalities :



1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : x + 2y = 8$ (as a solid line) that passes through the two points (0 , 4) and (8 , 0)

3 Draw the boundary line $L_2 : 3x + 2y = 12$ (as a solid line) that passes through the two points (0 , 6) and (4 , 0)

∴ The solution set of the inequalities is represented by the shaded region.

That is the ribbed region ABCO

To get the coordinates of the point B algebraically : Solve the two equations representing the two straight lines L_1 and L_2 simultaneously where :

$L_1 : x + 2y = 8$, $L_2 : 3x + 2y = 12$

, then we find that : B = (2 , 3)

Second Determine the vertices of the feasible region :

The vertices of the feasible region are : A (4 , 0) , B (2 , 3) , C (0 , 4) and O (0 , 0)

Third Determine the value of the objective function at each vertex :

∵ The objective function is : $P = 50x + 75y$

∴ $[P]_A = 50 \times 4 + 75 \times 0 = 200$, $[P]_B = 50 \times 2 + 75 \times 3 = 325$

, $[P]_C = 50 \times 0 + 75 \times 4 = 300$, $[P]_O = 50 \times 0 + 75 \times 0 = 0$

∴ The maximum value of the function P is 325 at the point B (2 , 3)

Life applications on linear programming

We can deal with the life problems which are related to the linear programming by the following steps :

1. Analyse the situation or the problem to determine the variables , the constraints and the available data and arrange them in a table.
2. Put the constraints in the form of a system of linear inequalities.
3. Write the objective function.
4. Represent the system of linear inequalities graphically and determine the feasible region.
5. Determine the vertices of the feasible region.
6. Find the objective function at each vertex of the previous vertices to determine the vertex where the required objective satisfied at it.

Example 2

A bakery produces two kinds of cake. The first kind of cake needs 200 gm. of flour and 25 gm. of butter and the second kind of cake needs 100 gm. of flour and 50 gm. of butter. If the quantity of the given flour is 4 kg. and the given butter is $1\frac{1}{4}$ kg. ,

find the greatest possible number of cakes that can be made.

Solution

- Let the number of cakes of the first kind be x
  and the number of cakes of the second kind be y
- **Arrange the available data of the problem as in the following table :**

	1st kind	2nd kind	Given quantity
Flour	200	100	4 000
Butter	25	50	1 250

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

2 $200\,x + 100\,y \leq 4\,000$ i.e. $2\,x + y \leq 40$

3 $25\,x + 50\,y \leq 1\,250$ i.e. $x + 2\,y \leq 50$

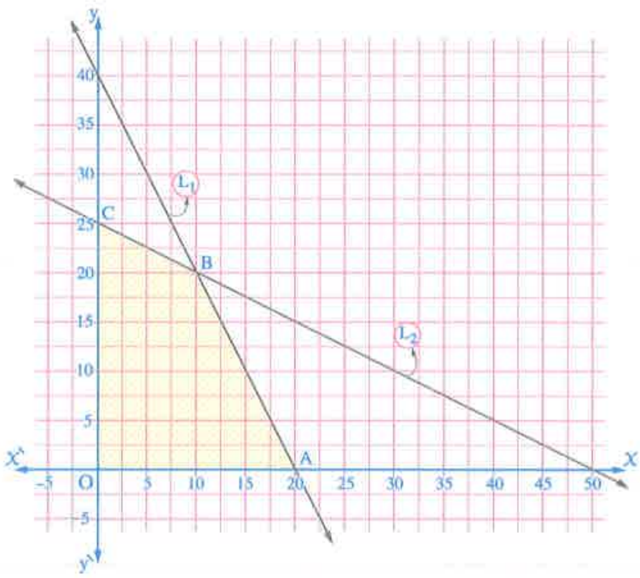
- Write the objective function : $P = x + y$, where P is maximum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : 2\,x + y = 40$ (as a solid line) that passes through the two points (0 , 40) and (20 , 0)

3 Draw the boundary line $L_2 : x + 2\,y = 50$ (as a solid line) that passes through the two points (0 , 25) and (50 , 0)



∴ The solution set of the inequalities is represented by the shaded region in the opposite graph and this is the ribbed region ABCO

- **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (20 , 0) , B (10 , 20) , C (0 , 25) and O (0 , 0)

- **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = x + y$

$[P]_O = 0 + 0 = 0$, $[P]_A = 20 + 0 = 20$

, $[P]_B = 10 + 20 = 30$, $[P]_C = 0 + 25 = 25$

$\therefore$ The greatest number of cakes is 30 ones , 10 of the first kind and 20 of the second kind.

Example 3

A factory produces 120 units at most of two different kinds of goods and achieves a profit in each unit of the first kind L.E. 15 and of the second kind L.E. 8 in each unit and the sold quantity of the second kind is not less than half the sold quantity of the first kind.

Find the number of produced units of each kind to satisfy the maximum profit.

Solution

- Let the number of produced units of the first kind be x and the number of produced units of the second kind be y
- **Arrange the available data of the problem as in the following table :**

	1^{st} kind	2^{nd} kind	The upper limit
The produced units	x	y	120
The profit	15	8	—

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

2 $x + y \leq 120$

3 $\because$ y is not less than $\frac{1}{2}x$

$\therefore y \geq \frac{1}{2}x$

$\therefore y - \frac{1}{2}x \geq 0$

$\therefore 2y - x \geq 0$

- Write the objective function : $P = 15x + 8y$, where P is maximum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 The two inequalities :

$x \geq 0$, $y \geq 0$ are represented by $\overrightarrow{Ox}$ $\cup$ $\overrightarrow{Oy}$ $\cup$ the first quadrant.

2 Draw the boundary line

$L_1 : x + y = 120$ (as a solid line)

that passes through the two points

(0 , 120) and (120 , 0)

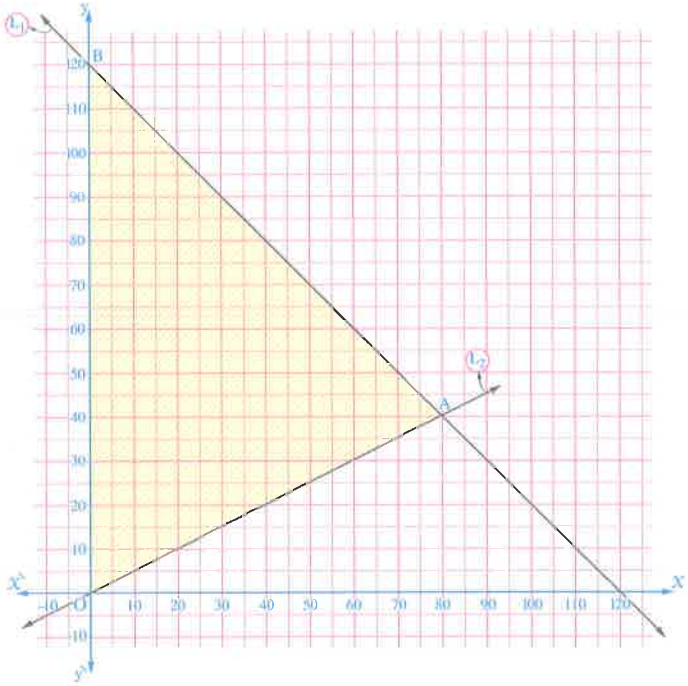
3 Draw the boundary line

$L_2 : 2y - x = 0$ (as a solid line)

that passes through the two points

(0 , 0) and (20 , 10)

$\therefore$ The solution set of the inequalities is represented by the shaded region in the opposite graph and this is the triangualr region OAB



- **Determine the vertices of the feasible region :**

The vertices of the feasible region are : O (0 , 0) , A (80 , 40) and B (0 , 120)

- **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = 15x + 8y$

$[P]_O = 0 + 0 = 0$, $[P]_A = 15 \times 80 + 8 \times 40 = 1\,520$

, $[P]_B = 15 \times 0 + 8 \times 120 = 960$

$\therefore$ The maximum profit that can be achieved is L.E. 1 520 that happens when the production is 80 units of the first kind and 40 units of the second kind.

Example 4

The required is forming a meal consisting of two kinds of food , if the piccc of thc first kind contains 3 calories , 6 units of vitamin "C" and the piece of the second kind contains 6 calories , 4 units of vitamin "C" Given that we need at least 36 calories and 48 units of vitamin "C" in the meal. If the price of the piece of the first kind is 3 pounds and of the second kind is 4 pounds , **then what is the number of pieces of the meal that satisfies the least limit with the least cost ?**

Solution

- Let the number of pieces of the first kind in the meal be x and the number of pieces of the second kind in the meal be y

- **Arrange the data in a table :**

	Pieces of the first kind	Pieces of the second kind	Least limit
Calories	3	6	36
Vitamin "C"	6	4	48

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

2 $3x + 6y \geq 36$ i.e. $x + 2y \geq 12$

3 $6x + 4y \geq 48$ i.e. $3x + 2y \geq 24$

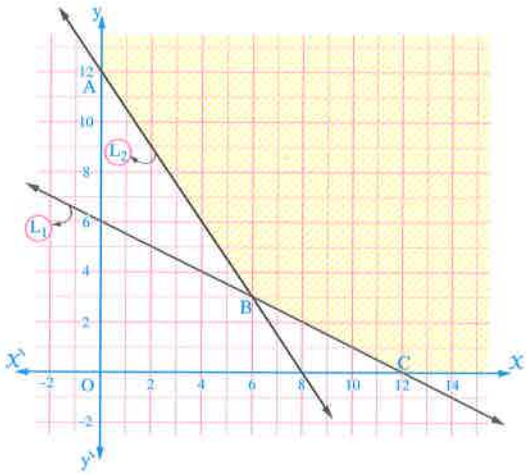
- Write the objective function : $P = 3x + 4y$, where P is minimum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : x + 2y = 12$ (as a solid line) that passes through the two points (0 , 6) and (12 , 0)

3 Draw the boundary line $L_2 : 3x + 2y = 24$ (as a solid line) that passes through the two points (0 , 12) and (8 , 0)



- **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (0 , 12) , B (6 , 3) and C (12 , 0)

- **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = 3x + 4y$

$\therefore [P]_A = 3 \times 0 + 4 \times 12 = 48$, $[P]_B = 3 \times 6 + 4 \times 3 = 30$

, $[P]_C = 3 \times 12 + 4 \times 0 = 36$

$\therefore$ The least cost of the meal is 30 pounds when it consists of 6 pieces of the first kind and 3 pieces of the second kind.

Example 5

A tourism company aims to rent a fleet of airplanes to transport 2800 passengers , 128 tons of luggage at least , and the available kinds of airplanes are A and B , and the number of available airplanes of kind (A) is 13 and of kind (B) is 12 , and the completed load of the airplane of kind (A) is 200 passengers , 8 tons of luggage and of kind (B) is 100 passengers , 6 tons of luggage , if the rent of airplane of kind (A) is 240 thousand pounds and of kind (B) is 100 thousand pounds , **then how many airplanes of each kind can be rented to satisfy the aim with the least cost ?**

Solution

- Let the number of airplanes of kind A be X and the number of airplanes of kind B be y
- **Arrange the available data of the problem in a table :**

	Kind (A)	Kind (B)	Least limit
Number of passengers	200	100	2800
Luggage in tons	8	6	128

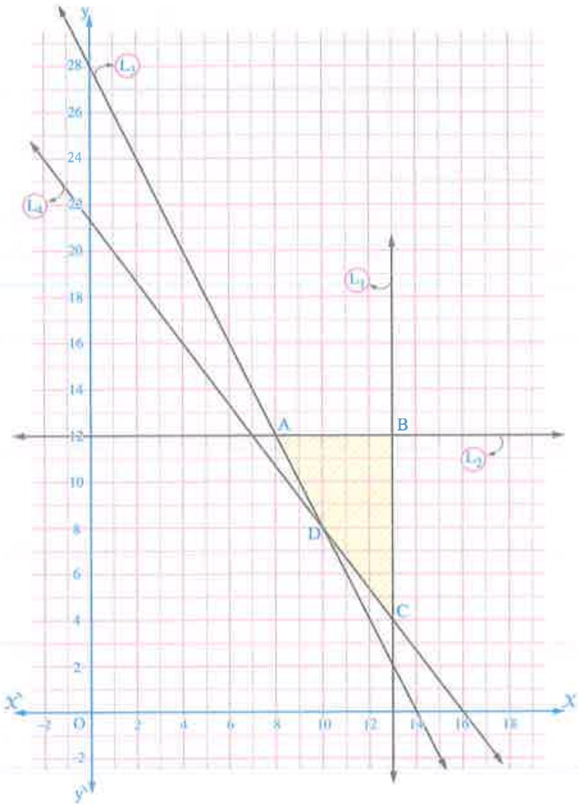
- **Translate the data and the constraints in the form of a system of inequalities :**

1. $X \leq 13$, $y \leq 12$
2. $200\,X + 100\,y \geq 2800$ i.e. $2\,X + y \geq 28$
3. $8\,X + 6\,y \geq 128$ i.e. $4\,X + 3\,y \geq 64$

- Write the objective function :

  $P = 240\,X + 100\,y$ where P is minimum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**



1. Draw the boundary line
   L_1 : $X = 13$ (as a solid line) that is parallel to the y-axis and cuts the X-axis at the point (13 , 0)

2. Draw the boundary line
   L_2 : $y = 12$ (as a solid line) that is parallel to the X-axis and cuts the y-axis at the point (0 , 12)

3 Draw the boundary line

$L_3 : 2x + y = 28$ (as a solid line) that passes through the two points (0 , 28) and (14 , 0)

4 Draw the boundary line

$L_4 : 4x + 3y = 64$ (as a solid line) that passes through the two points (1 , 20) and (16 , 0)

• **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (8 , 12) , B (13 , 12) , C (13 , 4) and D (10 , 8)

• **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = 240x + 100y$

$\therefore [P]_A = 240 \times 8 + 100 \times 12 = 3120$

, $[P]_B = 240 \times 13 + 100 \times 12 = 4320$

, $[P]_C = 240 \times 13 + 100 \times 4 = 3520$

, $[P]_D = 240 \times 10 + 100 \times 8 = 3200$

$\therefore$ The least cost that satisfies the aim is the rent of 8 airplanes of kind (A) , 12 airplanes of kind (B) , and the cost is 3120 thousand pounds.

TRY TO SOLVE

A factory produces two kinds of accessories A and B

To produce a piece of the kind A , the factory needs to run two machines , the first for one hour and the second for 2 hours and half. To produce a piece of the kind B , the factory needs running of the first machine for 4 hours and the second for 2 hours. If the first machine does not work more than 8 hours and the second does not work more than 21 hours daily and the profit of the factory is L.E. 24 and L.E. 40 in each piece of the two kinds A and B respectively.

Find the maximum profit the factory can achieve in one day.

Unit 3 TRIGONOMETRY

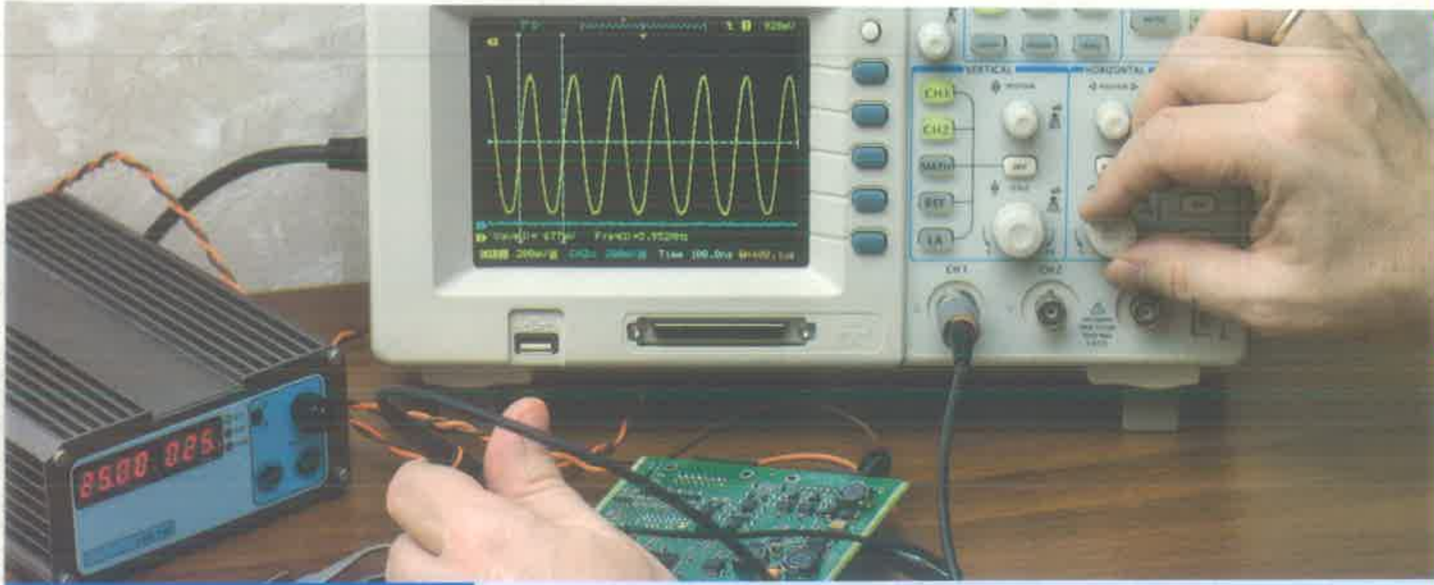
Unit Lessons

Lesson One	:	Trigonometric identities.
Lesson Two	:	Solving trigonometric equations.
Lesson Three	:	Solving the right-angled triangle.
Lesson Four	:	Angles of elevation and angles of depression.
Lesson Five	:	Circular sector.
Lesson Six	:	Circular segment.
Lesson Seven	:	Areas.

Learning outcomes

By the end of this unit, the student should be able to :

- Deduce the basic relations among trigonometric functions.
- Prove that the validity of identities on trigonometric functions.
- Determine the equality if it is identity or trigonometric equation.
- Solve simple trigonometric equations in the general form in the interval $[0 , 2\pi[$
- Recognize the general solution for the trigonometric equation.
- Solve the right-angled triangle.
- Solve applications that involve angles of elevation and depression.
- Recognize the circular sector and how to find its area.
- Recognize the circular segment and how to find its area.
- Find the area of the triangle , the area of the quadrilateral and the area of the regular polygon.
- Use activities for computer programs.



Lesson One — Trigonometric identities

Trigonometric identities and equations

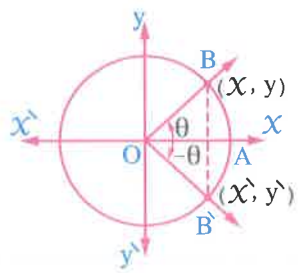
The identity

- It is a true equality for all real values of the variable , in which each of the two sides of the equality is known.

For example :

The equality : $\cos(-\theta) = \cos\theta$ is called identity because it is true for all real values of the variable θ because ,

In the opposite figure :



From our previous study for the related angles θ and $(-\theta)$, we find that :

The point $B^{`}(x^{`}, y^{`})$ is the image of the point $B(x, y)$ by the reflection in the $x$–axis $\overleftrightarrow{xx^{`}}$

i.e. $x^{`} = x$ , $\because \cos(-\theta) = x^{`}$, $\cos\theta = x$

$\therefore \cos(-\theta) = \cos\theta$ for all real values of θ

Remark

The trigonometric relations between the trigonometric functions of the related angles which we studied before are identities because all real values of the variable satisfy them.

For example :

$\sin(\pi - \theta) = \sin\theta$, $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$, ...

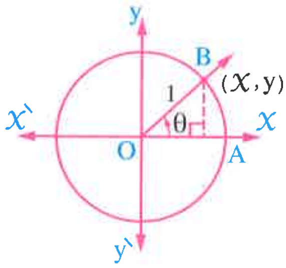
The equation

- It is a true equality for some real values of the variable which satisfy this equality , and it is not true for some others which do not satisfy it.

For example :

The equality : $\cos\theta = \sin\theta$ is called equation because it is true for some real values of the variable θ , not for all real values of the variable θ , and this because :

From the opposite figure :



From our previous study , we found that :

$\cos\theta = X$, $\sin\theta = y$

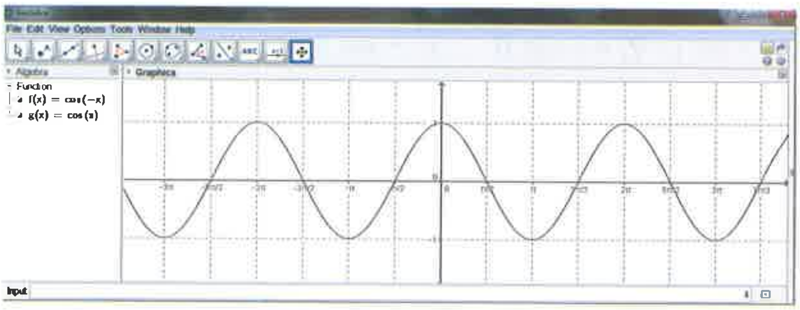
$\therefore$ $\cos\theta = \sin\theta$, when $X = y$ only and this happens when $\theta = 45°$ or $225°$ or any of the equivalent angles for them.

Remark

We can determine if the relation represents an identity or an equation and this by the graphical representation to the limits of the two determined functions, if the two functions are intersecting at all points (coincide), then the relation represents an identity and if the two functions are intersecting at some points only, then the relation represents an equation.

For example :

- **In the opposite figure :**



The two functions

$f_1 : f_1(\theta) = \cos(-\theta)$,

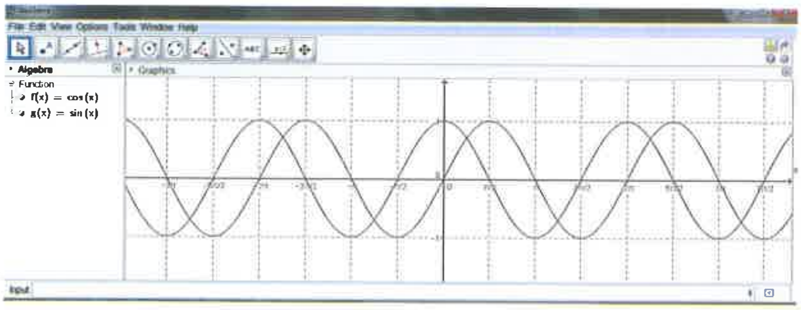
$f_2 : f_2(\theta) = \cos\theta$

are intersecting at all points (coincide)

then , the equality :

$\cos(-\theta) = \cos\theta$ is called identity.

- **In the opposite figure :**



The two functions

$f_1 : f_1(\theta) = \cos X$,

$f_2 : f_2(\theta) = \sin\theta$

are intersecting at some points.

then , the equality :

$\cos\theta = \sin\theta$ is called equation.

Unit 3

Basic trigonometric identities

We studied before the following trigonometric identities :

1 The identity of the trigonometric functions and their reciprocal :

- $\cos\theta = \dfrac{1}{\sec\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$
- $\sin\theta = \dfrac{1}{\csc\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

2 The expressing of tan θ and cot θ in terms of sin θ and cos θ :

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3 The identity of the trigonometric functions of two complementary angles :

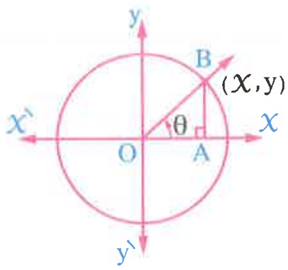
- $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$, • $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$, • $\csc\left(\dfrac{\pi}{2} - \theta\right) = \sec\theta$
- $\sec\left(\dfrac{\pi}{2} - \theta\right) = \csc\theta$, • $\cot\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta$

4 The identity of the trigonometric functions of the two angles (θ and (– θ)) :

- $\sin(-\theta) = -\sin\theta$, • $\cos(-\theta) = \cos\theta$
- $\csc(-\theta) = -\csc\theta$, • $\sec(-\theta) = \sec\theta$
- $\tan(-\theta) = -\tan\theta$, • $\cot(-\theta) = -\cot\theta$

5 Pythagorian identity :

For any directed angle of measure θ in the standard position ,
if its terminal side cuts the unit circle at the point (x , y) , then :



$x^2 + y^2 = 1$

$\because \cos\theta = x , \sin\theta = y$

$\therefore \cos^2\theta + \sin^2\theta = 1$ (1)

- Dividing both sides of the relation (1) by $\cos^2\theta$, we find that :

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \therefore\ 1 + \tan^2\theta = \sec^2\theta$$

- Dividing both sides of the relation (1) by $\sin^2\theta$, we find that :

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \therefore\ \cot^2\theta + 1 = \csc^2\theta$$

Remarks

1. From : $\sin^2\theta + \cos^2\theta = 1$, we get : $\boxed{\sin^2\theta = 1 - \cos^2\theta}$ and $\boxed{\cos^2\theta = 1 - \sin^2\theta}$
2. From : $1 + \tan^2\theta = \sec^2\theta$, we get : $\boxed{\tan^2\theta = \sec^2\theta - 1}$ and $\boxed{\sec^2\theta - \tan^2\theta = 1}$
3. From : $\cot^2\theta + 1 = \csc^2\theta$, we get : $\boxed{\cot^2\theta = \csc^2\theta - 1}$ and $\boxed{\csc^2\theta - \cot^2\theta = 1}$

Check your understanding

Choose the correct answer : $\sin^2\theta + \cos^2\theta \neq$ ....

(a) $\tan\theta\cot\theta$ (b) $\sin^2 2\theta + \cos^2 2\theta$ (c) $\cot^2\theta - \csc^2\theta$ (d) $\sec^2\theta - \tan^2\theta$

Simplifying the trigonometric expressions

We mean by simplifying the trigonometric expression is to put it in the simplest form , by using the basic trigonometric identities.

Example 1

Write each of the following expressions in the simplest form :

1. $\dfrac{1}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}$
2. $\sin\left(\dfrac{\pi}{2} - \theta\right)\csc\theta$
3. $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$
4. $\dfrac{1 + \cot^2\left(3\dfrac{\pi}{2} - \theta\right)}{1 + \tan^2\left(\dfrac{3\pi}{2} + \theta\right)}$

Solution

1. $\dfrac{1}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - \tan^2\theta = 1$

2. $\sin\left(\dfrac{\pi}{2} - \theta\right)\csc\theta = \cos\theta\csc\theta = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

Notice that

- $\dfrac{1}{\cos^2\theta} = \left(\dfrac{1}{\cos\theta}\right)^2 = \sec^2\theta$
- $\dfrac{\sin^2\theta}{\cos^2\theta} = \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta$

3. $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = \sin^2\theta + \cancel{2\sin\theta\cos\theta} + \cos^2\theta - \cancel{2\sin\theta\cos\theta}$

$= \sin^2\theta + \cos^2\theta = 1$

Remember that

$(a+b)^2 = a^2 + 2ab + b^2$

4 $\dfrac{1+\cot^2\left(\frac{3\pi}{2}-\theta\right)}{1+\tan^2\left(\frac{3\pi}{2}+\theta\right)} = \dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \dfrac{\sec^2\theta}{\csc^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

Notice that

$\dfrac{\sec^2\theta}{\csc^2\theta} = \dfrac{1}{\cos^2\theta} \div \dfrac{1}{\sin^2\theta}$

$= \dfrac{1}{\cos^2\theta} \times \sin^2\theta$

$= \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

TRY TO SOLVE

Put in the simplest form each of the following expressions :

1 $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta}$

2 $\sin\left(\dfrac{\pi}{2}-\theta\right)\sec(2\pi-\theta)$

3 $\dfrac{1-\sin^2\theta}{\cos^2\theta-1}$

Trigonometric identities

To prove the validity of the trigonometric identity , we follow one of the two methods :

1 Put one of the two sides of the identity in the form of the other side using the basic trigonometric identities.

2 Put the two sides of the trigonometric identity in the simplest form , to prove that the two sides have the same result when they are in the simplest form.

Example 2

Prove the validity of the identity : $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$

Solution

L.H.S. $= \sin^2\theta - \cos^2\theta = \sin^2\theta - (1-\sin^2\theta)$

$= \sin^2\theta - 1 + \sin^2\theta$

$= 2\sin^2\theta - 1 =$ R.H.S.

Notice that

$\cos^2\theta = 1 - \sin^2\theta$

Example 3

Prove the validity of the identity : $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$

Solution

L.H.S. $= \sin^4\theta - \cos^4\theta = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$

$= 1 \times (\sin^2\theta - \cos^2\theta)$

$= 1 - \cos^2\theta - \cos^2\theta$

$= 1 - 2\cos^2\theta$

Notice that:

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = 1 - \cos^2\theta$

Example 4

Prove the validity of the identity : $\dfrac{\sin^2\theta}{1-\cos\theta} = 1+\cos\theta$

Solution

$$\text{L.H.S.} = \frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{(1+\cos\theta)\cancel{(1-\cos\theta)}}{\cancel{1-\cos\theta}} = 1+\cos\theta = \text{R.H.S.}$$

TRY TO SOLVE

Prove the validity of the following identities :

1. $\dfrac{\cos^2\theta}{1+\sin\theta} = 1-\sin\theta$

2. $(\sin\theta+\cos\theta)^2 + (\sin\theta-\cos\theta)^2 = 2$

Example 5

Prove the validity of the identity : $\tan\theta + \cot\theta = \csc\theta\sec\theta$

Solution

$$\text{L.H.S.} = \tan\theta+\cot\theta = \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos\theta}$$
$$= \csc\theta\sec\theta$$
$$= \text{R.H.S.}$$

Notice that

To make the proof easy , we write the expression in terms of $\sin\theta$ and $\cos\theta$ only , using the following relations :

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$

Example 6

Prove the validity of the identity : $2\cos^2\theta - 1 = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution

$$\text{R.H.S.} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{\sec^2\theta} = \frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{1}{\cos^2\theta}} = \left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\times\cos^2\theta$$

$$= \cos^2\theta-\sin^2\theta = \cos^2\theta-(1-\cos^2\theta) = \cos^2\theta-1+\cos^2\theta$$

$$= 2\cos^2\theta - 1 = \text{L.H.S.}$$

Example 7

Prove the validity of the identity : $\sec^2\theta - \tan^2\theta \sin^2\theta = \cos^2\theta + 2\sin^2\theta$

Solution

$$\text{L.H.S.} = \sec^2\theta - \tan^2\theta\sin^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \times \sin^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^4\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^4\theta}{\cos^2\theta} = \frac{\cancel{(1-\sin^2\theta)}(1+\sin^2\theta)}{\cancel{1-\sin^2\theta}} = 1 + \sin^2\theta \qquad (1)$$

$$\text{R.H.S.} = \cos^2\theta + 2\sin^2\theta = 1 - \sin^2\theta + 2\sin^2\theta = 1 + \sin^2\theta \qquad (2)$$

From (1) and (2) , we get that : L.H.S. = R.H.S.

TRY TO SOLVE

Prove the validity of the following identity : $\frac{1-\cot^2\theta}{1+\cot^2\theta} = 2\sin^2\theta - 1$

Example 8

If $\sin\theta + \sin(270° - \theta) = \frac{1}{2}$, **find the value of :** $\sin\theta\cos\theta$, where $\theta \in]0 , \frac{\pi}{2}[$

Solution

$\because \sin\theta + \sin(270° - \theta) = \frac{1}{2}$

$\therefore \sin\theta - \cos\theta = \frac{1}{2}$ (squaring both sides)

$\therefore \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{4}$

$\therefore 1 - 2\sin\theta\cos\theta = \frac{1}{4}$

$\therefore -2\sin\theta\cos\theta = \frac{-3}{4}$

$\therefore \sin\theta\cos\theta = \frac{3}{8}$

Lesson TWO Solving trigonometric equations

- Solving trigonometric equation means finding the values of the variable in the equation which satisfy this equation using the trigonometric identities.

General solution of the trigonometric equation

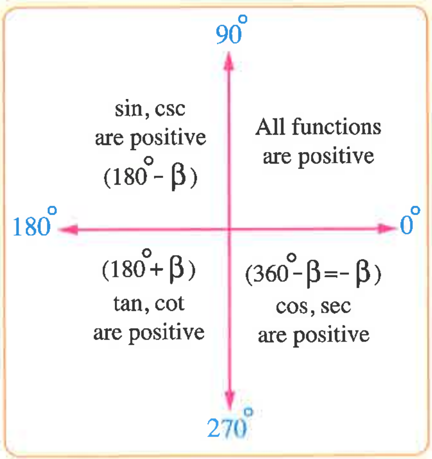
To find the general solution of the trigonometric equation in the form :

$\cos \theta = a$, $\sin \theta = a$ **or** $\tan \theta = a$ **, follow the following steps :**

1 Let β be the measure of the acute angle which satisfies the equation :

$\cos \theta = |a|$, $\sin \theta = |a|$, $\tan \theta = |a|$

2 Determine the quadrant in which the angle lies according to the sign of a "Look to the opposite figure" :



3 Find the values of the angle θ where :

- If θ lies in the first quadrant , then $\theta = \beta$
- If θ lies in the second quadrant , then $\theta = 180° - \beta$
- If θ lies in the third quadrant , then $\theta = 180° + \beta$
- If θ lies in the fourth quadrant , then $\theta = 360° - \beta$

4 We add a number of periods ($2 n \pi$) where $n \in \mathbb{Z}$ to the values of θ to get the general solution of the trigonometric equation.

Remark

- $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all real values of θ

  So, we find that the two equations : $\sin\theta = a$, $\cos\theta = a$ don't have solution in $\mathbb{R}$, if $a \notin [-1, 1]$

For example :

Each of the equations : $\sin\theta = 1.3$, $\cos\theta = 2.5$, $\sin\theta = -1.4$, $\sec\theta = 0.5$ and $\csc\theta = -0.7$ doesn't have real solutions.

i.e. It is not necessary that there real solutions for every trigonometric equations.

Example 1

Find the general solution of each of the following equations :

1 $\cos\theta = \frac{1}{2}$ **2** $2\sin\theta - \sqrt{2} = 0$ **3** $\sqrt{3}\tan\theta - 1 = 0$

Solution

1 $\cos\theta = \frac{1}{2}$ (positive)

$\therefore$ θ lies in the first quadrant $\therefore \theta = 60°$

or θ lies in the fourth quadrant.

$\therefore \theta = 360° - 60° = 300°$ and it is equivalent to $(-60°)$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{3} + 2n\pi$ or $\theta = -\frac{\pi}{3} + 2n\pi$

$\therefore$ The general solution of the equation is : $\pm\frac{\pi}{3} + 2n\pi$, where $n \in \mathbb{Z}$

2 $\sin\theta = \frac{\sqrt{2}}{2}$ (positive)

$\therefore$ θ lies in the first quadrant $\therefore \theta = 45°$

or θ lies in the second quadrant

$\therefore \theta = 180° - 45° = 135°$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{4} + 2n\pi$ or $\theta = \frac{3}{4}\pi + 2n\pi$

$\therefore$ The general solution of the equation is : $\theta = \frac{\pi}{4} + 2n\pi$ or $\theta = \frac{3}{4}\pi + 2n\pi$, where $n \in \mathbb{Z}$

3 $\tan\theta = \frac{1}{\sqrt{3}}$ (positive)

$\therefore$ θ lies in the first quadrant $\qquad \therefore \theta = 30°$

or θ lies in the third quadrant $\qquad \therefore \theta = 180° + 30° = 210°$

Adding $(2 n \pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{6} + 2 n \pi$ or $\theta = \frac{7}{6}\pi + 2 n \pi$

$\therefore$ The general solution of the equation is : $\theta = \frac{\pi}{6} + 2 n \pi$ or $\theta = \frac{7}{6}\pi + 2 n \pi$, where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution of the equation is : $\theta = \frac{\pi}{6} + n\pi$ where $n \in \mathbb{Z}$

and this by adding $n\pi$ to the smallest positive measure.

Remark

From the previous , we can deduce that :

If β is the smallest positive measure satisfies the equation , $n \in \mathbb{Z}$, then :

1. The general solution of the equation $\sin\theta = a$ is $\theta = \beta + 2\pi n$, $\theta = (\pi - \beta) + 2\pi n$
2. The general solution of the equation $\cos\theta = a$ is $\theta = \pm\beta + 2\pi n$
3. The general solution of the equation $\tan\theta = a$ is $\theta = \beta + \pi n$

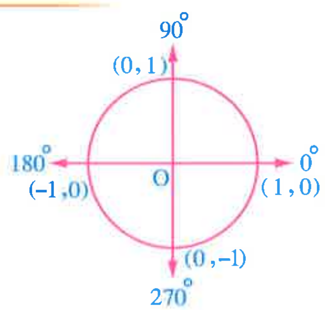
Example 2

Find the general solution of each of the following equations :

1 $\sin\theta = 0$ $\qquad$ **2** $\cos\theta = 0$

3 $\sin\theta = 1$ $\qquad$ **4** $\cos\theta = -1$

Solution



1 $\sin\theta = 0$

$\therefore \theta = 0°$ or $\theta = 180°$

Adding $(2\pi n)$ where $n \in \mathbb{Z}$ to the values of θ

$\therefore$ The general solution of the equation is :

$\theta = 2\pi n$ or $\theta = \pi + 2\pi n$ where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution of the equation is : $\theta = \pi n$ where $n \in \mathbb{Z}$

2 $\cos\theta = 0$ $\quad\therefore \theta = 90°$ or $\theta = 270°$

Adding $(2\pi n)$ where $n \in \mathbb{Z}$ to the values of θ

$\therefore$ The general solution is : $\theta = \frac{\pi}{2} + 2\pi n$

or $\theta = \frac{3}{2}\pi + 2\pi n$ where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution is : $\theta = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

3 $\sin\theta = 1$ $\quad\therefore \theta = 90°$

$\therefore$ The general solution is : $\theta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

4 $\cos\theta = -1$ $\quad\therefore \theta = 180°$

$\therefore$ The general solution is : $\theta = \pi + 2\pi n$ where $n \in \mathbb{Z}$

Remark

From the previous , we can deduce the general solution of the trigonometric equations of the quadrantal angles :

The equation	The general solution
• $\sin\theta = 0$	$\theta = \pi n$
• $\sin\theta = 1$	$\theta = \frac{\pi}{2} + 2\pi n$
• $\sin\theta = -1$	$\theta = \frac{3\pi}{2} + 2\pi n$

The equation	The general solution
• $\cos\theta = 0$	$\theta = \frac{\pi}{2} + \pi n$
• $\cos\theta = 1$	$\theta = 2\pi n$
• $\cos\theta = -1$	$\theta = \pi + 2\pi n$

Example 3

Find the general solution of each of the following equations :

1 $\cot\theta + 1 = 0$ **2** $\cos^2\theta - \cos\theta = 0$

Solution

1 $\cot\theta = -1$ $\quad\therefore \tan\theta = -1$ (negative)

$\therefore \theta$ lies in the second quadrant. $\therefore \theta = 180° - 45° = 135°$

or θ lies in the fourth quadrant. $\therefore \theta = 360° - 45° = 315°$

Adding $(n\pi)$ where $n \in \mathbb{Z}$ to the smallest positive measure which satisfies the equation "135°"

$\therefore$ The general solution is : $\frac{3}{4}\pi + n\pi$

> **Notice that**
>
> The measure of the acute angle which satisfies that : $\tan\theta = |-1|$ is 45°

2 $\cos\theta(\cos\theta - 1) = 0$

$\therefore \cos\theta = 0$

$\therefore \theta = 90°$ or $\theta = 270°$ and it is equivalent to $(-90°)$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \pm 90° + 2n\pi$

or $\cos\theta = 1$ $\qquad$ $\therefore \theta = 0$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$:

$\therefore \theta = 2n\pi$

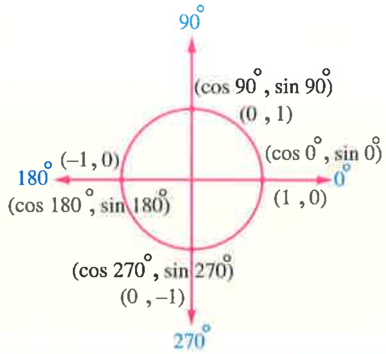
$\therefore$ The general solution is : $\theta = \pm\frac{\pi}{2} + 2n\pi$

or $\theta = 2n\pi$

Remark

Use the unit circle to determine the values of θ when :

$\cos\theta = 0$, $\cos\theta = 1$



TRY TO SOLVE

Find the general solution of each of the following equations :

1 $2\sin\theta - 1 = 0$ $\qquad$ **2** $2\cos\theta + \sqrt{3} = 0$ $\qquad$ **3** $\tan\theta - \sqrt{3} = 0$

Example 4

Find the general solution of the equation : $\sin\theta\cos\theta = \frac{1}{2}\sin\theta$

Solution

$\because \sin\theta\cos\theta - \frac{1}{2}\sin\theta = 0$ $\qquad$ $\therefore \sin\theta\left(\cos\theta - \frac{1}{2}\right) = 0$

$\therefore \sin\theta = 0$ $\qquad$ $\therefore \theta = 0°$ $\qquad$ or $\theta = 180°$

$\therefore \theta = n\pi$ where $n \in \mathbb{Z}$

or $\cos\theta - \frac{1}{2} = 0$ $\qquad$ $\therefore \cos\theta = \frac{1}{2}$(positive)

$\therefore \theta$ lies in the first quadrant. $\qquad$ $\therefore \theta = 60°$

or θ lies in the fourth quadrant. $\qquad$ $\therefore \theta = 360° - 60° = 300°$ and it is equivalent to $(-60°)$

$\therefore \theta = \pm\frac{\pi}{3} + 2n\pi$ where $n \in \mathbb{Z}$

$\therefore$ The general solution is : $\theta = n\pi$ or $\theta = \pm\frac{\pi}{3} + 2n\pi$ where $n \in \mathbb{Z}$

TRY TO SOLVE

Find the general solution of the equation : $2\sin\theta\cos\theta - \sqrt{3}\sin\theta = 0$

Solving the trigonometric equation in the interval $[0, 2\pi[$

Example 5

If $\theta \in [0, 2\pi[$ Find the solution set of each of the following equations :

1. $2\cos\theta + 1 = 0$
2. $\sqrt{2}\sec\theta - 2 = 0$

Solution

1. $\because 2\cos\theta + 1 = 0 \quad \therefore \cos\theta = -\frac{1}{2}$ (negative)

$\therefore \theta$ lies in the second quadrant or in the third quadrant.

$, \because$ the acute angle of cosine $= \frac{1}{2}$, its measure is $60°$

$\therefore \theta = 180° - 60° = 120°$ or $\theta = 180° + 60° = 240°$

$\therefore$ The S.S. $= \{120°, 240°\}$

2. $\because \sqrt{2}\sec\theta - 2 = 0 \qquad \therefore \sec\theta = \frac{2}{\sqrt{2}}$

$\therefore \cos\theta = \frac{\sqrt{2}}{2}$ (positive)

$\therefore \theta$ lies in the first quadrant or in the fourth quadrant.

$, \because$ the acute angle of cosine $= \frac{\sqrt{2}}{2}$, its measure is $45°$

$\therefore \theta = 45°$ or $\theta = 360° - 45° = 315°$

$\therefore$ The S.S. $= \{45°, 315°\}$

Example 6

Find the solution set of the equation : $4\cos^2\theta - 3 = 0$, where $\theta \in [0, 2\pi[$

Solution

$\because 4\cos^2\theta - 3 = 0 \qquad \therefore 4\cos^2\theta = 3 \qquad \therefore \cos^2\theta = \frac{3}{4}$

$\therefore \cos\theta = \pm\frac{\sqrt{3}}{2} \qquad \therefore \cos\theta = \frac{\sqrt{3}}{2}$ (positive)

$\therefore \theta$ lies in the first quadrant or in the fourth quadrant.

$, \because$ the acute angle of cosine $= \frac{\sqrt{3}}{2}$, its measure is $30°$

$\therefore \theta = 30°$ or $\theta = 360° - 30° = 330°$

or $\cos\theta = \frac{-\sqrt{3}}{2}$ (negative)

$\therefore \theta$ lies in the second quadrant or in the third quadrant.

$\therefore \theta = 180° - 30° = 150°$ or $\theta = 180° + 30° = 210°$

$\therefore$ The S.S. $= \{30°, 150°, 210°, 330°\}$

TRY TO SOLVE

Find the solution set of each of the following equations where $\theta \in [0 , 2\pi[$

1 $\sqrt{2} \csc \theta - 2 = 0$ **2** $\tan^2 \theta = 1$

Example 7

Find the solution set of the equation : $2 \sin \theta \cos \theta + 3 \cos \theta = 0$, where $\theta \in [0 , \pi[$

Solution

$\because 2 \sin \theta \cos \theta + 3 \cos \theta = 0$

$\therefore \cos \theta (2 \sin \theta + 3) = 0$

$\therefore \cos \theta = 0$

$\therefore \theta = 90°$ or $\theta = 270°$ (refused because $\theta \in [0 , \pi[$)

or $2 \sin \theta + 3 = 0$

$\therefore \sin \theta = \frac{-3}{2}$ (this equation has no solution because $-1 \le \sin \theta \le 1$)

$\therefore$ The S.S. $= \{90°\}$

Example 8

Find the solution set of the equation : $4 \sin^2 \theta - 3 \sin \theta \cos \theta = 0$, where $\theta \in [0 , 2\pi[$

Solution

$\because 4 \sin^2 \theta - 3 \sin \theta \cos \theta = 0$

$\therefore \sin \theta (4 \sin \theta - 3 \cos \theta) = 0$

$\therefore \sin \theta = 0$

$\therefore \theta = 0°$ or $\theta = 180°$

or $4 \sin \theta - 3 \cos \theta = 0$

$\therefore 4 \sin \theta = 3 \cos \theta$

$\therefore \frac{\sin \theta}{\cos \theta} = \frac{3}{4}$

$\therefore \tan \theta = \frac{3}{4}$ (positive)

$\therefore \theta$ lies in the first quadrant or in the third quadrant.

, $\because$ the acute angle of tangent $= \frac{3}{4}$, its measure is $36° 52'$

$\therefore \theta = 36° 52'$ or $\theta = 180° + 36° 52' = 216° 52'$

$\therefore$ The S.S. $= \{0° , 36° 52' , 180° , 216° 52'\}$

TRY TO SOLVE

If $0° < \theta < 360°$ Find the solution set of the equation : $2 \sin \theta \cos \theta = 3 \cos^2 \theta$

Example 9

Find the solution set of the equation : $2 \sin^2 \theta - \cos \theta - 1 = 0$, where $\theta \in [0 , 2\pi[$

Solution

$\because \sin^2 \theta = 1 - \cos^2 \theta$

$\therefore 2 (1 - \cos^2 \theta) - \cos \theta - 1 = 0$

$\therefore 2 - 2 \cos^2 \theta - \cos \theta - 1 = 0$

$\therefore 1 - \cos \theta - 2 \cos^2 \theta = 0$

$\therefore (1 + \cos \theta) (1 - 2 \cos \theta) = 0$

$\therefore 1 + \cos \theta = 0$

$\therefore \cos \theta = -1$

$\therefore \theta = 180°$

or $1 - 2 \cos \theta = 0$

$\therefore \cos \theta = \frac{1}{2}$ (positive)

$\therefore$ θ lies in the first quadrant or in the fourth quadrant.

, $\because$ The acute angle of cosine $= \frac{1}{2}$, its measure is $60°$

$\therefore \theta = 60°$ or $\theta = 360° - 60° = 300°$

$\therefore$ The S.S. $= \{60° , 180° , 300°\}$

Using the technology

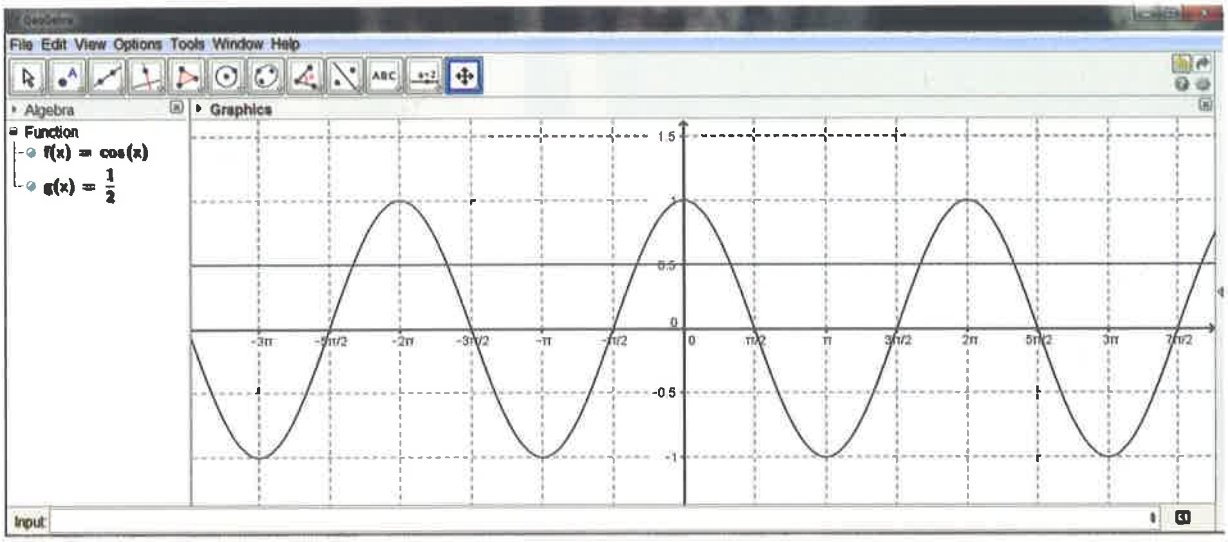
From example (1) , we found that :

The general solution of the equation : $\cos \theta = \frac{1}{2}$ is $\theta = \pm \frac{\pi}{3} + 2 n \pi$, where $n \in \mathbb{Z}$

We can verify the solution by drawing the two function :

$f_1 : f_1 (\theta) = \cos \theta$, $f_2 : f_2 (\theta) = \frac{1}{2}$

by using one of the drawing programs and determining the corresponding values of θ for the intersection points of the two functions , and compare them with the values of θ in the general solution at $n = ... , -2 , -1 , 0 , 1 , 2 , ...$



From the graph , we notice that the two functions intersect at the points :

$\ldots , \left(\frac{-5}{3}\pi , \frac{1}{2}\right) , \left(\frac{-1}{3}\pi , \frac{1}{2}\right) , \left(\frac{1}{3}\pi , \frac{1}{2}\right) , \left(\frac{5}{3}\pi , \frac{1}{2}\right) , \ldots$

i.e. $\theta = \ldots , \frac{-5}{3}\pi , \frac{-1}{3}\pi , \frac{1}{3}\pi , \frac{5}{3}\pi , \ldots$

and they are the same values we can get from the general solution by substituting $n = \ldots , -1 , 0 , 1 , 2 , \ldots$

Lesson Three — Solving the right-angled triangle

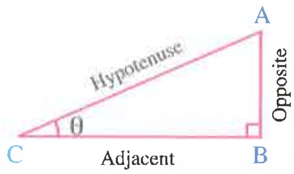
- Any triangle contains six elements (three sides and three angles) , and solving the triangle means evaluating the unknown measures of its angles and the unknown lengths of its sides.
- To solve the right-angled triangle , we must be given :

  the lengths of two sides of its sides or the length of one of its sides and the measure of one of its two acute angles.
- The trigonometric ratios of the acute angle and Pythagoras' theorem are used in solving the right-angled triangle. If ABC is a right-angled triangle at B then :

1 $\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{AB}{AC}$

$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{BC}{AC}$

$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{AB}{BC}$



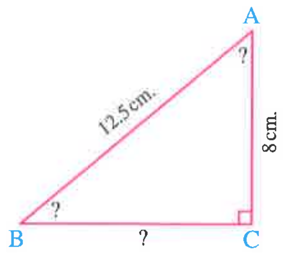
2 $(AC)^2 = (AB)^2 + (BC)^2$

First Solving the right-angled triangle given the lengths of two sides

Example 1

ABC is a right-angled triangle at C , in which AC = 8 cm. and AB = 12.5 cm. Solve this triangle.

Solution



- $\because \frac{AC}{AB} = \sin B$ $\quad\quad\quad\quad \therefore \sin B = \frac{8}{12.5}$

  Using the calculator , we find that : $m(\angle B) \approx 39°\ 47'\ 31''$

- $\because m(\angle B) = 39°\ 47'\ 31''$

  $\therefore m(\angle A) = 90° - 39°\ 47'\ 31'' = 50°\ 12'\ 29''$

- $\because \frac{BC}{AB} = \cos B$ $\quad\quad\quad\quad \therefore \frac{BC}{12.5} = \cos(39°\ 47'\ 31'')$

  $\therefore BC = 12.5 \cos(39°\ 47'\ 31'')$

  Using the calculator , we find that : $BC \approx 9.6$ cm.

  Notice that : We can find BC by using Pythagoras' theorem where :

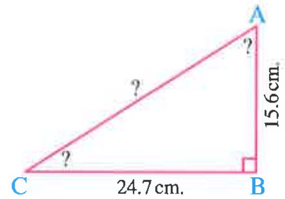
  $(BC)^2 = (AB)^2 - (AC)^2$

  So , $BC = \sqrt{(12.5)^2 - (8)^2} \approx 9.6$ cm.

Example 2

Solve the right-angled triangle ABC , in which $m(\angle B) = 90°$, AB = 15.6 cm. and BC = 24.7 cm.

Solution



- $\because \frac{AB}{BC} = \tan C$ $\quad\quad\quad\quad \therefore \tan C = \frac{15.6}{24.7}$

  Using the calculator , we find that : $m(\angle C) \approx 32°\ 16'\ 28''$

- $\therefore m(\angle A) = 90° - 32°\ 16'\ 32'' = 57°\ 43'\ 28''$

- $\because \frac{AB}{AC} = \sin C$ $\quad\quad\quad\quad \therefore \frac{15.6}{AC} = \sin(32°\ 16'\ 32'')$

  $\therefore AC = \frac{15.6}{\sin(32°\ 16'\ 32'')}$

  Using the calculator , we find that : $AC \approx 29.21$ cm.

  Notice that : We can find AC using Pythagoras' theorem where :

  $(AC)^2 = (AB)^2 + (BC)^2$

  $\therefore AC = \sqrt{(15.6)^2 + (24.7)^2} \approx 29.21$ cm.

TRY TO SOLVE

Solve the right-angled triangle ABC at B in each of the following two cases :

1. AB = 6 cm. , AC = 8.6 cm.
2. AB = 5.4 cm. , BC = 7.3 cm.

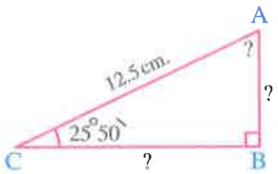
Unit 3

Second Solving the right-angled triangle given the length of one side and the measure of one of its two acute angles

Example 3

Solve Δ ABC where m ($\angle$ B) = 90° , AC = 12.5 cm. and m ($\angle$ C) = 25° 50`

Solution



- m ($\angle$ A) = 90° – 25° 50` = 64° 10`
- $\because \frac{AB}{AC} = \sin C$ $\therefore \frac{AB}{12.5} = \sin 25° 50`$

i.e. AB = 12.5 sin 25° 50`

Using the calculator , we get : AB ≈ 5.45 cm.

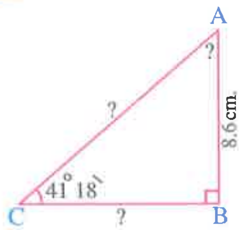
- $\because \frac{BC}{AC} = \cos C$ $\therefore \frac{BC}{12.5} = \cos 25° 50`$ $\therefore$ BC = 12.5 cos 25° 50`

Using the calculator , we get : BC ≈ 11.25 cm.

Example 4

Solve the right-angled triangle ABC at B in which AB = 8.6 cm. and m ($\angle$ C) = 41° 18`

Solution



- m ($\angle$ A) = 90° – 41° 18` = 48° 42`
- $\because \frac{AB}{BC} = \tan C$ $\therefore \frac{8.6}{BC} = \tan 41° 18`$

$\therefore BC = \frac{8.6}{\tan 41° 18`}$

Using the calculator , we find that : BC ≈ 9.79 cm.

Notice that : We can find the length of $\overline{BC}$ using m ($\angle$ A) where $\frac{BC}{AB} = \tan A$

$\therefore \frac{BC}{8.6} = \tan 48° 42`$ $\therefore$ BC = 8.6 tan 48° 42` ≈ 9.79 cm.

- $\because \frac{AB}{AC} = \sin C$ $\therefore \frac{8.6}{AC} = \sin 41° 18`$ $\therefore AC = \frac{8.6}{\sin 41° 18`}$

Using the calculator , we find that : AC ≈ 13.03 cm.

Notice that : We can find the length of $\overline{AC}$ using m ($\angle$ A) where $\frac{AB}{AC} = \cos A$

$\therefore \frac{8.6}{AC} = \cos 48° 42`$ $\therefore AC = \frac{8.6}{\cos 48° 42`} ≈ 13.03$ cm.

TRY TO SOLVE

Solve the triangle ABC in which m ($\angle$ B) = 90° , if :

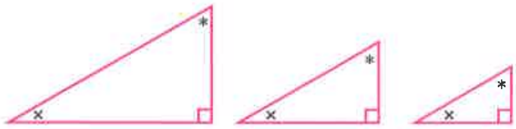
1. AB = 10 cm. , m ($\angle$ C) = 54°
2. AC = 32 cm. , m ($\angle$ A) = 32° 24`

Critical thinking

Can you solve the right-angled triangle given the measures of its acute angles ?

Answer : No

Because there is an infinite number of the right-angled triangle which have the same measures of the acute angles (i.e. Similar triangles)



So , we can not determine which of the triangles is the required to solve by finding its side lengths (i.e. To solve the right-angled triangle , we must be given one of its side lengths at least.)

Example 5

Solve the right-angled triangle ABC in B approximating the measures of angles to the nearest thousandth in radian measure and the lengths to the nearest thousandth in cm. , if :

1 $m(\angle A) = 1.254^{rad}$, $BC = 10.6$ cm. **2** $m(\angle C) = 0.715^{rad}$, $AC = 23$ cm.

Solution

Note that : We should convert the system of the calculator from the system (Deg) to the system (Rad) before performing the mathematical operations which contain trigonometric functions for angles measured in radian , and this by pressing on SHIFT then MODE and 4

Notice that

90° is equivalent to $\frac{\pi}{2}^{rad}$

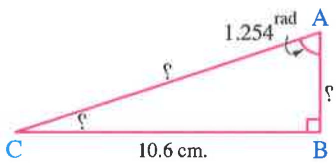
1 • $\because m(\angle C) = \frac{\pi}{2} - 1.254^{rad} \approx 0.317^{rad}$

• $\because \sin A = \frac{BC}{AC}$ $\therefore \sin 1.254^{rad} = \frac{10.6}{AC}$

$\therefore AC = \frac{10.6}{\sin 1.254^{rad}} \approx 11.155$ cm.

• $\because \tan A = \frac{BC}{AB}$ $\therefore \tan 1.254^{rad} = \frac{10.6}{AB}$

$\therefore AB = \frac{10.6}{\tan 1.254^{rad}} \approx 3.475$ cm.



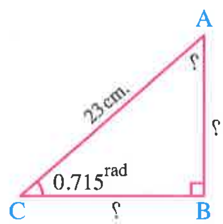
2 • $\because m(\angle A) = \frac{\pi}{2} - 0.715^{rad} \approx 0.856^{rad}$

• $\because \sin C = \frac{AB}{AC}$ $\therefore \sin 0.715^{rad} = \frac{AB}{23}$

$\therefore AB = 23 \sin 0.715^{rad} \approx 15.079$ cm.

• $\because \cos C = \frac{BC}{AC}$ $\therefore \cos 0.715^{rad} = \frac{BC}{23}$

$\therefore BC = 23 \cos 0.715^{rad} \approx 17.367$ cm.



TRY TO SOLVE

Solve the right-angled triangle ABC in B approximating the measure of angles to the nearest thousandth in radian measure and the lengths to the nearest thousandth in cm. , if :

1 m ($\angle$ A) = 0.623^{rad} , BC = 10 cm. **2** m ($\angle$ C) = 1.073^{rad} , AC = 37.5 cm.

Example 6

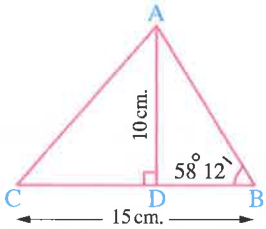
ABC is a triangle in which m ($\angle$ B) = $58°\ 12'$ and BC = 15 cm.

Draw $\overline{AD} \perp \overline{BC}$, where D $\in \overline{BC}$ if AD = 10 cm. , **then find :** m ($\angle$ C)

Solution

In Δ ADB : $\tan B = \dfrac{AD}{DB}$

$\therefore \tan 58°\ 12' = \dfrac{10}{DB}$ $\therefore DB = \dfrac{10}{\tan 58°\ 12'} \approx 6.2$ cm.



$\therefore$ DC = 15 – 6.2 = 8.8 cm.

In Δ ADC : $\tan C = \dfrac{AD}{DC} = \dfrac{10}{8.8}$

Using the calculator , we get m ($\angle$ C) = $48°\ 39'\ 8''$

Example 7

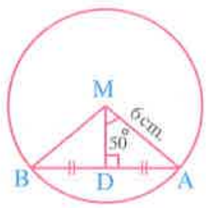
A circle of radius length 6 cm. , a chord was drawn in it opposite to a central angle of measure 100°. **Calculate the length of this chord to the nearest thousandth.**

Solution

Draw $\overline{MD} \perp \overline{AB}$, and cuts it at D

$\because \overline{MD} \perp \overline{AB}$ $\therefore$ D is the midpoint of $\overline{AB}$

, $\because$ MA = MB = r $\therefore \overrightarrow{MD}$ bisects $\angle$ AMB



$\therefore$ m ($\angle$ AMD) = 100° ÷ 2 = 50°

In Δ ADM : m ($\angle$ ADM) = 90°

$\therefore \sin (\angle AMD) = \dfrac{AD}{AM}$ $\therefore \sin 50° = \dfrac{AD}{6}$

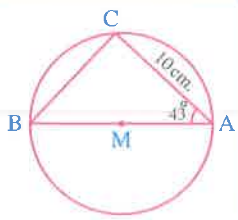
$\therefore AD = 6 \sin 50° \approx 4.596$ cm. $\therefore AB = 2\,AD \approx 9.193$ cm.

TRY TO SOLVE

In the opposite figure :

$\overline{AB}$ is a diameter in the circle M , m ($\angle$ A) = 43° and AC = 10 cm.

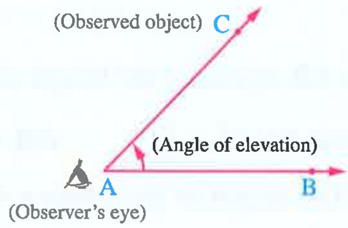
Find the radius length of the circle M to the nearest hundredth.



Lesson Four — Angles of elevation and angles of depression

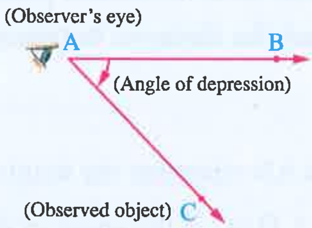
Angle of elevation

If an observer A observed an object C **above** his horizontal sight , then the angle between the horizontal ray $\overrightarrow{AB}$ and the ray $\overrightarrow{AC}$ connecting the observed object and the observer's eye is called **angle of elevation** of C with respect to A



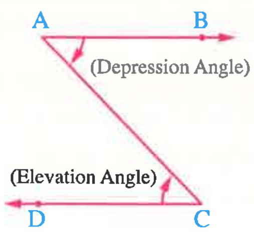
Angle of depression

If an observer A observed an object C **down** his horizontal sight , then the angle between the horizontal ray $\overrightarrow{AB}$ and the ray $\overrightarrow{AC}$ connecting the observed object and the observer's eye is called **angle of depression** of C with respect to A



Remark

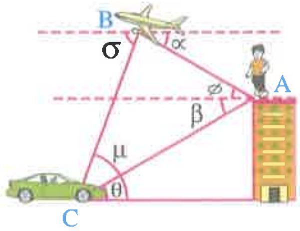
The measure of the depression angle of C with respect to A equals the measure of the elevation angle of A with respect to C because m (∠ A) = m (∠ C) (alternate angles)



Unit 3

Check your understanding

Using the opposite figure , complete the following :

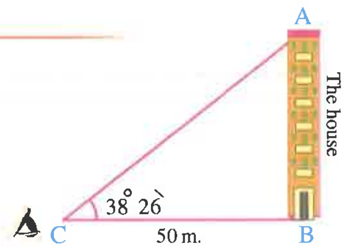


1. The elevation angle of the person A with respect to the car C is ....
2. The depression angle of the car C with respect to the plane B is ....
3. The elevation angle of the plane B with respect to the person A is ....
4. The depression angle of the person A with respect to the plane B is ....

Example 1

A man found that the measure of the angle of elevation of the top of a house at a point 50 metres far from its base is $38° 26'$ **Find the height of the house to the nearest metre.**

Solution



Let AB represent the height of the house.

$\therefore \tan 38° 26' = \frac{AB}{50}$ $\quad \therefore AB = 50 \tan 38° 26' \approx 40$ m.

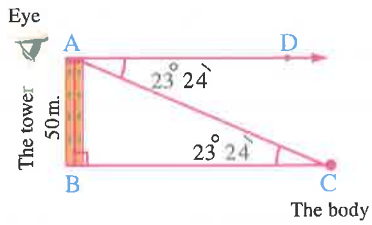
$\therefore$ The height of the house ≈ 40 m.

Example 2

From the top of a tower of height 50 m. , the measure of the angle of depression of a body in the same horizontal plane with its base is $23° 24'$

Find the distance between this body and the base of the tower to the nearest metre.

Solution



Let AB represent the height of the tower.

$\therefore \angle DAC$ is the angle of depression of the body.

$\therefore m(\angle C) = m(\angle DAC) = 23° 24'$ (for $\overrightarrow{AD} \parallel \overline{BC}$)

$\therefore \tan 23° 24' = \frac{50}{BC}$ $\quad \therefore BC = \frac{50}{\tan 23° 24'} \approx 116$ m.

$\therefore$ The distance between the body and the base of the tower ≈ 116 m.

TRY TO SOLVE

From a point on the ground 50 metres far from the base of a vertical pole , it is found that the measure of the elevation angle of the top of the pole is $18° 32'$.

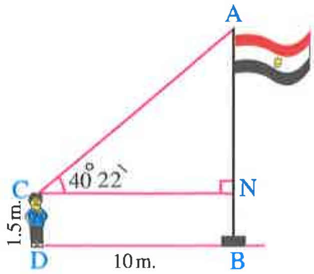
Find to the nearest metre the height of the pole from the ground.

Example 3

A man of height 1.5 m. was standing on the ground at a point which is 10 m. far from a flagpole. He found that the measure of angle of elevation of the top of the flagpole is $40^\circ 22'$.

Find the height of the flagpole to the nearest metre.

Solution



Let AB represent the height of the flagpole and CD represent the height of the man.

Draw $\overline{CN} \parallel \overline{DB}$ where $N \in \overline{AB}$

$\therefore \tan 40^\circ 22' = \frac{AN}{10}$

$\therefore AN = 10 \tan 40^\circ 22' \approx 8.5$ m.

$\therefore AB = AN + NB = 8.5 + 1.5 = 10$ m.

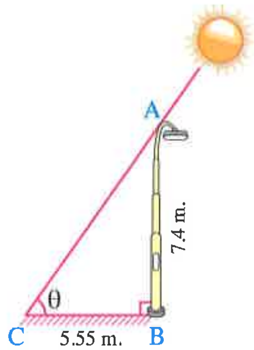
$\therefore$ The height of the flagpole ≈ 10 m.

Example 4

A light pole of height 7.4 m. gives a shade on the ground of length 5.55 m.

Find in radian the measure of the elevation angle of the sun at that moment.

Solution



Let $\overline{AB}$ represent the light pole

, $\overline{BC}$ represent its shade on the ground

, θ the measure of the elevation angle of the sun at that moment.

$\therefore \tan \theta = \frac{AB}{BC} = \frac{7.4}{5.55}$

$\therefore \theta \approx 53^\circ 7' 48''$

$\therefore$ The measure of the elevation angle of the sun in radian $= 53^\circ 7' 48'' \times \frac{\pi}{180^\circ} \approx 0.927^{rad}$

TRY TO SOLVE

From the top of a rock 200 m. high from the sea level , the depression angle of a boat 300 m. apart from the base of the rock was measured.

What is the radian measure of the depression angle ?

Example 5

From the top of a rock of 50 m. high , the measures of the two angles of depression of two sailboats are $32° 10'$ and $49° 30'$

Find the distance between the two sailboats , knowing that the two sailboats and the base of the rock are collinear.

Solution

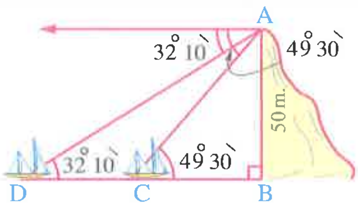
Let AB represent the height of the rock and CD represent the distance between the two sailboats

$\therefore$ In Δ ABD : $\tan 32° 10' = \frac{50}{BD}$

$\therefore BD = \frac{50}{\tan 32° 10'} \approx 79.5$ m.

In Δ ABC : $\tan 49° 30' = \frac{50}{BC}$

$\therefore BC = \frac{50}{\tan 49° 30'} \approx 42.7$ m.



$\therefore CD = 79.5 - 42.7 = 36.8$ m.

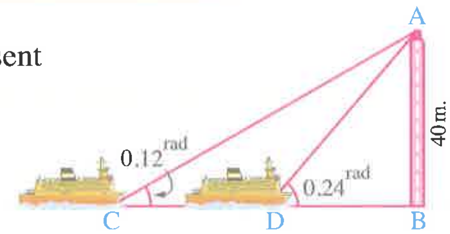
Example 6

A ship approaches a lighthouse 40 m. high from the sea level. At a moment, it was found that the measure of the elevation angle of the top of the lighthouse is 0.12^{rad} , after 5 minutes , it was found again that its elevation angle measure is 0.24^{rad}

Calculate the uniform velocity of the ship.

Solution

Let AB represent the height of the lighthouse and CD represent the distance covered by the ship during 5 minutes.



$\therefore$ In Δ ABC :

$\because \tan C = \frac{AB}{BC}$ $\qquad \therefore \tan 0.12^{rad} = \frac{40}{BC}$

$\therefore BC = \frac{40}{\tan 0.12^{rad}} \approx 331.73$ m.

, in Δ ABD : $\because \tan(\angle ADB) = \frac{AB}{BD}$

$\therefore \tan 0.24^{rad} = \frac{40}{BD}$

$\therefore BD = \frac{40}{\tan 0.24^{rad}} \approx 163.45$ m.

$\therefore DC = 331.73 - 163.45 = 168.28$ m.

$\therefore$ The ship covered 168.28 m. during 5 minutes.

$\therefore$ The uniform velocity of the ship $= \frac{\text{distance}}{\text{time}} = \frac{168.28}{5} = 33.656$ m./min.

Remember that

When calculating the length of $\overline{BC}$ or $\overline{BD}$, we must convert the calculator from (Deg) into (Rad) by pressing :

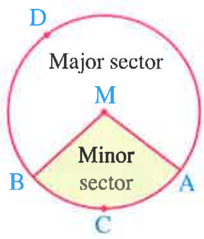
SHIFT → MODE → 4

Lesson Five — Circular sector

Definition :

A circular sector is that part of the circle bounded by an arc and the two radii through the ends of that arc.

In the circle M , if we draw the two radii $\overline{MA}$ and $\overline{MB}$, in the opposite figure , then the circle surface is divided by the two radii into two parts each of them is called a "circular sector".



- The part MACB is called the minor sector while the part MADB is called the major sector.
- $\angle$ AMB is called the angle of the minor sector and the reflex $\angle$ AMB is called the angle of the major sector.
- $\overset{\frown}{ACB}$ is called the arc of the minor sector and $\overset{\frown}{ADB}$ is called the arc of the major sector.

The area of the circular sector

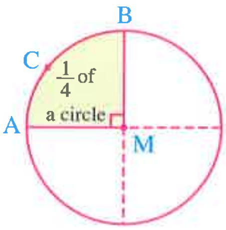


Fig. (1)

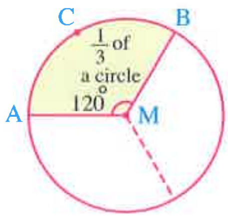


Fig. (2)

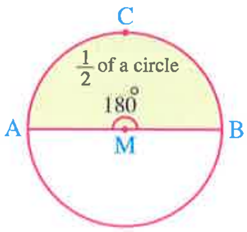


Fig. (3)

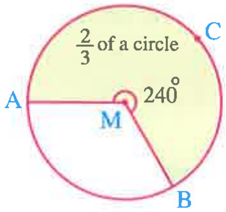


Fig. (4)

Noticing the previous figures , we get that :

Figure (1) : $\dfrac{\text{The area of the sector (MACB)}}{\text{The area of the circle M}} = \dfrac{1}{4}$, $\dfrac{m(\angle AMB)}{\text{Measure of the circle M}} = \dfrac{90°}{360°} = \dfrac{1}{4}$

Figure (2) : $\frac{\text{The area of the sector (MACB)}}{\text{The area of the circle M}} = \frac{1}{3}$, $\frac{m(\angle AMB)}{\text{Measure of the circle M}} = \frac{120°}{360°} = \frac{1}{3}$

Figure (3) : $\frac{\text{The area of the sector (MACB)}}{\text{The area of the circle M}} = \frac{1}{2}$, $\frac{m(\angle AMB)}{\text{Measure of the circle M}} = \frac{180°}{360°} = \frac{1}{2}$

Figure (4) : $\frac{\text{The area of the sector (MACB)}}{\text{The area of the circle M}} = \frac{2}{3}$, $\frac{m(\text{reflex } \angle AMB)}{\text{Measure of the circle M}} = \frac{240°}{360°} = \frac{2}{3}$

i.e. The ratio between the area of the sector and the area of the circle is the same ratio between the measure of the angle of the sector and the measure of the circle.

$$\frac{\text{Area of the sector}}{\text{Area of the circle}} = \frac{\text{Measure of the angle of the sector}}{\text{Measure of the circle}}$$

- If we symbolized the measure of the angle of the sector in radian measure by the symbol θ^{rad} and its measure in degree measure by the symbol $x°$, the radius length of the circle by the symbol r and the length of the arc of the sector by the symbol ℓ , then :

1 $\frac{\text{Area of the circular sector}}{\pi r^2} = \frac{\theta^{rad}}{2\pi}$

$\therefore$ The area of the circular sector $= \frac{\theta^{rad}}{2\pi} \times \pi r^2$

i.e. The area of the circular sector $= \frac{1}{2} \theta^{rad} r^2$

2 $\frac{\text{Area of the circular sector}}{\pi r^2} = \frac{x°}{360°}$

$\therefore$ The area of the circular sector $= \frac{x°}{360°} \times \pi r^2$

i.e. The area of the circular sector $= \frac{x°}{360°} \times$ the area of the circle

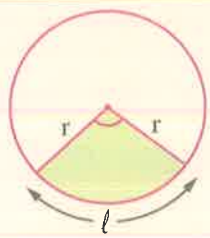
3 $\because \theta^{rad} = \frac{\ell}{r}$, $\because$ the area of the circular sector $= \frac{1}{2} \theta^{rad} r^2$

$\therefore$ The area of the circular sector $= \frac{1}{2} \times \frac{\ell}{r} \times r^2$

i.e. The area of the circular sector $= \frac{1}{2} \ell r$

Remarks

1 We can consider that the circle is a circular sector of angle measure = 360° and the area of the circular sector = the area of the circle = πr^2



2 The perimeter of the circular sector = $2r + \ell$

Example 1

Find the area of the circular sector where the length of its arc is ℓ in a circle of radius length r , if the measure of its angle is θ^{rad} in radian measure and $x°$ in degree measure in each of the following :

1. $r = 10$ cm. , $\theta^{rad} = 1.5^{rad}$
2. $r = 10.5$ cm. , $x° = 144°$
3. $r = 6$ cm. , $\ell = 4$ cm.

Solution

1. The area of the sector $= \frac{1}{2}\,\theta^{rad}\,r^2 = \frac{1}{2} \times 1.5 \times (10)^2 = 75$ cm^2.
2. The area of the sector $= \frac{x°}{360°} \times \pi\,r^2 = \frac{144°}{360°} \times \pi \times (10.5)^2 \approx 138.5$ cm^2.
3. The area of the sector $= \frac{1}{2}\,\ell\,r = \frac{1}{2} \times 4 \times 6 = 12$ cm^2.

TRY TO SOLVE

Find :

1. The area of the sector in which the length of the radius of its circle = 7 cm. and the measure of its central angle is 2.1^{rad}
2. The area of the sector in which the length of the radius of its circle = 6.5 cm. and the length of its arc is 8 cm.
3. The area of the sector in which the measure of its angle is 60° in a circle of radius length 5 cm.

Example 2

The perimeter of a circular sector is 55 cm. and the length of the radius of its circle is 12 cm. **Find its area.**

Solution

$\because$ r = 12 cm. , the perimeter of the sector = 55 cm.

, $\because$ the perimeter of the sector $= 2\,r + \ell$

$\therefore 55 = 2\,(12) + \ell$ $\qquad\qquad$ $\therefore \ell = 31$ cm.

$\therefore$ The area of the sector $= \frac{1}{2}\,\ell\,r = \frac{1}{2}\,(31)\,(12) = 186$ cm^2.

Example 3

The area of a circular sector is 270 cm^2 and the length of the radius of its circle is 15 cm.

Find : 1 The length of the arc of this sector.

2 The measure of its central angle in radians and in degrees.

Solution

1 $\because$ r = 15 cm. , the area of the sector = 270 cm^2

, $\because$ the area of the sector $= \frac{1}{2}\ell r$ $\quad \therefore 270 = \frac{1}{2}\ell(15)$ $\quad \therefore \ell = 36$ cm.

2 $\because \ell = 36$ cm. and r = 15 cm.

$\therefore \theta^{rad} = \frac{\ell}{r} = \frac{36}{15} = 2.4^{rad}$ $\quad \therefore x° = 2.4^{rad} \times \frac{180°}{\pi} \approx 137° 31'$

Example 4

The area of a circular sector is 75 cm^2 and its perimeter is 35 cm. **Find the radius length of its circle and the measure of its central angle in degree measure.**

Solution

$\because$ The area of the sector = 75 $\quad \therefore \frac{1}{2}\ell r = 75$

$\therefore \ell r = 150$ $\quad$ (1)

, $\because$ the perimeter of the sector = 35 $\quad \therefore \ell + 2r = 35$

$\therefore \ell = 35 - 2r$ $\quad$ (2)

Substituting from (2) in (1) : $\quad \therefore (35 - 2r)r = 150$

$\therefore 2r^2 - 35r + 150 = 0$ $\quad \therefore (r - 10)(2r - 15) = 0$

$\therefore r = 10$ cm. $\quad$ or $\quad r = 7\frac{1}{2}$ cm. substituting in (1)

$\therefore \ell = 15$ cm. $\quad \therefore \ell = 20$ cm.

, $\because \theta^{rad} = \frac{\ell}{r} = \frac{15}{10} = 1.5^{rad}$ $\quad$, $\because \theta^{rad} = \frac{\ell}{r} = \frac{20}{7.5} = \frac{8}{3}^{rad}$

$\therefore x° = 1.5^{rad} \times \frac{180°}{\pi} \approx 85° 56' 37''$ $\quad \therefore x° = \frac{8}{3}^{rad} \times \frac{180°}{\pi} \approx 152° 47' 19''$

TRY TO SOLVE

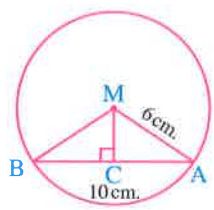
The area of a circular sector is 120 cm^2 and the length of its arc is 20 cm.

Find the measure of its angle in radian measure and in degree measure , then find the perimeter of the sector.

Example 5

A circle M is of radius length 6 cm. , the two radii $\overline{MA}$ and $\overline{MB}$ are drawn in this circle such that AB = 10 cm. **Find the area of the minor sector MAB to the nearest cm^2**

Solution



Draw $\overline{MC} \perp \overline{AB}$ and cuts it at C , then C is the midpoint of $\overline{AB}$

$\therefore$ AC = CB = 5 cm.

$\because$ Δ ACM in which : m ($\angle$ ACM) = 90°

$\therefore \sin(\angle AMC) = \frac{AC}{AM} = \frac{5}{6}$ $\qquad \therefore m(\angle AMC) \approx 56° \, 26' \, 34''$

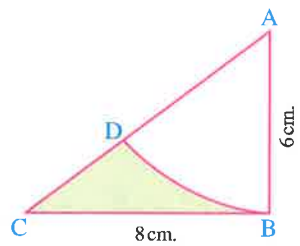
$\therefore m(\angle AMB) = 2 \times 56° \, 26' \, 34'' = 112° \, 53' \, 8''$

$\therefore$ The area of the minor sector MAB $= \frac{x°}{360°} \times \pi r^2 = \frac{112° \, 53' \, 8''}{360°} \times \pi \times (6)^2 \approx 35.46$ cm^2

Example 6

ABC is a right-angled triangle at B , in which AB = 6 cm. and BC = 8 cm.
Draw a circular arc with centre A and with radius AB to intersect $\overline{AC}$ at D
Find to the nearest cm^2 the area of the zone bounded by $\overline{BC}$ and $\overline{CD}$ and $\overset{\frown}{BD}$

Solution



The required area = the area of Δ ABC – the area of the sector ABD

The area of $\Delta ABC = \frac{1}{2}(AB)(BC) = \frac{1}{2} \times 6 \times 8 = 24$ cm^2

$\because$ r = AB = 6 cm. , $\tan(\angle BAD) = \frac{BC}{AB} = \frac{8}{6}$

$\therefore m(\angle BAD) \approx 53° \, 7' \, 48''$

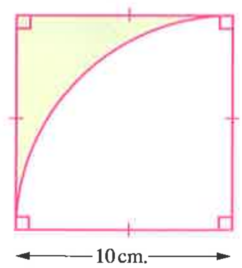
$\therefore$ The area of the sector ABD $= \pi r^2 \times \frac{x°}{360°} = \pi (6)^2 \times \frac{53° \, 7' \, 48''}{360°} \approx 17$ cm^2

$\therefore$ The required area = 24 – 17 = 7 cm^2

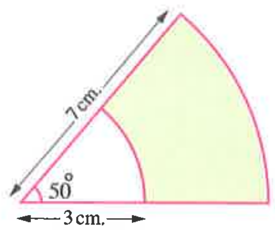
TRY TO SOLVE

Find the area of the shaded part in each of the following in terms of π :

1



2



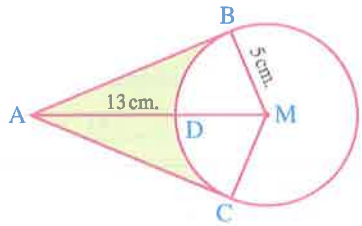
Unit 3

Example 7

$\overline{AB}$ and $\overline{AC}$ are two tangents from A to the circle M to touch it at B and C, where the length of the radius of the circle is 5 cm. and AM = 13 cm.

Find to the nearest cm² the area of the zone bounded by $\overline{AB}$, $\overline{AC}$ and $\overset{\frown}{BC}$

Solution



The area of the required zone

= the area of the figure ABMC – the area of the sector BMCD

$\because$ $\overline{AB}$, $\overline{AC}$ are tangents, $\overline{BM}$, $\overline{CM}$ are radii

$\therefore$ m ($\angle$ MBA) = m ($\angle$ ACM) = 90°

$\therefore$ AB = AC = $\sqrt{(13)^2-(5)^2}$ = 12 cm.

$\therefore$ The area of ABMC = 2 × the area of Δ ABM

$= 2 \times \frac{1}{2} \times 5 \times 12 = 60 \text{ cm}^2$ (1)

$\because$ m ($\angle$ B) = 90° in Δ MBA

$\therefore \cos(\angle BMA) = \frac{5}{13}$

$\therefore$ m ($\angle$ BMA) $\approx 67° \, 22' \, 48''$

$\therefore$ m ($\angle$ BMC) $= 2 \times 67° \, 22' \, 48'' \approx 134° \, 45' \, 36''$

$\therefore$ The area of the sector BMCD

$= \pi r^2 \times \frac{x°}{360°} = \pi (5)^2 \times \frac{134° \, 45' \, 36''}{360°} \approx 29 \text{ cm}^2$ (2)

From (1) and (2) :

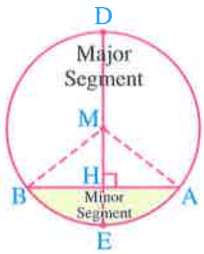
$\therefore$ The area of the required zone = 60 – 29 = 31 cm²

Lesson Six

Circular segment

Definition :

The circular segment is a part of the surface of the circle bounded by an arc and a chord passing by the ends of this arc.



- If we draw the chord $\overline{AB}$ in the circle M , then the surface of the circle is divided into two parts , each of them is called a "circular segment".
- The central angle $\angle$ AMB is called the angle of the circular segment.
- The angle AMB in the given figure is the angle of the minor segment AEB , while the reflex angle $\angle$ AMB is the angle of the major segment ADB
- If $\overline{DE}$ is a diameter such that $\overline{DE} \perp \overline{AB}$ and $\overline{DE} \cap \overline{AB} = \{H\}$, then EH is called the height of the minor segment.
- **Notice that :**

- The area of the minor segment = the area of the circular sector MAEB – the area of Δ MAB
- The area of the major segment = the area of the sector MADB + the area of Δ MAB

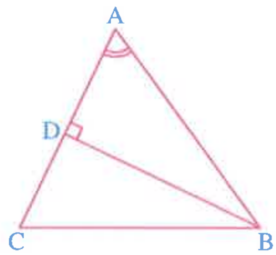
Hence , to find the area of the circular segment , we must find the area of the triangle , whose base is the chord of the circular segment and its vertex is the centre of the circle.

The area of a triangle knowing the lengths of two of its sides and the measure of the included angle between them :

Suppose we have Δ ABC , in which the length of $\overline{AB}$, the length of $\overline{AC}$ and m ($\angle$ A) are known.

Draw $\overline{BD} \perp \overline{AC}$

$\therefore$ The area of Δ ABC $= \frac{1}{2}$ (AC) (BD) (1)



But m ($\angle$ BDA) = 90° in Δ ABD

$\therefore \frac{BD}{AB} = \sin A$ **i.e.** BD = AB sin A (2)

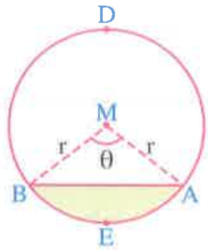
Substituting from (2) in (1) , we get :

The area of Δ ABC $= \frac{1}{2}$ (AC) (AB sin A)

$\therefore$ The area of the triangle $= \frac{1}{2}$ the product of the lengths of two of its sides × sin (the included angle between them)

The area of the circular segment

Let us get the area of the minor circular segment AEB ,
in a circle of radius length r and its central angle is of measure θ^{rad}



$\therefore$ The area of the circular sector MAEB $= \frac{1}{2} \theta^{rad} r^2$

and the area of Δ MAB $= \frac{1}{2}$ (MA) (MB) $\sin (\angle AMB) = \frac{1}{2} r^2 \sin \theta$

$\therefore$ The area of the circular segment AEB

= the area of the circular sector MAEB – the area of Δ MAB

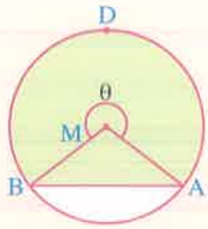
$= \frac{1}{2} r^2 \theta^{rad} - \frac{1}{2} r^2 \sin \theta = \frac{1}{2} r^2 (\theta^{rad} - \sin \theta)$

$\therefore$ The area of the circular segment $= \frac{1}{2} r^2 (\theta^{rad} - \sin \theta)$

Remarks

1 The area of the major segment ADB

= the area of the circular sector MADB + the area of Δ MAB



$= \frac{1}{2} r^2 \theta^{rad} + \frac{1}{2} r^2 \sin (2\pi - \theta)$

$= \frac{1}{2} r^2 \theta^{rad} - \frac{1}{2} r^2 \sin \theta$ $\left[\text{for } \sin (2\pi - \theta) = -\sin \theta\right]$

$= \frac{1}{2} r^2 (\theta^{rad} - \sin \theta)$

2 We can find the area of the major circular segment by subtracting the area of the minor circular segment from the area of the circle.

3 The perimeter of the circular segment = the length of its arc + the length of its chord.

Example 1

Find the area of a circular segment if the measure of its central angle is 120° and the length of the radius of its circle is 8 cm.

Solution

$\because \theta^{rad} = 120° \times \dfrac{\pi}{180°} \approx 2.0944^{rad}$

$\therefore$ The area of the circular segment $= \dfrac{1}{2} r^2 (\theta^{rad} - \sin\theta)$

$$= \frac{1}{2} (8)^2 (2.0944 - \sin 120°) \approx 39.3 \text{ cm}^2.$$

Remark

In the previous example , we can use the radian measure of the central angle to find the area of the circular segment instead of the degree measure :

The area of the circular segment $= \dfrac{1}{2} (8)^2 (2.0944^{rad} - \sin 2.0944) \approx 39.3 \text{ cm}^2.$ noticing that we must convert the calculator system from (Deg) into (Rad) before finding the area by pressing SHIFT → MODE → 4

Example 2

Find the area of the circular segment whose length of the radius of its circle is 10 cm. and the measure of its central angle is 1.02^{rad} approximating the result to the nearest hundredth.

Solution

The area of the circular segment $= \dfrac{1}{2} r^2 (\theta^{rad} - \sin\theta)$

$$= \frac{1}{2} (10)^2 (1.02^{rad} - \sin 1.02^{rad}) \approx 8.39 \text{ cm}^2.$$

TRY TO SOLVE

Find the area of the circular segment whose length of the radius of its circle is r and the measure of its central angle is θ if :

1. $r = 12$ cm. , $\theta = 150°$
2. $r = 8$ cm. , $\theta^{rad} = 2.02^{rad}$

Example 3

A circular segment where the radius length of its circle is 10 cm. and the length of its arc is 26.19 cm. **Find its area.**

Solution

$\because \theta^{rad} = \frac{\ell}{r} = \frac{26.19}{10} = 2.619^{rad}$

$\therefore$ The area of the circular segment

$= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta) = \frac{1}{2} (10)^2 (2.619^{rad} - \sin 2.619^{rad}) \approx 105.99 \text{ cm}^2$

Example 4

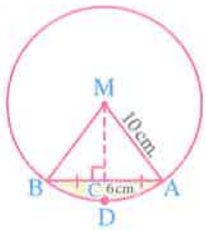
Find the area of a minor circular segment whose chord is of length 12 cm. in a circle of radius length 10 cm.

Solution

Let $\overline{AB}$ represent the chord of the segment and M be the centre of the circle.

Draw $\overline{MC} \perp \overline{AB}$, thus C is the midpoint of $\overline{AB}$

i.e. AC = CB = 6 cm.



From Δ AMC, we have :

$\sin(\angle AMC) = 0.6$

$\therefore m(\angle AMC) \approx 36° 52'$

$\therefore m(\angle AMB) = 2(36° 52') = 73° 44'$

$\therefore \theta^{rad} = 73° 44' \times \frac{\pi}{180°} \approx 1.2869^{rad}$

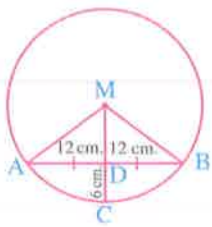
$\therefore$ The area of the circular segment ADB

$= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta) = \frac{1}{2} (100)(1.2869^{rad} - \sin 73° 44') \approx 16.35 \text{ cm}^2$

Example 5

Find the area of the minor circular segment in which the length of its chord is 24 cm. and its height is 6 cm.

Solution



Let $\overline{AB}$ be the chord of the segment in the circle M and draw $\overline{MC} \perp \overline{AB}$ cutting $\overline{AB}$ at D and the circle at C

Thus, CD is the height of the segment. $\therefore$ CD = 6 cm.

, $\because \overline{MD} \perp \overline{AB}$ $\therefore$ AD = DB = 12 cm.

, $\because$ MC = MA = r $\qquad \therefore$ MD = r – 6

$\because$ Δ ADM in which m ($\angle$ ADM) = 90° $\qquad \therefore (AM)^2 = (AD)^2 + (MD)^2$

$\therefore r^2 = 144 + (r - 6)^2 \qquad \therefore r^2 = 144 + r^2 - 12\,r + 36$

$\therefore 12\,r = 180 \qquad \therefore$ r = 15 cm.

, $\because \sin(\angle AMD) = \frac{AD}{AM} = \frac{12}{15} = \frac{4}{5} \qquad \therefore m(\angle AMD) \approx 53^\circ\ 7'\ 48''$

$\therefore m(\angle AMB) = 2 \times 53^\circ\ 7'\ 48'' = 106^\circ\ 15'\ 36''$

, $\because \theta^{\underline{rad}} = 106^\circ\ 15'\ 36'' \times \frac{\pi}{180^\circ} \approx 1.85^{\underline{rad}}$

$\therefore$ The area of the minor circular segment $= \frac{1}{2} \times r^2\,(\theta^{\underline{rad}} - \sin\theta)$

$= \frac{1}{2} \times (15)^2\,(1.85^{\underline{rad}} - \sin 106^\circ\ 15'\ 36'')$

$= 100.125\ cm^2$.

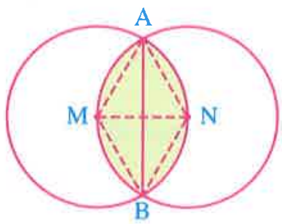
TRY TO SOLVE

Find the area of the circular segment in which its height is 3 cm. and the radius length of its circle is 10 cm.

Example 6

Two congruent circles in which the radius length of each one is 6 cm. and one of them passes through the centre of the other. **Find the area of the included part between them.**

Solution



Let the two circles intersect at A and B

$\therefore \overline{AB}$ divides the included part between the two circles into two equal circular segments in area.

, $\because$ Δ AMN is an equilateral triangle in which :

MA = MN = AN = 6 cm.

, Δ BMN is an equilateral triangle in which :

MB = MN = NB = 6 cm.

$\therefore m(\angle AMN) = m(\angle BMN) = 60^\circ \qquad \therefore m(\angle AMB) = m(\angle ANB) = 120^\circ$

, $\because \theta^{\underline{rad}} = 120^\circ \times \frac{\pi}{180^\circ} = \frac{2}{3}\pi$

$\therefore$ The area of the minor circular segment ANB $= \frac{1}{2} r^2\,(\theta^{\underline{rad}} - \sin\theta)$

$= \frac{1}{2} \times 6^2 \left(\frac{2}{3}\pi - \sin 120^\circ\right) \approx 22.11\ cm^2$

$\therefore$ The area of the included part between the two circles $= 2 \times 22.11 = 44.22\ cm^2$

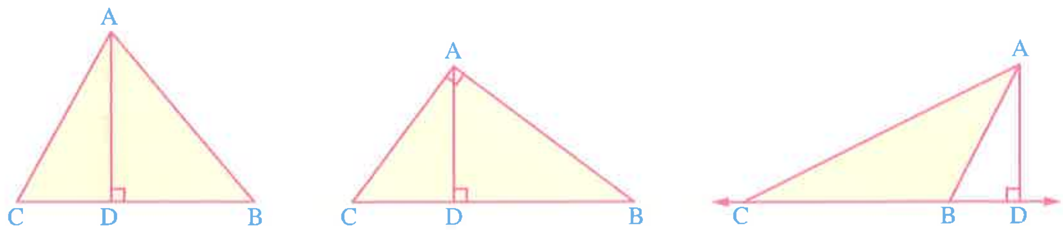
Lesson Seven — Areas

First — The area of the triangle

You have previously studied the area of the triangle and known that :

First The area of the triangle = $\frac{1}{2}$ the length of the base × corresponding height

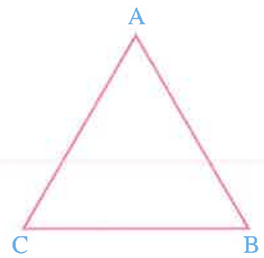
i.e. In any triangle ABC , if $\overline{AD} \perp \overleftrightarrow{BC}$, then :



The area of Δ ABC = $\frac{1}{2}$ BC × AD

Second The area of the triangle = $\frac{1}{2}$ the product of the lengths of two sides × sine of the included angle between them.

i.e. In any triangle ABC



$$\text{The area of } \Delta ABC = \frac{1}{2} AB \times AC \times \sin A$$

$$= \frac{1}{2} AB \times BC \times \sin B$$

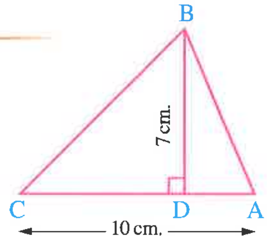
$$= \frac{1}{2} AC \times BC \times \sin C$$

Example 1

Calculate the area of the triangle ABC in each of the following cases :

1. AC = 10 cm. and the length of the perpendicular drawn from B to $\overline{AC}$ equals 7 cm.
2. AB = 12 cm. , BC = 15 cm. and m (∠ B) = 90°
3. AB = 11 cm. , BC = 10 cm. and m (∠ B) = 47° approximating the result to the nearest hundredth.
4. AB = 25 cm. , BC = 17 cm. and AC = 26 cm.

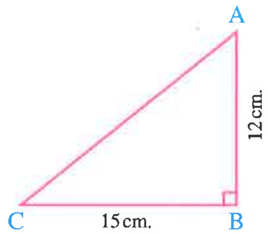
Solution



1 The area of the triangle ABC $= \frac{1}{2}$ AC × BD

$= \frac{1}{2} \times 10 \times 7$

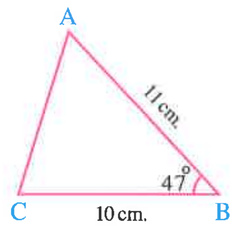
$= 35$ cm².



2 The area of the triangle ABC $= \frac{1}{2}$ BC × AB

$= \frac{1}{2} \times 15 \times 12$

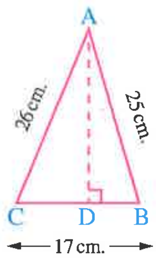
$= 90$ cm².



3 The area of the triangle ABC $= \frac{1}{2}$ AB × BC × sin B

$= \frac{1}{2} \times 11 \times 10 \times \sin 47°$

≈ 40.22 cm².



4 Draw $\overline{AD} \perp \overline{BC}$, let BD = x cm.

, thus DC = $(17 - x)$ cm.

In Δ ABD : ∵ m (∠ ADB) = 90° ∴ $(AD)^2 = (AB)^2 - (BD)^2$

∴ $(AD)^2 = 625 - x^2$ (1)

In Δ ADC : ∵ m (∠ ADC) = 90° ∴ $(AD)^2 = (AC)^2 - (DC)^2$

∴ $(AD)^2 = 676 - (17 - x)^2$ (2)

From (1) and (2) , we get that : $625 - x^2 = 676 - (17 - x)^2$

∴ $625 - \cancel{x^2} = 676 - 289 + 34\,x - \cancel{x^2}$ ∴ $34\,x = 238$ ∴ $x = 7$ cm.

Substituting in (1) :

∴ $(AD)^2 = 625 - 49 = 576$ ∴ AD = 24 cm.

∴ The area of the triangle ABC $= \frac{1}{2}$ BC × AD $= \frac{1}{2} \times 17 \times 24 = 204$ cm².

Unit 3

Hero's formula for calculating the area of the triangle

If we symbolized the perimeter of the triangle ABC (the sum of the side lengths of the triangle) by 2 S , then the area of the triangle ABC = $\sqrt{S(S-AB)(S-BC)(S-AC)}$

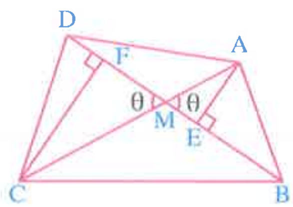
Check the solution of the previous example by using Hero's formula.

TRY TO SOLVE

Calculate the area of the triangle ABC in each of the following cases approximating the result to the nearest hundredth :

1. The triangle ABC is equilateral and its side length is 6 cm.
2. AB = 12 cm. , BC = 15 cm. and m ($\angle$ B) = 62°

Second The area of the convex quadrilateral



In the opposite figure :

ABCD is a quadrilateral in which its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at M and include between them an angle of measure θ , if $\overline{AE} \perp \overline{BD}$, $\overline{CF} \perp \overline{BD}$

, then the area of the quadrilateral ABCD = the area of Δ ABD + the area of Δ BCD

$$= \frac{1}{2} BD \times AE + \frac{1}{2} BD \times CF$$

$$= \frac{1}{2} BD (AE + CF)$$

$\because$ In Δ AEM : m ($\angle$ AEM) = 90° $\therefore \sin\theta = \frac{AE}{AM}$ $\therefore AE = AM \sin\theta$

, in Δ CFM : m ($\angle$ CFM) = 90° $\therefore \sin\theta = \frac{CF}{CM}$ $\therefore CF = CM \sin\theta$

$\therefore$ The area of the quadrilateral ABCD $= \frac{1}{2} BD (AM \sin\theta + CM \sin\theta)$

$$= \frac{1}{2} BD \sin\theta (AM + CM)$$

$$= \frac{1}{2} BD \sin\theta \times AC = \frac{1}{2} BD \times AC \times \sin\theta$$

i.e. The area of the quadrilateral = $\frac{1}{2}$ the product of the lengths of its diagonals × sine of the included angle between them.

Remark

If we used $\angle$ AMD of measure (180° – θ) that is the supplementary angle of $\angle$ AMB of measure θ , then the area of the quadrilateral ABCD does not change because sin (180° – θ) = sin θ

Example 2

Find the area of the quadrilateral in which the lengths of its diagonals are 10 cm. , 12 cm. and the measure of the included angle between them is 62°

Solution

The area of the quadrilateral = $\frac{1}{2}$ the product of the lengths of its diagonals × sine of the included angle between them

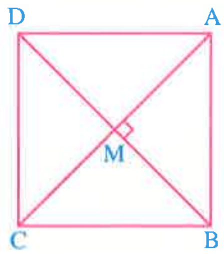
$= \frac{1}{2} \times 10 \times 12 \times \sin 62° \approx 52.98 \text{ cm}^2$.

Remark

We can use the previous law for calculating the areas of some special quadrilaterals as :

1 Square :

In the opposite figure :



ABCD is a square ∴ AC = BD , $\overline{AC} \perp \overline{BD}$

∴ The area of the square ABCD $= \frac{1}{2} \times AC \times BD \times \sin 90°$

$= \frac{1}{2} \times AC \times AC \times 1 = \frac{1}{2} (AC)^2$

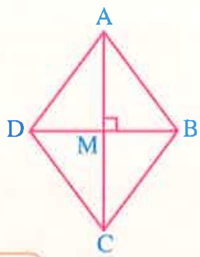
∴ The area of the square = $\frac{1}{2}$ the square of its diagonal length

For example :

The area of the square whose diagonal length is 6 cm. $= \frac{1}{2} \times (6)^2 = 18 \text{ cm}^2$.

2 Rhombus :

In the opposite figure :



ABCD is a rhombus ∴ $\overline{AC} \perp \overline{BD}$

∴ The area of the rhombus ABCD $= \frac{1}{2} \times AC \times BD \times \sin 90°$

$= \frac{1}{2} \times AC \times BD$

∴ The area of the rhombus = $\frac{1}{2}$ the product of its diagonal lengths

For example :

The area of the rhombus whose diagonal lengths are 6 cm. and 8 cm. $= \frac{1}{2} \times 6 \times 8 = 24 \text{ cm}^2$.

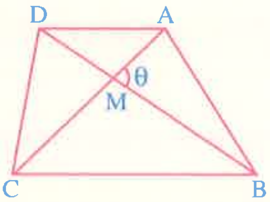
TRY TO SOLVE

Find :

1. The area of the square whose diagonal length is 8 cm.
2. The area of the rhombus whose diagonal lengths are 12 cm. and 16 cm.
3. The area of the quadrilateral whose diagonal lengths are 6 cm. , 8 cm. and the measure of the included angle between them is 120°

Remark

In the quadrilateral ABCD , if its two diagonals intersect at M , then



Area (Δ ABM) $\times$ Area (Δ DCM)

$= $ Area (Δ ADM) $\times$ Area (Δ BCM)

Proof

Area (Δ ABM) $\times$ Area (Δ DCM)

$$= \left[\frac{1}{2} (MA)(MB) \sin\theta\right] \times \left[\frac{1}{2} (MD)(MC) \sin\theta\right]$$

$$= \left[\frac{1}{2} (MA)(MD) \sin(180° - \theta)\right] \times \left[\frac{1}{2} (MB)(MC) \sin(180° - \theta)\right]$$

$= $ Area (Δ ADM) $\times$ Area (Δ BCM)

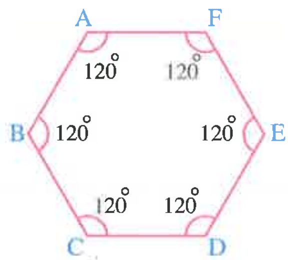
Third The area of the regular polygon

- **The regular polygon :** It is a polygon in which all interior angles are equal in measure and all sides are equal in length.

- The measure of the vertex angle of a regular polygon in which the number of its sides is n sides $= \dfrac{(n-2) \times 180°}{n}$

For example :

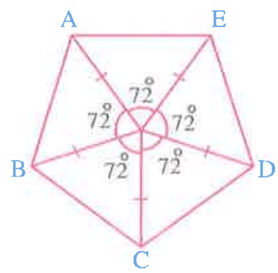
The measure of the vertex angle of the regular hexagon is $\dfrac{(6-2) \times 180°}{6} = 120°$



- We can divide the regular polygon whose the number of its sides is n sides into a number n of the congruent isosceles triangles and the measure of the vertex angle of each of them $= \dfrac{2\pi}{n}$

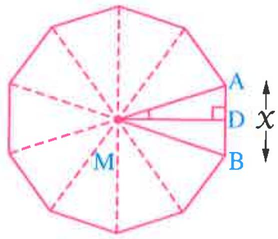
For example :

The regular pentagon is divided into 5 congruent triangles , each one of them is isosceles and the measure of its vertex angle is $\dfrac{2\pi}{5} = 72°$



The area of the regular polygon :

In the opposite figure :



A regular polygon in which the number of its sides is n sides and the length of its side = x length unit.

Then the area of the polygon = the area of Δ AMB $\times$ n

$\because$ Δ AMB is an isosceles triangle in which : MA = MB , m ($\angle$ AMB) = $\frac{2\pi}{n}$

, $\because$ $\overline{MD} \perp \overline{AB}$ $\therefore$ m ($\angle$ AMD) = $\frac{\pi}{n}$

, $\because$ cot ($\angle$ AMD) = $\frac{MD}{AD}$ $\therefore$ $\cot\frac{\pi}{n} = \frac{MD}{\frac{1}{2}x}$ $\therefore$ $MD = \frac{1}{2}x\cot\frac{\pi}{n}$

, $\because$ the area of Δ AMB $= \frac{1}{2}$ base length $\times$ height $= \frac{1}{2} \times AB \times MD$

$$= \frac{1}{2} \times x \times \frac{1}{2}x\cot\frac{\pi}{n} = \frac{1}{4}x^2\cot\frac{\pi}{n}$$

$\therefore$ The area of the polygon $= \left(\frac{1}{4}x^2\cot\frac{\pi}{n}\right) \times n = \frac{1}{4}n x^2\cot\frac{\pi}{n}$

i.e. The area of the regular polygon whose the number of its sides is n sides and the length of its side is $x = \frac{1}{4}n x^2\cot\frac{\pi}{n}$

Example 3

Find the area of each of :

1 a regular octagon of side length 7 cm. (to the nearest hundredth)

2 a regular polygon whose the number of its sides is 12 sides and its side length = 10 cm. (to the nearest centimetre square)

3 an equilateral triangle of side length 9 cm. (to the nearest thousandths)

Solution

1 The area of the regular octagon $= \frac{1}{4}n x^2\cot\frac{\pi}{n} = \frac{1}{4} \times 8 \times 7^2 \times \cot\frac{\pi}{8} \approx 236.59\ cm^2$.

2 The area of the polygon of 12 sides $= \frac{1}{4}n x^2\cot\frac{\pi}{n} = \frac{1}{4} \times 12 \times 10^2 \times \cot\frac{\pi}{12} \approx 1120\ cm^2$.

3 The area of the equilateral triangle $= \frac{1}{4}n x^2\cot\frac{\pi}{n} = \frac{1}{4} \times 3 \times 9^2 \times \cot\frac{\pi}{3} \approx 35.074\ cm^2$.

Another solution :

The area of the triangle $= \frac{1}{2}$ the product of two side lengths $\times$ sine of the included angle between them $= \frac{1}{2} \times 9 \times 9 \times \sin 60° \approx 35.074\ cm^2$.

Remarks

- The equilateral triangle is a regular trilateral , so we can use the law of calculating the area of the regular polygon to find its area as the previous example , then :

The area of the equilateral triangle $= \frac{1}{4} \times 3 \times x^2 \times \cot \frac{\pi}{3} = \frac{3}{4} x^2 \times \cot 60°$

$$= \frac{3}{4} x^2 \times \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{4} x^2$$

i.e. The area of the equilateral triangle $= \frac{\sqrt{3}}{4} x^2$ where x is the side length of the triangle.

- Using the same way , we can find the area of the regular hexagon :

The area of the regular hexagon $= \frac{1}{4} \times 6 \times x^2 \times \cot \frac{\pi}{6} = \frac{3}{2} x^2 \times \cot 30° = \frac{3\sqrt{3}}{2} x^2$

i.e. The area of the regular hexagon $= \frac{3\sqrt{3}}{2} x^2$ where x is its side length.

TRY TO SOLVE

Use the law of calculating the area of the regular polygon to find the area of each of :

1. an equilateral triangle of side length 15 cm. (approximating the result to the nearest hundredth)
2. a square of side length 6 cm.
3. a regular pentagon of side length 12 cm. (approximating the result to the nearest thousandth)

Second

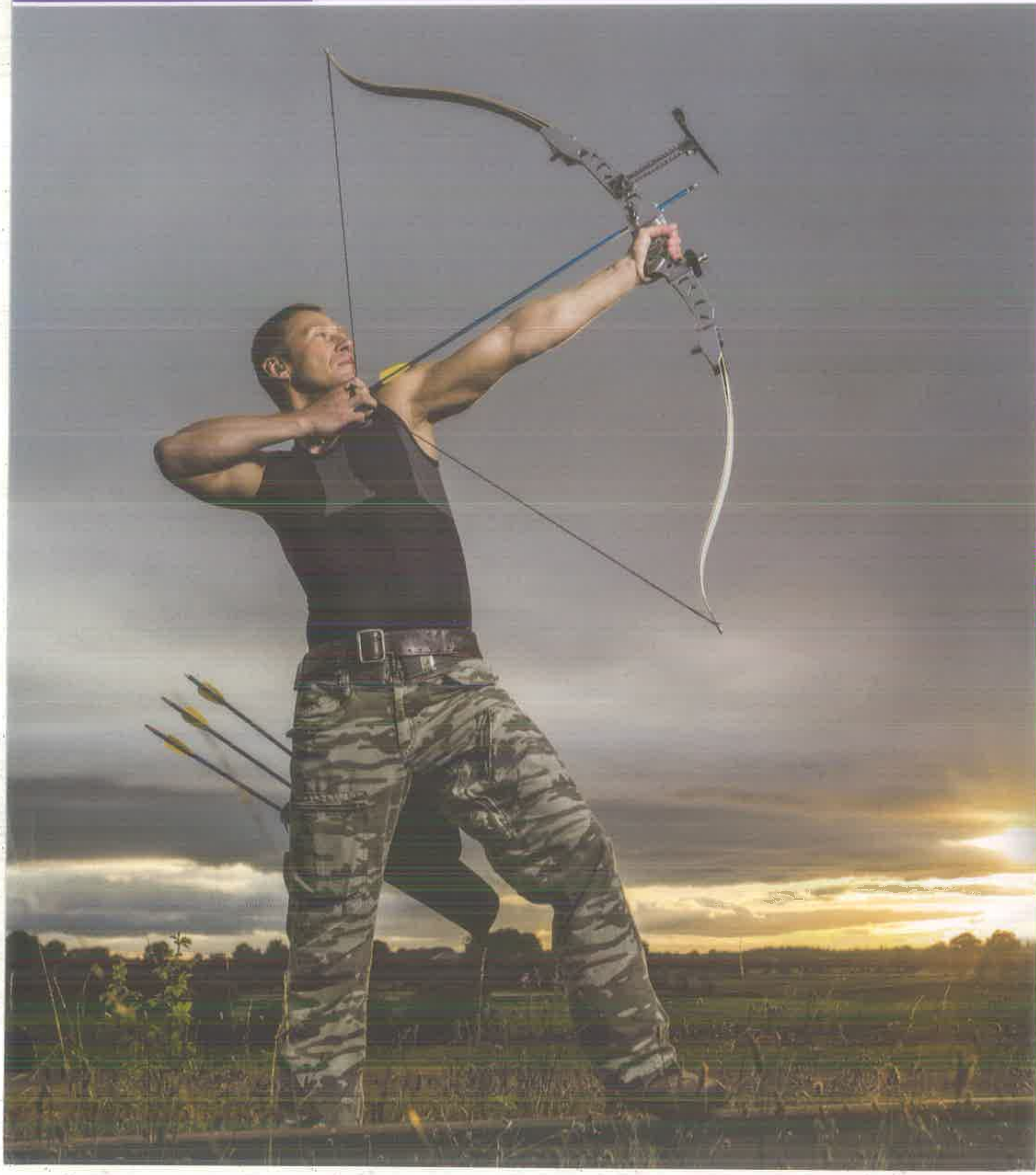
Analytic Geometry

Unit Four Vectors

Unit Five Straight line

Unit 4

VECTORS



Unit Lessons

Lesson One	:	Scalars , vectors and directed line segment.
Lesson Two	:	Vectors.
Lesson Three	:	Operations on vectors.
Lesson Four	:	Applications on vectors.

Learning outcomes

By the end of this unit, the student should be able to :

- Recognize the scalar quantity , the vector quantity and the directed line segment and express it in terms of its two ends in the coordinate plane.
- Recognize the position vector and put it in the polar form.
- Find the norm of the vector and zero vector.
- Recognize and solve exercises on equivalent vectors.
- Recognize the unit vector and express the vector in terms of the fundamental unit vectors.
- Recognize parallel and perpendicular vectors.
- Multiply a vector by a real number.
- Add two vectors using the triangle rule (Coordinates - Parallelogram rule) - Subtract two vectors.
- Prove some geometric theorems using vectors.
- Solve applications in the plane geometry on vectors.

Lesson One

Scalars , vectors and directed line segment

Scalar quantity and vector quantity

• The quantities which we deal with in our life are two kinds :

1 Scalar quantity :

It is a quantity determined completely by a real number which is the magnitude of this quantity.

For example : Length - mass - time - temperature - volume - distance.

2 Vector quantity :

It is a quantity determined completely by a real number which is the magnitude of this quantity , and the direction.

For example : Force - displacement - velocity.

• To show the difference between the scalar quantity and the vector quantity , we show - for example - the difference between distance as a scalar quantity and displacement as a vector quantity.

1. **Distance :** It is the length of the actual path covered during movement from a position to another , and it is a scalar quantity because it is determined completely by its magnitude only without direction.

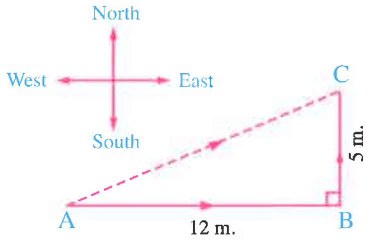
2. **Displacement :** It is the smallest distance between the starting point and the ending point , and in the direction from the starting point to the ending point.

   i.e. It is the distance covered in a certain direction , and it is a vector quantity because it is determined completely by its magnitude and its direction.

For example :

In the opposite figure :

If a body moved from the point A a distance 12 m. east , then changed its direction and moved 5 m. north and stopped at the point C , then :



the distance covered by the body during the movement = AB + BC = 12 + 5 = 17 m.

and the displacement resulted during the movement is the length of $\overline{AC}$ in the direction from A to C

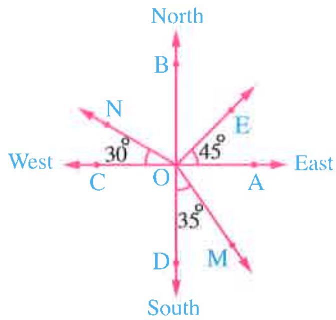
i.e. The displacement = $\sqrt{(12)^2+(5)^2}$ = 13 m. in the direction $\overrightarrow{AC}$

Direction

- Each ray in the plane determines a certain direction.

For example :

In the opposite figure :

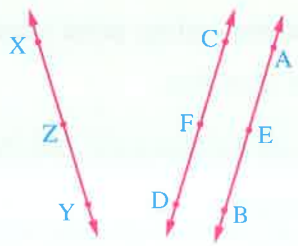


- $\overrightarrow{OA}$ determines east direction.
- $\overrightarrow{OE}$ determines north east direction.
- $\overrightarrow{ON}$ determines the direction 30° north of west.
- $\overrightarrow{OM}$ determines the direction 35° east of south.

Notice that

In the opposite figure :

If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and each of them is not parallel to $\overleftrightarrow{XY}$, E $\in \overleftrightarrow{AB}$, F $\in \overleftrightarrow{CD}$, Z $\in \overleftrightarrow{XY}$, then :



- $\overrightarrow{EA}$ and $\overrightarrow{BA}$ have the same direction and are carried on one straight line.
- $\overrightarrow{EA}$ and $\overrightarrow{FC}$ have the same direction and are carried on two parallel straight lines.
- $\overrightarrow{EA}$ and $\overrightarrow{EB}$ are in opposite directions and carried on one straight line.
- $\overrightarrow{EA}$ and $\overrightarrow{FD}$ are in opposite directions and carried on two parallel straight lines.
- $\overrightarrow{EA}$ and $\overrightarrow{ZX}$ are in different directions and carried on two not parallel straight lines.

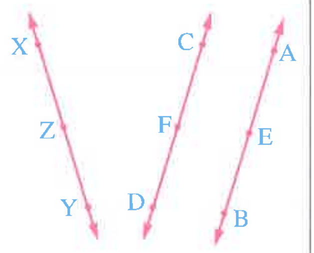
Unit 4

Generally

1. The two rays which have the same direction or in opposite directions are carried on one straight line or two parallel straight lines.
2. The two rays different in directions are not carried on one straight line or two parallel straight lines.

TRY TO SOLVE

In the opposite figure :



$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and each of them is not parallel to $\overleftrightarrow{XY}$,

$E \in \overline{AB}$, $F \in \overline{CD}$ and $Z \in \overline{XY}$

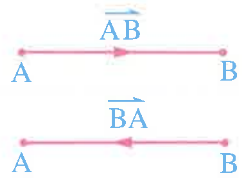
Show whether the two rays have the same , opposite or different directions in each of the following :

1. $\overrightarrow{AB}$ and $\overrightarrow{DF}$
2. $\overrightarrow{AB}$ and $\overrightarrow{XY}$
3. $\overrightarrow{CD}$ and $\overrightarrow{EB}$
4. $\overrightarrow{ZY}$ and $\overrightarrow{ZX}$
5. $\overrightarrow{CF}$ and $\overrightarrow{ZX}$

The directed line segment

- If we determined to the line segment $\overline{AB}$ a starting point A and an ending point B , then follow from that the line segment has a direction from A to B and is called a directed line segment and is denoted by the symbol $\overrightarrow{AB}$, with noticing that :

  $\overrightarrow{AB} \neq \overrightarrow{BA}$ because they are different in the starting point and the ending point , and this lead to their opposite in the direction.



- From the previous , we deduce that the directed line segment is determined by three elements :
  1. Starting point.
  2. Ending point.
  3. Direction from starting point to ending point.

Definitions

1 The directed line segment :

It is a line segment which has a starting point , an ending point and a direction.

2 The norm of the directed line segment "norm of $\overrightarrow{AB}$" **:**

It is the length of $\overline{AB}$ and is denoted by the symbol $\|\overrightarrow{AB}\|$

and notice that : $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\| = AB$

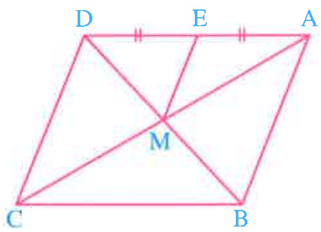
3 Equivalent directed line segments :

Two directed line segments are equivalent if :

1. They have the same length "norm".
2. They have the same direction.

Example 1

In the opposite figure :



ABCD is a parallelogram , its diagonals intersecting at M ,

and E is the midpoint of $\overline{AD}$

First Determine the directed line segments (if existed) which are equivalent to :

1 $\overrightarrow{AB}$	2 $\overrightarrow{DA}$	3 $\overrightarrow{MB}$
4 $\overrightarrow{AM}$	5 $\overrightarrow{AE}$	6 $\overrightarrow{ME}$

Second Show why the following directed line segments are not equivalent :

1 $\overrightarrow{DM}$ and $\overrightarrow{DB}$	2 $\overrightarrow{AD}$ and $\overrightarrow{CB}$	3 $\overrightarrow{AM}$ and $\overrightarrow{CM}$

Solution

First

1 $\overrightarrow{DC}$	2 $\overrightarrow{CB}$	3 $\overrightarrow{DM}$
4 $\overrightarrow{MC}$	5 $\overrightarrow{ED}$	6 None.

Second

1 Because $\|\overrightarrow{DM}\| \neq \|\overrightarrow{DB}\|$

2 Because $\overrightarrow{AD}$ and $\overrightarrow{CB}$ are in opposite direction.

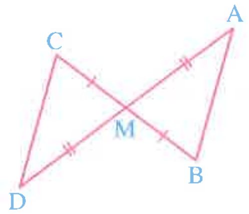
3 Because $\overrightarrow{AM}$ and $\overrightarrow{CM}$ are in opposite direction.

TRY TO SOLVE

In the opposite figure :

If $\overline{AD} \cap \overline{BC} = \{M\}$, MA = MD and MB = MC

, then complete the following with "is equivalent to" or "is not equivalent to" , showing the reason :

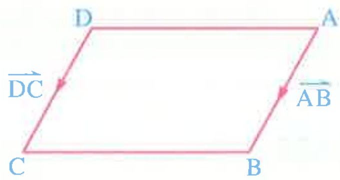


1. $\overrightarrow{AM}$ ………… $\overrightarrow{MD}$ because they are …………
2. $\overrightarrow{MB}$ ………… $\overrightarrow{MC}$ because they are …………
3. $\overrightarrow{AB}$ ………… $\overrightarrow{CD}$ because they are …………
4. $\overrightarrow{BM}$ ………… $\overrightarrow{BC}$ because they are …………

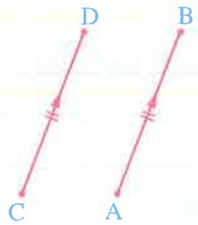
Remarks

1 $\overrightarrow{AB}$, $\overrightarrow{CD}$ can not be equivalent unless two parallel straight lines carrying them or one straight line as in the following figures :

2 If A , B , C and D are not collinear and $\overrightarrow{AB}$ is equivalent to $\overrightarrow{DC}$, then ABCD is a parallelogram.



3 From a point in the plane , for example C , we can draw a unique directed line segment $\overrightarrow{CD}$ equivalent to another line segment $\overrightarrow{AB}$ in the same plane.



4 There are an infinte number of directed line segments can be drawn in the plane and each of them is equivalent to another directed line segment.

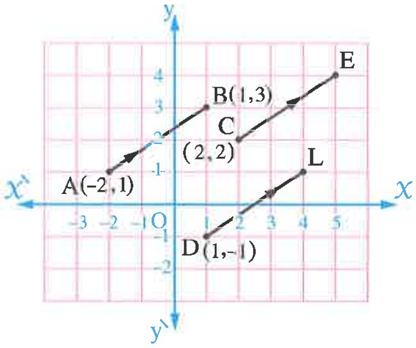
Example 2

In an orthogonal coordinate plane , determine the points A (–2 , 1) , B (1 , 3) , C (2 , 2) and D (1 , – 1) , then draw $\overrightarrow{CE}$ and $\overrightarrow{DL}$ each of them is equivalent to $\overrightarrow{AB}$

Find the coordinates of each of : E and L

Solution

To draw $\overrightarrow{CE}$ equivalent to $\overrightarrow{AB}$, then $\overrightarrow{CE}$ and $\overrightarrow{AB}$ should have the same direction and the same norm.



- Draw $\overrightarrow{CE} \parallel \overline{AB}$ $\left(\text{the slope of } \overleftrightarrow{CE} = \text{the slope of } \overleftrightarrow{AB} = \frac{2}{3}\right)$
- Use the compasses to determine the length of $\overline{EC}$ = the length of $\overline{AB}$ or by calculating the number of horizontal and vertical squares ,
  then we get that : E = (5 , 4)
- Similarly , draw $\overrightarrow{DL}$, we get that : L = (4 , 1)

Another Solution :

$\because$ The translation preserves parallelism and lengths of line segments.

$\therefore$ The point C is the image of the point A by the translation $[(2 , 2) - (-2 , 1)] = (4 , 1)$ and to draw $\overrightarrow{CE}$ equivalent to $\overrightarrow{AB}$, we get that : $\overrightarrow{CE}$ is the image of $\overrightarrow{AB}$ by the translation (4 , 1)

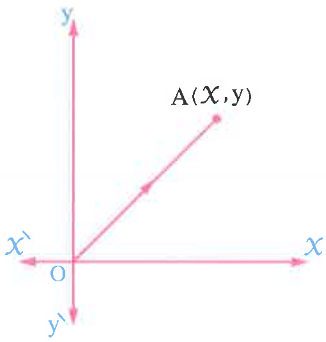
$\therefore$ The point E is the image of the point B by the translation (4 , 1)

$\therefore$ The point E = (1 + 4 , 3 + 1) = (5 , 4)

Similarly , we can find the coordinates of the point L

Lesson TWO — Vectors

The position vector

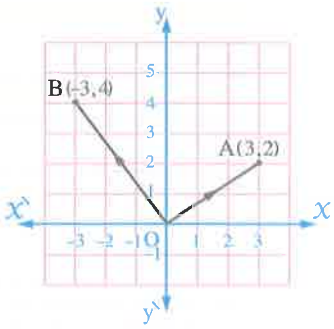


We know that each point A in the orthogonal coordinates plane is determined by a unique ordered pair (x , y) , so it has a unique position with respect to the origin point O which is determined by the directed line segment $\overrightarrow{OA}$ and it is called the position vector of the point A and is written as $\overrightarrow{OA} = (x , y)$

Definition :

The position vector of a given point A with respect to the origin point O is the directed line segment $\overrightarrow{OA}$ which its starting point is the origin point O and the given point A is its ending point.

In the opposite figure :



* $\overrightarrow{OA}$ is the position vector of the point (A) with respect to the origin point O and is written as $\overrightarrow{OA} = (3 , 2)$
* $\overrightarrow{OB}$ is the position vector of the point (B) with respect to the origin point O and is written as $\overrightarrow{OB} = (-3 , 4)$

Remark

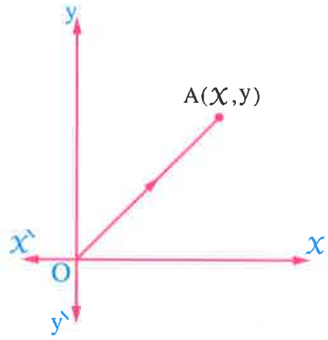
All position vectors have the same starting point (O) , then we denote the position vector $\overrightarrow{OA}$ by the symbol $\overrightarrow{A}$

In the previous figure :

We write $\overrightarrow{A} = (3 , 2)$, $\overrightarrow{B} = (-3 , 4)$

The norm of the vector

It is the length of the line segment that represents the vector.



If $\overrightarrow{A} = (x , y)$

, then $\|\overrightarrow{A}\|$ = the length of $\overline{OA}$ and if we used the law of the distance between two points to find the length of $\overline{OA}$, then :

The length of $\overline{OA} = \sqrt{(x-0)^2 + (y-0)^2}$

$\therefore \boxed{\|\overrightarrow{A}\| = \sqrt{x^2 + y^2}}$

For example :

- If $\overrightarrow{A} = (3 , -4)$

  , then $\|\overrightarrow{A}\| = \sqrt{(3)^2 + (-4)^2} = 5$ length units.

- If $\overrightarrow{B} = \left(-3 , 3\sqrt{3}\right)$

  , then $\|\overrightarrow{B}\| = \sqrt{(-3)^2 + (3\sqrt{3})^2} = 6$ length units.

- If $\overrightarrow{C} = (-3 , K)$ and $\|\overrightarrow{C}\| = 3\sqrt{2}$

  , then $\sqrt{(-3)^2 + K^2} = 3\sqrt{2}$

  $\therefore 9 + K^2 = 18$

  $\therefore K^2 = 9$

  $\therefore K = \pm 3$

The unit vector

It is a vector whose norm is unity.

For example :

$\overrightarrow{A} = \left(\frac{3}{5} , \frac{4}{5}\right)$ is a unit vector because

$\|\overrightarrow{A}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$ length unit.

The zero vector

It is a vector whose norm equals zero and denoted by $\overrightarrow{O}$ or $\overrightarrow{0}$, where $\overrightarrow{O} = (0 , 0)$ and it has no direction.

Unit 4

Check your understanding

1 If $\overrightarrow{A} = (6 , -8)$, **find :** $\|\overrightarrow{A}\|$

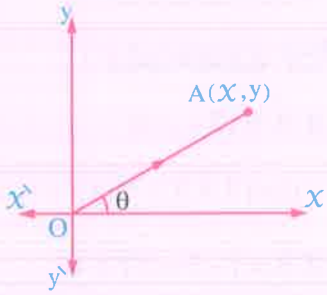
2 Is the vector $\overrightarrow{A} = \left(\frac{-\sqrt{3}}{2} , \frac{1}{2}\right)$ a unit vector ? **and why** ?

3 If $\left(k , \frac{-3}{5}\right)$ is a unit vector , then **find the value of k**

The polar form of the position vector

If the position vector $\overrightarrow{OA}$ makes an angle of measure θ with the positive direction of x-axis , then the polar form of the position vector

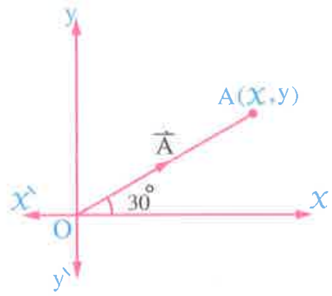
$\overrightarrow{OA} = \left(\|\overrightarrow{OA}\| , \theta\right)$



For example :

If $\overrightarrow{OA}$ makes an angle of measure 30° with the positive direction of x-axis and $\|\overrightarrow{A}\| = 6$ length units , then the polar form of the vector $\overrightarrow{OA} = (6 , 30°)$

i.e. $\overrightarrow{OA} = \left(6 , \frac{\pi}{6}\right)$



Remark

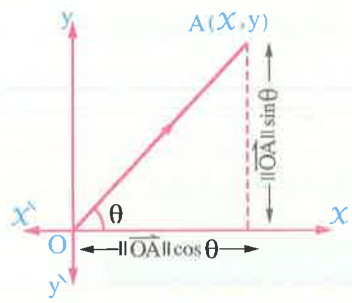
If the position vector of the point A (x , y) is in the polar form $\overrightarrow{OA} = \left(\|\overrightarrow{OA}\| , \theta\right)$, then

$x = \|\overrightarrow{OA}\| \cos\theta$

, $y = \|\overrightarrow{OA}\| \sin\theta$ where $\tan\theta = \frac{y}{x}$

and the cartesian form of the vector $\overrightarrow{OA}$ is

$\overrightarrow{OA} = \left(\|\overrightarrow{OA}\| \cos\theta , \|\overrightarrow{OA}\| \sin\theta\right)$



Example 1

If $\overrightarrow{OA}$ is the position vector of the point A with respect to the origin point , then find the coordinates of the point A in each of the following :

1 $\overrightarrow{OA} = \left(10\sqrt{3} , 60°\right)$ **2** $\overrightarrow{OA} = \left(6\sqrt{2} , 135°\right)$ **3** $\overrightarrow{OA} = \left(8 , \frac{4\pi}{3}\right)$

Solution

1 $x = 10\sqrt{3} \cos 60° = 5\sqrt{3}$, $y = 10\sqrt{3} \sin 60° = 15$ $\therefore A = \left(5\sqrt{3} , 15\right)$

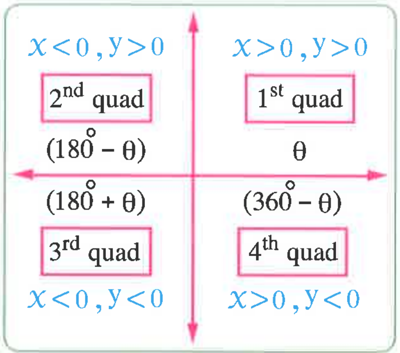
2 $x = 6\sqrt{2} \cos 135° = -6$, $y = 6\sqrt{2} \sin 135° = 6$ $\therefore A = (-6 , 6)$

3 $x = 8 \cos \frac{4\pi}{3} = -4$, $y = 8 \sin \frac{4\pi}{3} = -4\sqrt{3}$ $\therefore A = \left(-4 , -4\sqrt{3}\right)$

Example 2

If $\overrightarrow{OA}$ is the position vector of the point A with respect to the origin point , then find the polar form of the vector $\overrightarrow{OA}$ in each of the following :

1 $\overrightarrow{OA} = \left(4 , 4\sqrt{3}\right)$ **2** $\overrightarrow{OA} = \left(5\sqrt{3} , -5\right)$



Solution

1 $\because \overrightarrow{OA} = \left(4 , 4\sqrt{3}\right)$

$\therefore \|\overrightarrow{OA}\| = \sqrt{(4)^2 + \left(4\sqrt{3}\right)^2} = 8$ length units.

, $\tan\theta = \frac{4\sqrt{3}}{4} = \sqrt{3}$

, $\because$ the measure of the acute angle whose tan is $\sqrt{3}$ is $\tan^{-1}\left(\sqrt{3}\right) = 60°$

, $\because x > 0 , y > 0$ $\therefore \theta = 60°$

$\therefore \overrightarrow{OA} = (8 , 60°)$

2 $\because \overrightarrow{OA} = \left(5\sqrt{3} , -5\right)$

$\therefore \|\overrightarrow{OA}\| = \sqrt{\left(5\sqrt{3}\right)^2 + (-5)^2} = 10$ length units.

, $\tan\theta = \frac{-5}{5\sqrt{3}} = \frac{-1}{\sqrt{3}}$

, $\because$ the measure of the acute angle whose tan is $\frac{1}{\sqrt{3}}$ is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

, $\because x > 0 , y < 0$ $\therefore \theta = 360° - 30° = 330°$

$\therefore \overrightarrow{OA} = (10 , 330°)$

Unit 4

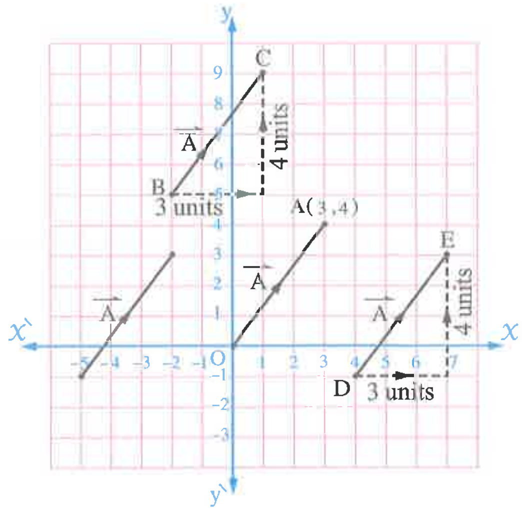
TRY TO SOLVE

1. If the position vector $\overrightarrow{OA} = \left(5\sqrt{2}\ ,\ 225^\circ\right)$, then find the coordinates of the point A
2. Write in the polar form the position vector $\overrightarrow{OA} = \left(-12\sqrt{3}\ ,\ 12\right)$

Equivalent vectors

Every vector $\vec{A} = (x\ ,\ y)$ can be represented by an infinite number of equivalent directed line segments , where each of them is equivalent to the position vector of the point $A = (x\ ,\ y)$

In the opposite figure :



$\vec{A} = (3\ ,\ 4)$ is a position vector of the point A

, $\overrightarrow{BC} = \overrightarrow{DE} = \cdots = \overrightarrow{OA}$ because

$\|\overrightarrow{BC}\| = \|\overrightarrow{DE}\| = \cdots = \|\overrightarrow{OA}\| = \sqrt{3^2 + 4^2}$

$= 5$ length units.

, $\overrightarrow{BC}$, $\overrightarrow{DE}$, ⋯ , $\overrightarrow{OA}$ are in the same direction

, so each of $\overrightarrow{BC}$, $\overrightarrow{DE}$, ⋯ consider

a geometrical representation of the vector $\vec{A}$

i.e. $\overrightarrow{BC} = \overrightarrow{DE} = \cdots = \overrightarrow{OA} = (3\ ,\ 4)$

- From the previous , we notice that the vectors are related by the ordered pairs.

i.e. The elements of $\mathbb{R} \times \mathbb{R}$ $(\mathbb{R}^2)$, so we can deduce the definition of the vectors with the mathematical concept or the algebraic concept as the following :

Definition :

The vectors are the elements of the set $\mathbb{R}^2$ with addition and multiplication by a real number defined on it , and denoted by one of the symbols $\vec{A}$, $\vec{B}$, $\vec{C}$, ...

Where $\mathbb{R}^2$ = the set of the ordered pairs of the cartesian product $\mathbb{R} \times \mathbb{R}$

$= \{(x\ ,\ y) : x \in \mathbb{R}\ ,\ y \in \mathbb{R}\}$

Adding two vectors algebraically

If $\vec{A} = (x_1\ ,\ y_1) \in \mathbb{R}^2$, $\vec{B} = (x_2\ ,\ y_2) \in \mathbb{R}^2$

, then $\vec{A} + \vec{B} = (x_1 + x_2\ ,\ y_1 + y_2)$

For example : If $\vec{A} = (3\ ,\ 5)$, $\vec{B} = (2\ ,\ 1)$

, then $\vec{A} + \vec{B} = (3 + 2\ ,\ 5 + 1) = (5\ ,\ 6)$

Properties of addition on vectors

1 **Closure property :** For every $\overrightarrow{A}$, $\overrightarrow{B} \in \mathbb{R}^2$, then $\overrightarrow{A} + \overrightarrow{B} \in \mathbb{R}^2$

2 **Commutative property :** For every $\overrightarrow{A}$, $\overrightarrow{B}$, then $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$

3 **Associative property :** For every $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, then $\overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C}) = (\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$

4 **Identity element property :** For every $\overrightarrow{A}$ there exists the zero vector $\overrightarrow{O} = (0 , 0)$ where
$\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{O} + \overrightarrow{A} = \overrightarrow{A}$

5 **Additive inverse property :** For every $\overrightarrow{A}\,(x , y)$ there exists $(-\overrightarrow{A}) = (-x , -y)$where
$\overrightarrow{A} + (-\overrightarrow{A}) = (-\overrightarrow{A}) + (\overrightarrow{A}) = \overrightarrow{O}$ (zero vector)
the vector $(-\overrightarrow{A})$ is called the additive inverse of $\overrightarrow{A}$

6 **Elimination property :** For every $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ if $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{A} + \overrightarrow{C}$, then $\overrightarrow{B} = \overrightarrow{C}$

Multiplying a vector by a real number

If $\overrightarrow{A} = (x , y) \in \mathbb{R}^2$, $k \in \mathbb{R}$, then $k\,\overrightarrow{A} = k\,(x , y) = (k\,x , k\,y)$

For example : If $\overrightarrow{A} = (2 , -5)$
, then $3\,\overrightarrow{A} = 3\,(2 , -5) = (6 , -15)$

Properties of multiplying a vector by a real number

1 **Distributive property :**

First : For every $\overrightarrow{A}$, $\overrightarrow{B}$, $k \in \mathbb{R}$, then $k\,(\overrightarrow{A} + \overrightarrow{B}) = k\,\overrightarrow{A} + k\,\overrightarrow{B}$

Second : For every $\overrightarrow{A}$, k_1 , $k_2 \in \mathbb{R}$, then $(k_1 + k_2)\,\overrightarrow{A} = k_1\,\overrightarrow{A} + k_2\,\overrightarrow{A}$

2 **Associative property :** For every $\overrightarrow{A}$, k_1 , $k_2 \in \mathbb{R}$, then $(k_1\,k_2)\,\overrightarrow{A} = k_1\,(k_2\,\overrightarrow{A})$

3 **Elimination property :** For every $\overrightarrow{A}$, $\overrightarrow{B}$, $k \in \mathbb{R}^*$ if $k\,\overrightarrow{A} = k\,\overrightarrow{B}$, then $\overrightarrow{A} = \overrightarrow{B}$

Example 3

If $\overrightarrow{A} = (3 , -1)$, $\overrightarrow{B} = (2 , 5)$, $\overrightarrow{C} = (-4 , 2)$, find each of the following vectors :

1 $2\,\overrightarrow{A} - 3\,\overrightarrow{B}$

2 $\frac{1}{2}\,(4\,\overrightarrow{A} + 2\,\overrightarrow{B} - \overrightarrow{C})$

3 $2\,\overrightarrow{O} - 3\,(\overrightarrow{C} + \overrightarrow{A})$ Where $\overrightarrow{O}$ is the zero vector

Solution

1 $2\,\overrightarrow{A} - 3\,\overrightarrow{B} = 2\,(3 , -1) - 3\,(2 , 5)$
$= (6 , -2) + (-6 , -15) = (0 , -17)$

2 $\frac{1}{2}\,(4\,\overrightarrow{A} + 2\,\overrightarrow{B} - \overrightarrow{C}) = 2\,\overrightarrow{A} + \overrightarrow{B} - \frac{1}{2}\,\overrightarrow{C}$
$= 2\,(3 , -1) + (2 , 5) - \frac{1}{2}\,(-4 , 2)$
$= (6 , -2) + (2 , 5) + (2 , -1) = (10 , 2)$

3 $2\overrightarrow{O} - 3(\overrightarrow{C} + \overrightarrow{A}) = 2(0, 0) - 3[(-4, 2) + (3, -1)]$

$= (0, 0) - 3(-1, 1) = (0, 0) + (3, -3) = (3, -3)$

Example 4

If $\overrightarrow{A} = (6, 1)$, $\overrightarrow{B} = (1, 2)$, find : $\| \overrightarrow{A} - 2\overrightarrow{B} \|$

Solution

$\because \overrightarrow{A} - 2\overrightarrow{B} = (6, 1) - 2(1, 2) = (6, 1) - (2, 4) = (4, -3)$

$\therefore \| \overrightarrow{A} - 2\overrightarrow{B} \| = \sqrt{16 + 9} = \sqrt{25} = 5$ length units.

TRY TO SOLVE

If $\overrightarrow{A} = (3, -1)$, $\overrightarrow{B} = (2, -5)$, $\overrightarrow{C} = (-5, 5)$, then write in the polar form the vector $\overrightarrow{M}$, where $\overrightarrow{M} = \overrightarrow{A} - 2\overrightarrow{B} - \overrightarrow{C}$

Equality of two vectors

For any two vectors $\overrightarrow{A} = (x_1, y_1)$, $\overrightarrow{B} = (x_2, y_2)$
, then $\overrightarrow{A} = \overrightarrow{B}$ if and only if $x_1 = x_2$, $y_1 = y_2$

Example 5

If $\overrightarrow{A} = (2, -3)$, $\overrightarrow{B} = (3, 5)$
, express $\overrightarrow{C} = (12, 1)$ in terms of $\overrightarrow{A}$ and $\overrightarrow{B}$

Solution

Let $\overrightarrow{C} = k\overrightarrow{A} + \ell\overrightarrow{B}$ where $k, \ell \in \mathbb{R}$

$\therefore \overrightarrow{C} = k(2, -3) + \ell(3, 5) = (2k, -3k) + (3\ell, 5\ell)$

$= (2k + 3\ell, -3k + 5\ell)$

$\therefore (2k + 3\ell, -3k + 5\ell) = (12, 1)$

$\therefore 2k + 3\ell = 12$ (Multiplying by 3) $\quad \therefore 6k + 9\ell = 36 \quad (1)$

, $-3k + 5\ell = 1$ (Multiplying by 2) $\quad \therefore -6k + 10\ell = 2 \quad (2)$

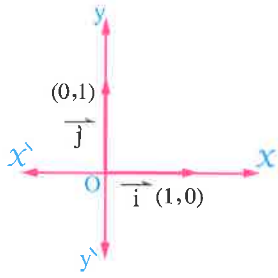
Adding (1) and (2) : $\therefore 19\ell = 38 \quad \therefore \ell = 2$

Substituting in (1) : $\therefore 6k = 18 \therefore k = 3$

$\therefore \overrightarrow{C} = 3\overrightarrow{A} + 2\overrightarrow{B}$

The fundamental unit vectors $\vec{i}$, $\vec{j}$

If O is the origin point of an orthogonal coordinate plane , then :



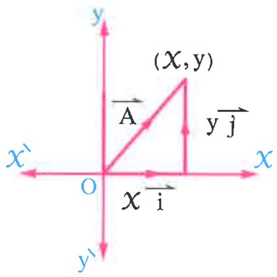
1 The fundamental unit vector $\vec{i} = (1 , 0)$ is the position vector of the point (1 , 0) , its norm is the unity and its direction is the positive direction of the x-axis

2 The fundamental unit vector $\vec{j} = (0 , 1)$ is the position vector of the point (0 , 1) , its norm is the unity and its direction is the positive direction of the y-axis

* **Note that :** $\|\vec{i}\| = \|\vec{j}\| = 1$

Expressing the vector in terms of the fundamental unit vectors

If $\vec{A}$ is a vector in the plane where $\vec{A} = (x , y)$, then it is possible to express $\vec{A}$ in terms of the fundamental unit vectors as follows :



$\vec{A} = (x , y) = (x , 0) + (0 , y)$ (from definition of addition)

$= x(1 , 0) + y(0 , 1)$ (from definition of multiplying by a real number)

$\therefore \vec{A} = x\vec{i} + y\vec{j}$

and this rule is used directly to express the ordered pair representing $\vec{A}$ in terms of the fundamental unit vectors $\vec{i}$, $\vec{j}$

For example : $\vec{A} = (2 , 3) = 2\vec{i} + 3\vec{j}$, $\vec{B} = (-5 , 1) = -5\vec{i} + \vec{j}$

Example 6

Express each of the following vectors in terms of the fundamental unit vectors , then find its norm :

1 $\vec{A} = (-8 , 6)$ 2 $\vec{L} = (0 , -2)$ 3 $\vec{M} = \left(\frac{3}{2} , 0\right)$ 4 $\vec{C} = (2 , -6)$

Solution

1 $\vec{A} = -8\vec{i} + 6\vec{j}$ $\therefore \|\vec{A}\| = \sqrt{(-8)^2 + (6)^2} = 10$ length units.

2 $\vec{L} = -2\vec{j}$ $\therefore \|\vec{L}\| = \sqrt{(0)^2 + (-2)^2} = 2$ length units.

3 $\vec{M} = \frac{3}{2}\vec{i}$ $\therefore \|\vec{M}\| = \sqrt{\left(\frac{3}{2}\right)^2 + (0)^2} = \frac{3}{2}$ length units.

4 $\vec{C} = 2\vec{i} - 6\vec{j}$ $\therefore \|\vec{C}\| = \sqrt{(2)^2 + (-6)^2} = 2\sqrt{10}$ length units.

Unit 4

TRY TO SOLVE

Express each of the following vectors in terms of the fundamental unit vectors , then find its norm :

1 $\overrightarrow{A} = (7 , -24)$ **2** $\overrightarrow{D} = (-6 , 0)$ **3** $\overrightarrow{L} = (0 , -12)$ **4** $\overrightarrow{E} = (-12 , -16)$

Example 7

Write each of the following vectors in its polar and cartesian forms , then express it in terms of the fundamental unit vectors :

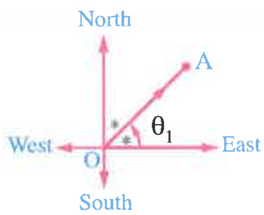
1 A force of magnitude 12 newtons acts in the direction North East.

2 A uniform speed of a car covers 8 metres per second in the direction 30° North of West.

3 The displacement of a body a distance 24 metres in the North direction.

4 A force of magnitude 4 kg.wt. in the direction 30° East of South.

Solution



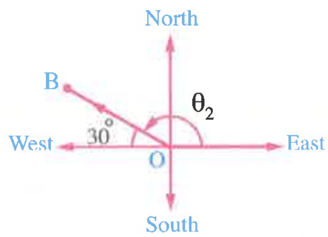
1 Let the position vector of the force = $\overrightarrow{A}$

$\because$ The direction North East bisects the angle between the North and the East. $\therefore \theta_1 = \frac{90^\circ}{2} = 45°$

* The polar form of $\overrightarrow{A} = (12 , 45°)$

* The cartesian form of $\overrightarrow{A} = (12 \cos 45° , 12 \sin 45°) = \left(6\sqrt{2} , 6\sqrt{2}\right)$

* $\overrightarrow{A} = 6\sqrt{2}\ \overrightarrow{i} + 6\sqrt{2}\ \overrightarrow{j}$



2 Let the position vector of the speed = $\overrightarrow{B}$

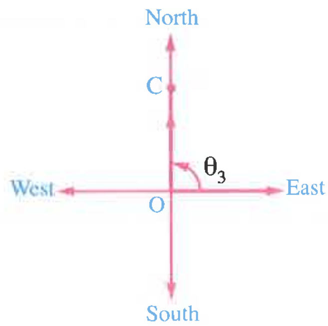
$\theta_2 = 180° - 30° = 150°$

* The polar form of $\overrightarrow{B} = (8 , 150°)$

* The cartesian form of $\overrightarrow{B} = (8 \cos 150° , 8 \sin 150°)$

$= \left(-4\sqrt{3} , 4\right)$

* $\overrightarrow{B} = -4\sqrt{3}\ \overrightarrow{i} + 4\ \overrightarrow{j}$



3 Let the position vector of the displacement = $\overrightarrow{C}$

$\therefore \theta_3 = 90°$

* The polar form of $\overrightarrow{C} = (24 , 90°)$

* The cartesian form of $\overrightarrow{C} = (24 \cos 90° , 24 \sin 90°)$

$= (0 , 24)$

* $\overrightarrow{C} = 24\ \overrightarrow{j}$

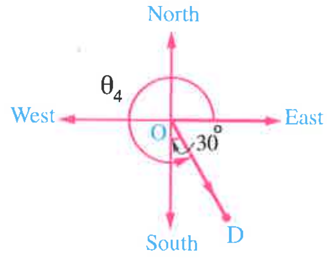
4 Let the position vector of the force = $\overrightarrow{D}$

$\therefore \theta_4 = 270° + 30° = 300°$

* The polar form of $\overrightarrow{D} = (4 , 300°)$
* The cartesian form of $\overrightarrow{D} = (4 \cos 300° , 4 \sin 300°)$

$= \left(2 , -2\sqrt{3}\right)$

* $\overrightarrow{D} = 2\overrightarrow{i} - 2\sqrt{3}\ \overrightarrow{j}$



TRY TO SOLVE

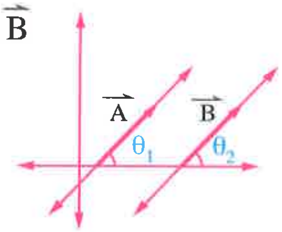
Write each of the following vectors in the polar and cartesian forms , then express it in terms of the fundamental unit vectors :

1. A force of magnitude 63 newtons acts in the East direction.
2. The displacement of a body a distance 3 metres in the South direction.
3. A uniform speed of a car covers 50 metres per second in the direction North West.

Parallel and perpendicular vectors

For every non zero vectors $\overrightarrow{A} , \overrightarrow{B}$ where $\overrightarrow{A} = (x_1 , y_1) , \overrightarrow{B} = (x_2 , y_2)$

1 If $\overrightarrow{A} // \overrightarrow{B}$

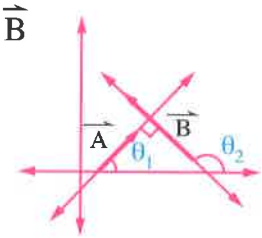


, then $\tan \theta_1 = \tan \theta_2$

$\therefore \frac{y_1}{x_1} = \frac{y_2}{x_2}$

$\therefore\ x_1 y_2 - x_2 y_1 = 0$ and vice versa.

2 If $\overrightarrow{A} \perp \overrightarrow{B}$



, then $\tan \theta_1 \times \tan \theta_2 = -1$

$\therefore \frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$

$\therefore\ x_1 x_2 + y_1 y_2 = 0$ and vice versa.

For example : If $\overrightarrow{A} = (3 , 4) , \overrightarrow{B} = (8 , -6) , \overrightarrow{C} = (9 , 12)$

, then $\overrightarrow{A} \perp \overrightarrow{B}$ because $[3 \times 8 + 4 \times (-6) = 0]$

, $\overrightarrow{A} // \overrightarrow{C}$ because $[3 \times 12 - 4 \times 9 = 0]$

* **Note that :** The slope of $\overrightarrow{A} = \frac{4}{3}$

, the slope of $\overrightarrow{B} = \frac{-6}{8} = \frac{-3}{4}$, the slope of $\overrightarrow{C} = \frac{12}{9} = \frac{4}{3}$

$\therefore$ The slope of $\overrightarrow{A}$ = the slope of $\overrightarrow{C}$

$\therefore \overrightarrow{A} // \overrightarrow{C}$, the slope of $\overrightarrow{A}$ × the slope of $\overrightarrow{B} = -1$ $\therefore \overrightarrow{A} \perp \overrightarrow{B}$

Remark

If $\overrightarrow{A} = (x , y)$

, then the slope of $\overrightarrow{A} = \frac{y}{x}$

Example 8

If $\overrightarrow{A} = (-2 , 3)$, $\overrightarrow{B} = (-4 , m)$, find the value of m in each of the following :

1 $\overrightarrow{A} // \overrightarrow{B}$ 2 $\overrightarrow{A} \perp \overrightarrow{B}$

Solution

1 $\because \overrightarrow{A} // \overrightarrow{B}$ $\therefore -2\,m - 3\,(-4) = 0$ $\therefore 2\,m = 12$ $\therefore m = 6$

2 $\because \overrightarrow{A} \perp \overrightarrow{B}$ $\therefore (-2)\,(-4) + 3\,m = 0$ $\therefore 3\,m = -8$ $\therefore m = \frac{-8}{3}$

Example 9

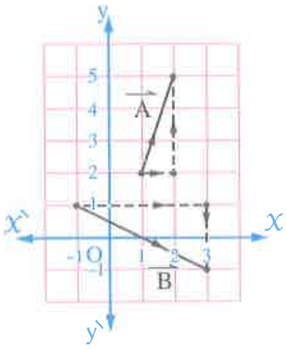
Draw an orthogonal coordinate plane where O is the origin point , then represent on it each of the following :

1 The vector $\overrightarrow{A} = (1 , 3)$ by a directed line segment whose starting point is (1 , 2)

2 The vector $\overrightarrow{B} = (4 , -2)$ by a directed line segment whose starting point is (– 1 , 1)

* Then find the ending point in each case

Solution



1 To represent the vector $\overrightarrow{A} = (1 , 3)$

* Start from the point (1 , 2) , then move to the right one unit in the positive direction of the X-axis.
* Then move upwards 3 units in the positive direction of the y-axis.

$\therefore$ The ending point = (2 , 5)

2 To represent the vector $\overrightarrow{B} = (4 , -2)$

* Start from the point (– 1 , 1) , then move to the right 4 units in the positive direction of the X-axis.
* Then move downwards 2 units in the negative direction of the y-axis

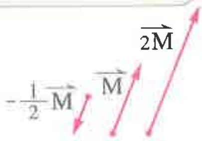
$\therefore$ The ending point = (3 , – 1)

Remark

If $\overrightarrow{M}$ is a non zero vector , $k \neq 0$, then $\overrightarrow{M} // k\,\overrightarrow{M}$ and $\|k\,\overrightarrow{M}\| = |k| . \|\overrightarrow{M}\|$, where the direction of $k\,\overrightarrow{M}$ is the same direction of $\overrightarrow{M}$ for every $k > 0$ and the direction of $k\,\overrightarrow{M}$ is the opposite direction of $\overrightarrow{M}$ for every $k < 0$

For example : • $\overrightarrow{M}$, $2\,\overrightarrow{M}$ are parallel and in the same directions.

• $\overrightarrow{M}$, $-\frac{1}{2}\,\overrightarrow{M}$ are parallel and in the opposite directions.



Example 10

If $\overrightarrow{A}$ is a non-zero vector. Find the value of K in each of the following cases :

1 $K \| \overrightarrow{A} \| = \| -2 \overrightarrow{A} \|$ **2** $\| 2 K \overrightarrow{A} \| = \| -3 \overrightarrow{A} \|$

Solution

1 $\because K \| \overrightarrow{A} \| = \| -2 \overrightarrow{A} \|$ $\therefore K \| \overrightarrow{A} \| = 2 \| \overrightarrow{A} \|$ $\therefore K = 2$

2 $\because \| 2 K \overrightarrow{A} \| = \| -3 \overrightarrow{A} \|$

$\therefore 2 | K | \| \overrightarrow{A} \| = 3 \| \overrightarrow{A} \|$

$\therefore 2 | K | = 3$

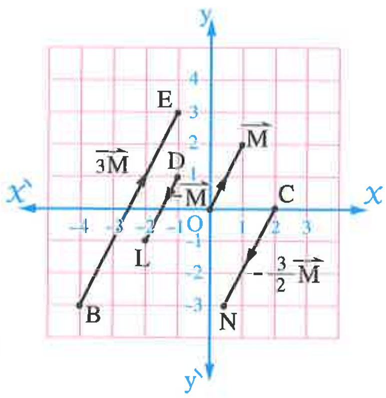
$\therefore | K | = \frac{3}{2}$

$\therefore K = \pm \frac{3}{2}$

Example 11

Represent the vector $\overrightarrow{M} = (1 , 2)$, then draw from the points B $(-4 , -3)$, C $(2 , 0)$, D $(-1 , 1)$ the directed line segments which are equivalent to $3 \overrightarrow{M} , -\frac{3}{2} \overrightarrow{M} , -\overrightarrow{M}$ respectively.

Solution



1 Represent the vector $\overrightarrow{M} = (1 , 2)$ starting from the point $(0 , 0)$

2 Represent the vector $3 \overrightarrow{M} = 3 (1 , 2) = (3 , 6)$ starting from the point B $(-4 , -3)$

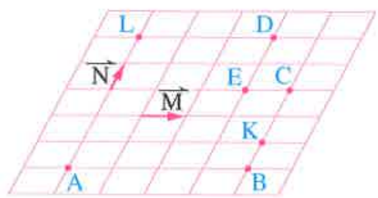
3 Represent the vector $-\frac{3}{2} \overrightarrow{M} = -\frac{3}{2} (1 , 2) = \left(-\frac{3}{2} , -3\right)$ starting from the point C $= (2 , 0)$

4 Represent the vector $-\overrightarrow{M} = (-1 , -2)$ starting from the point D $= (-1 , 1)$

Example 12

The opposite lattice represents congruent parallelograms.

Express each of the following directed line segments in terms of the vectors $\overrightarrow{M}$ and $\overrightarrow{N}$:



1 $\overrightarrow{AB}$ **2** $\overrightarrow{CB}$ **3** $\overrightarrow{CE}$

4 $\overrightarrow{BC}$ **5** $\overrightarrow{BA}$ **6** $\overrightarrow{KC}$

7 $\overrightarrow{DL}$ **8** $\overrightarrow{DE}$ **9** $\overrightarrow{LA}$

Solution

1 $4\overrightarrow{M}$

2 $-3\overrightarrow{N}$

3 $-\overrightarrow{M}$

4 $3\overrightarrow{N}$

5 $-4\overrightarrow{M}$

6 $2\overrightarrow{N}$

7 $-3\overrightarrow{M}$

8 $-2\overrightarrow{N}$

9 $-5\overrightarrow{N}$

Remark

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are non zero vectors , $\overrightarrow{A} = k\,\overrightarrow{B}$, $k \neq 0$, then $\overrightarrow{A} \,/\!/\, \overrightarrow{B}$

For example : If $\overrightarrow{A} = (3 , 2)$, $\overrightarrow{B} = (15 , 10)$

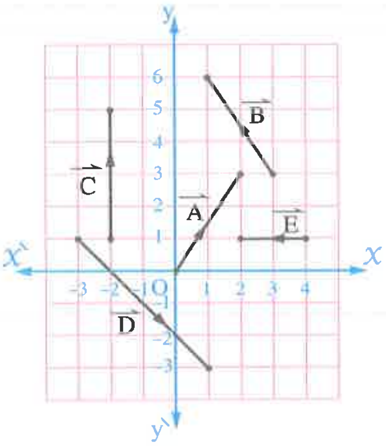
$\therefore \overrightarrow{B} = 5\,(3 , 2) = 5\,\overrightarrow{A}$ $\qquad \therefore \overrightarrow{A} \,/\!/\, \overrightarrow{B}$

TRY TO SOLVE

In the opposite figure :

Some vectors in the orthogonal coordinate plane.

Write each of the following vectors in terms of the fundamental unit vectors :



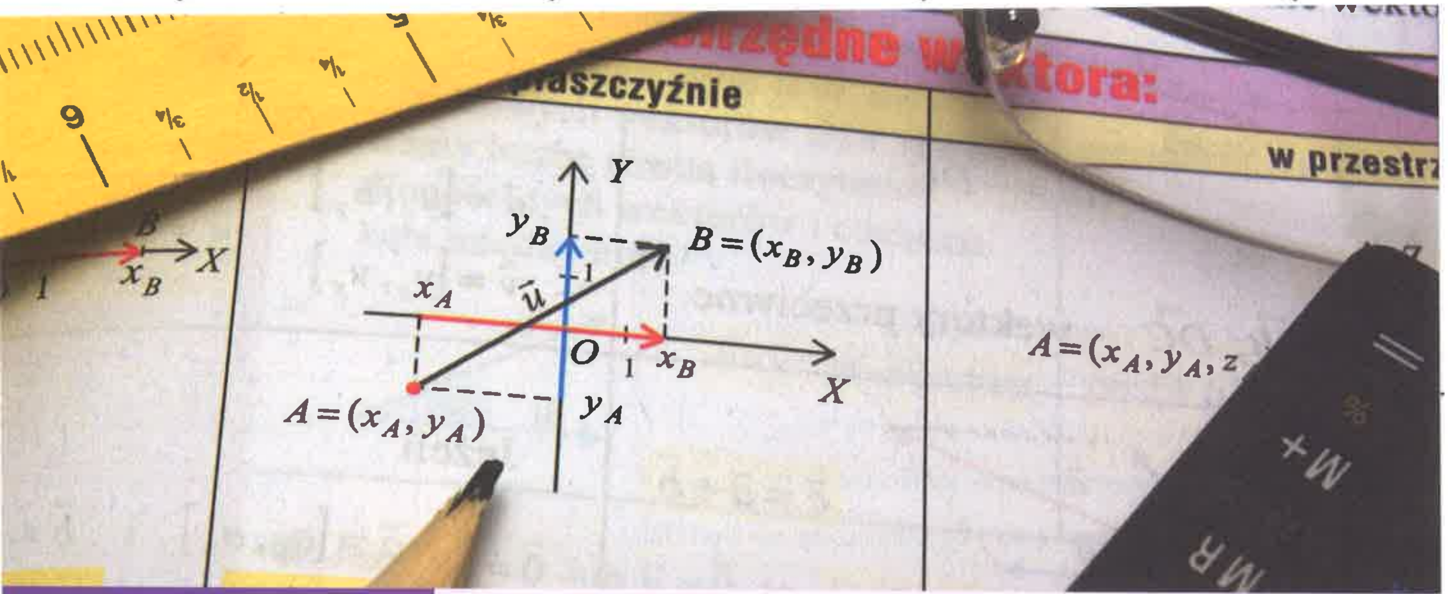
1 $\overrightarrow{A}$

2 $\overrightarrow{B}$

3 $\overrightarrow{C}$

4 $\overrightarrow{D}$

5 $\overrightarrow{E}$

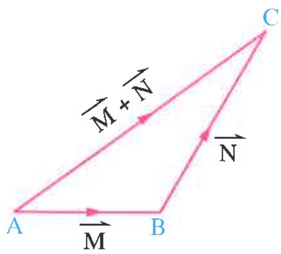


Lesson Three Operations on vectors

First Adding vectors geometrically

The first method : Trianlge rule "Shal relation"

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$, $\overrightarrow{BC}$ represents the vector $\overrightarrow{N}$, where the point B (the ending point of the first vector $\overrightarrow{M}$) is the starting point of the vector $\overrightarrow{N}$, then $\overrightarrow{AC}$ represents the vector $\overrightarrow{M} + \overrightarrow{N}$



i.e. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

i.e. The displacement $\overrightarrow{AB}$ followed by another displacement $\overrightarrow{BC}$ is equivalent to a unique displacement $\overrightarrow{AC}$

Example 1

If a ship moved from the position (A) in the given directions till it reached the position (B) Draw the path of the trip with a suitable drawing scale using your geometric tools , then find from the drawing the magnitude and the direction of the ship's displacement ($\overrightarrow{AB}$) if the directions are :

1 A distance 600 metres towards East , then a distance 800 metres towards North.

2 A distance 20 km. towards West , then a distance 30 km. in the direction 60° North West.

Solution

1 Let the drawing scale be every 200 metres in fact are represented by 1 cm. in the drawing.

∴ 600 metres are represented by 3 cm. ,
800 metres are represented by 4 cm.

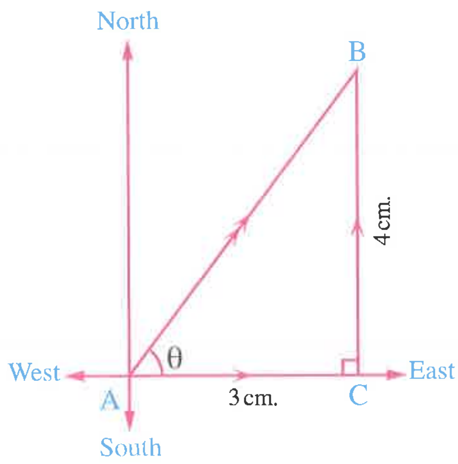
From the drawing and by measuring we find that AB = 5 cm.

∴ The norm of the displacement = 5 × 200
= 1000 metres.

and the direction of the displacement θ = 53°

(using the protractor) or $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53°$

∴ The ship is at a distance 1000 metres apart from the position A in the direction 53° North East.



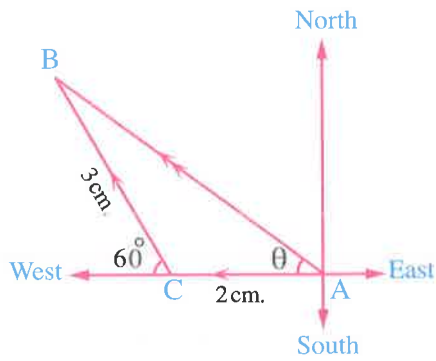
2 Let the drawing scale be each 10 km. in fact are represented by 1 cm. in drawing

∴ 20 km. are represented by 2 cm. and 30 km. are represented by 3 cm.

From drawing and by measuring we find that AB ≈ 4.4 cm.

∴ The norm of the displacement = 4.4 × 10 = 44 km.
and the direction of the displacement θ ≈ 37° (using thc protractor)

∴ The ship is at a distance 44 km. from the position A in the direction 37° North West.



TRY TO SOLVE

If a car moved from the position A in the given directions till it arrived the position B , draw the path of the trip using a suitable drawing scale using your geometric tools , from the drawing , find the norm and the directions of the displacement of the car ($\overrightarrow{AB}$) if the directions are :

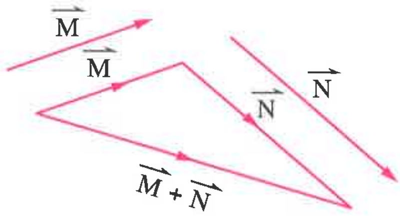
1 A distance 1200 metres in East , then a distance 1600 metres in North.

2 A distance 25 km. in East , then 30 km. in the direction 60° North East.

3 A distance 50 km. in West , then a distance 40 km. in the direction Eastern North.

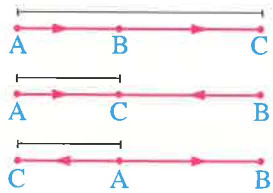
Important notes

1 Any two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ could be added (finding their resultant) by constructing two consecutive vectors equivalent to the two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ as in the opposite figure.



2 Shal rule of adding two vectors is true if the points A , B and C belong to the same straight line

In the three opposite figures : $\boxed{\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}}$



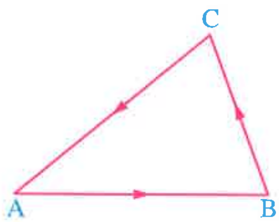
3 $\boxed{\overrightarrow{AB} = -\overrightarrow{BA}}$, where $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{O}$ (Zero vector)

4 In any triangle ABC :

$$\boxed{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{O}}$$

, because $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{O}$

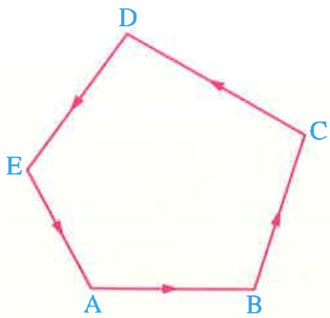
So , we can generalize this to any polygon.



For example :

In the pentagon ABCDE , we get :

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = \overrightarrow{O}$$

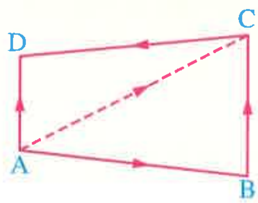


5 In any quadrilateral ABCD :

$$\boxed{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}}$$

, because $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

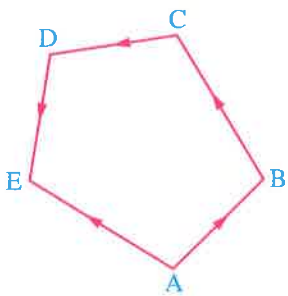
So , we can generalize this to any polygon.



For example :

In the pentagon ABCDE , we get :

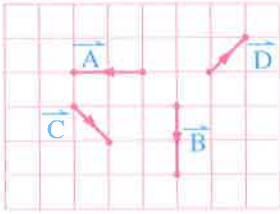
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$$



Example 2

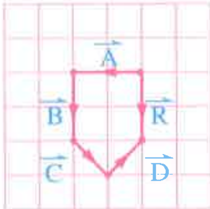
In the opposite figure :

4 vectors $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$. Represent graphically vector $\overrightarrow{R}$ where $\overrightarrow{R} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}$



Solution

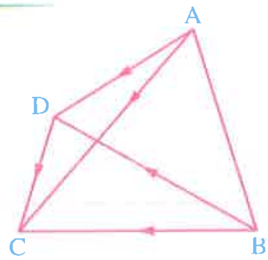
Draw the vector $\overrightarrow{A}$ as it is , then from its end point draw a vector equivalent to $\overrightarrow{B}$, from its end point draw a vector equivalent to $\overrightarrow{C}$, from its end point draw a vector equivalent to $\overrightarrow{D}$ then draw from the starting point of $\overrightarrow{A}$ to the ending point of $\overrightarrow{D}$ the vector $\overrightarrow{R}$ which is the resultant of these vectors.



Example 3

In any quadrilateral ABCD , prove that : $\overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AD} - \overrightarrow{BD}$

Solution



$\overrightarrow{AC} - \overrightarrow{BC} = (\overrightarrow{AD} + \overrightarrow{DC}) - (\overrightarrow{BD} + \overrightarrow{DC})$

$= \overrightarrow{AD} + \overrightarrow{DC} - \overrightarrow{BD} - \overrightarrow{DC} = \overrightarrow{AD} - \overrightarrow{BD}$

Another Solution :

L.H.S. $= \overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$ (1)

, R.H.S. $= \overrightarrow{AD} - \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$ (2)

From (1) and (2) : ∴ The two sides are equal.

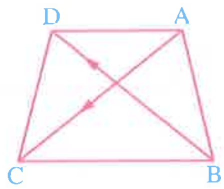
Example 4

ABCD is a quadrilateral in which $2\,\overrightarrow{BC} = 3\,\overrightarrow{AD}$, prove that :

1 ABCD is a trapezium.

2 $\overrightarrow{AC} + \overrightarrow{BD} = \frac{5}{2}\,\overrightarrow{AD}$

Solution



1 ∵ $\overrightarrow{BC} = \frac{3}{2}\,\overrightarrow{AD}$ ∴ $\overline{BC} \parallel \overline{AD}$

, $BC = \frac{3}{2}\,AD$ i.e. $BC \neq AD$

∴ The quadrilateral ABCD is a trapezium.

2 In Δ ADC : $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ (1)

In Δ DBC : $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$ (2)

Adding (1) and (2) :

$\therefore \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{CD}$

$= \overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DC} - \overrightarrow{DC}$

$= \overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AD} = \frac{5}{2}\overrightarrow{AD}$

Remember that

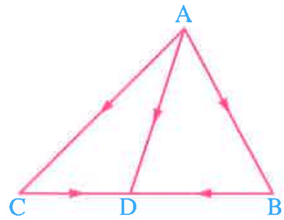
To prove that the quadrilateral is a trapezium , we prove that there exist two opposite sides parallel and not equal in length.

Example 5

ABC is a triangle , D $\in \overline{BC}$, where 3 BD = 4 DC

Prove that : $3\overrightarrow{AB} + 4\overrightarrow{AC} = 7\overrightarrow{AD}$

Solution



$\because \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\therefore 3\overrightarrow{AB} + 3\overrightarrow{BD} = 3\overrightarrow{AD}$ (1)

$\because \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\therefore 4\overrightarrow{AC} + 4\overrightarrow{CD} = 4\overrightarrow{AD}$ (2)

Adding (1) and (2) :

$\therefore 3\overrightarrow{AB} + 4\overrightarrow{AC} + 3\overrightarrow{BD} + 4\overrightarrow{CD} = 7\overrightarrow{AD}$

$\therefore 3\overrightarrow{AB} + 4\overrightarrow{AC} + 3\overrightarrow{BD} - 4\overrightarrow{DC} = 7\overrightarrow{AD}$

but $3\overrightarrow{BD} = 4\overrightarrow{DC}$

$\therefore 3\overrightarrow{AB} + 4\overrightarrow{AC} = 7\overrightarrow{AD}$

Example 6

If $4\overrightarrow{M} - 3\overrightarrow{XY} = 4\overrightarrow{ZY} + 7\overrightarrow{YX}$

, prove that : $\overrightarrow{M} = \overrightarrow{ZX}$

Solution

$4\overrightarrow{M} = 4\overrightarrow{ZY} + 7\overrightarrow{YX} + 3\overrightarrow{XY} = 4\overrightarrow{ZY} + 7\overrightarrow{YX} - 3\overrightarrow{YX}$

$= 4\overrightarrow{ZY} + 4\overrightarrow{YX} = 4(\overrightarrow{ZY} + \overrightarrow{YX}) = 4\overrightarrow{ZX}$

$\therefore \overrightarrow{M} = \overrightarrow{ZX}$

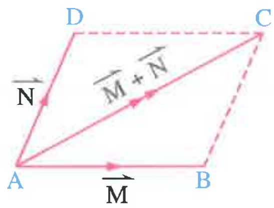
TRY TO SOLVE

1 ABCD is a quadrilateral. If $\overrightarrow{AB} = \frac{5}{3}\ \overrightarrow{DC}$, prove that : $\overrightarrow{AD} - \overrightarrow{BC} = \frac{2}{3}\ \overrightarrow{DC}$

2 If $5\ \overrightarrow{N} - 4\ \overrightarrow{BA} = 9\ \overrightarrow{AB} + 5\ \overrightarrow{BC}$, prove that : $\overrightarrow{N} = \overrightarrow{AC}$

The second method : The parallelogram rule

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$ and $\overrightarrow{AD}$ represents the vector $\overrightarrow{N}$ where the two vectors $\overrightarrow{M}$ and $\overrightarrow{N}$ have the same starting point A



- To find $\overrightarrow{M} + \overrightarrow{N}$, complete the parallelogram ABCD and draw its diagonal $\overrightarrow{AC}$, then $\overrightarrow{AC}$ represents the vector $\overrightarrow{M} + \overrightarrow{N}$

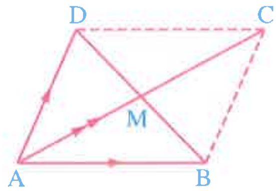
i.e. $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ this because $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Notice that : $\overrightarrow{AD}$ is equivalent to $\overrightarrow{BC}$

In the opposite figure :

If M is the point of intersection of the two diagonals of the parallelogram , then $\overrightarrow{AC} = 2\ \overrightarrow{AM}$

thus it will be $\overrightarrow{AB} + \overrightarrow{AD} = 2\ \overrightarrow{AM}$



We can get the same result if we notice that : $\overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AM}$, $\overrightarrow{AD} + \overrightarrow{DM} = \overrightarrow{AM}$

Adding we find that : $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{BM} + \overrightarrow{DM} = 2\ \overrightarrow{AM}$

, $\because\ \overrightarrow{BM} = \overrightarrow{MD} = -\ \overrightarrow{DM}$

$\therefore\ \overrightarrow{AB} + \overrightarrow{AD} - \overrightarrow{DM} + \overrightarrow{DM} = 2\ \overrightarrow{AM}$

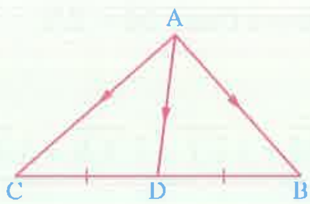
$\therefore\ \overrightarrow{AB} + \overrightarrow{AD} = 2\ \overrightarrow{AM}$ **, so we can deduce the following notice.**

Notice that

In the opposite figure :

If $\overline{AD}$ is a median in Δ ABC

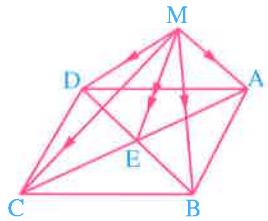
, then $\overrightarrow{AB} + \overrightarrow{AC} = 2\ \overrightarrow{AD}$



Example 7

ABCD is a parallelogram , M is a certain point in its plane , E is the point of intersection of its diagonals $\overline{AC}$ and $\overline{BD}$ Prove that : $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = 4\,\overrightarrow{ME}$

Solution



$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = (\overrightarrow{MA} + \overrightarrow{MC}) + (\overrightarrow{MB} + \overrightarrow{MD})$

, but $\overrightarrow{MA} + \overrightarrow{MC} = 2\,\overrightarrow{ME}$

, where E is the midpoint of $\overline{AC}$

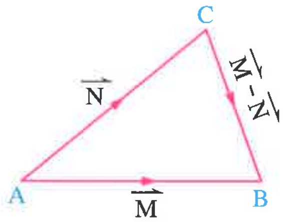
, $\overrightarrow{MB} + \overrightarrow{MD} = 2\,\overrightarrow{ME}$, where E is the midpoint of $\overline{BD}$

$\therefore\ \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = 2\,\overrightarrow{ME} + 2\,\overrightarrow{ME} = 4\,\overrightarrow{ME}$

Second Subtracting two vectors geometrically

If $\overrightarrow{AB}$ represents the vector $\overrightarrow{M}$, $\overrightarrow{AC}$ represents the vector $\overrightarrow{N}$

, then $\overrightarrow{CB}$ represents the vector $\overrightarrow{M} - \overrightarrow{N}$



i.e. $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$

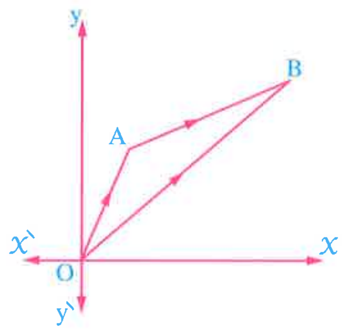
this because $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{CA} = \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$

Expressing the directed line segment $\overrightarrow{AB}$ in terms of the position vectors of its ends

If $A = (x_1 , y_1)$, $B = (x_2 , y_2)$

, then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where $\overrightarrow{OB}$ and $\overrightarrow{OA}$ are the two position vectors of the point B and A respectively

$\therefore\ \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$



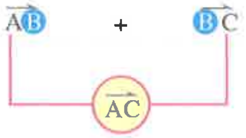
For example :

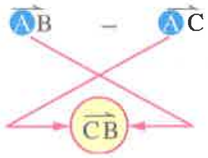
If $A = (5 , 3)$, $B = (-2 , 4)$

, then $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (-2 , 4) - (5 , 3) = (-7 , 1)$

Remember that

As applying the previous rules of adding and subtracting vectors on two directed line segments , we should notice that :





1 In the case of adding , the starting point of the second directed segment is the ending point of the first directed segment.

2 In the case of subtracting , the starting point is the same for the two directed segments.

Example 8

ABCD is a parallelogram in which A = (2 , – 2) , B = (4 , – 2) , C = (2 , 3) Find the co-ordinates of the point D

Solution

$\because$ ABCD is a parallelogram

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$

$\therefore \vec{D} - \vec{A} = \vec{C} - \vec{B}$

$\therefore \vec{D} = \vec{A} + \vec{C} - \vec{B} = (2 , -2) + (2 , 3) - (4 , -2) = (0 , 3)$

$\therefore$ The point D is (0 , 3)

TRY TO SOLVE

If ABCD is a parallelogram in which A = (x , 1) , B = (5 , 2) , C = (– 3 , – 4) , D = (2 , y) , **find the values of x and y**

Example 9

ABCD is a trapezium in which : A = (– 1 , 1) , B = (3 , 3) , C = (5 , – 1) , D = (– 5 , k)

1 If $\overline{AB} \parallel \overline{DC}$, find : the value of k

2 Prove that : $\overrightarrow{CB} \perp \overrightarrow{AB}$

3 Find the area of the trapezium ABCD

Solution

$\because \overrightarrow{AB} = \vec{B} - \vec{A} = (3 , 3) - (-1 , 1) = (4 , 2)$

, $\overrightarrow{DC} = \overrightarrow{C} - \overrightarrow{D} = (5 , -1) - (-5 , k) = (10 , -1 - k)$

, $\because$ $\overrightarrow{AB} \parallel \overrightarrow{DC}$

$\therefore 4(-1 - k) - 2 \times 10 = 0$

$\therefore -4 - 4k = 20$ $\therefore k = -6$ (First req.)

, $\because$ $\overrightarrow{CB} = \overrightarrow{B} - \overrightarrow{C} = (3 , 3) - (5 , -1) = (-2 , 4)$, $\overrightarrow{AB} = (4 , 2)$

$\therefore \overrightarrow{CB} \perp \overrightarrow{AB}$ because $-2 \times 4 + 4 \times 2 = 0$ (Second req.)

$\because \|\overrightarrow{AB}\| = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$ length unit

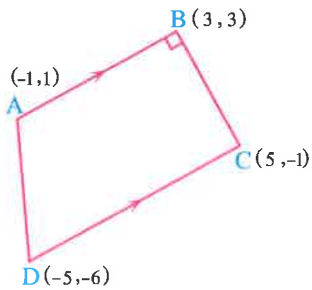
, $\because k = -6$ $\therefore \overrightarrow{DC} = (10 , 5)$

$\therefore \|\overrightarrow{DC}\| = \sqrt{100 + 25} = 5\sqrt{5}$ length unit.

, $\because \|\overrightarrow{CB}\| = \sqrt{4 + 16} = 2\sqrt{5}$ length unit.

$\therefore$ The area of the trapezium ABCD

$= \dfrac{\|\overrightarrow{AB}\| + \|\overrightarrow{DC}\|}{2} \times \|\overrightarrow{CB}\| = \dfrac{2\sqrt{5} + 5\sqrt{5}}{2} \times 2\sqrt{5} = 35$ square unit. (Third req.)



TRY TO SOLVE

ABC is a triangle in which : A = (3 , 2) , B = (2 , – 1) and C = (– 4 , 1)

1 **Prove that :** $\overrightarrow{AB} \perp \overrightarrow{BC}$

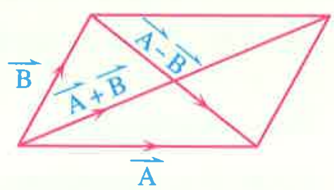
2 **Find :** The area of Δ ABC

Remark

In the opposite figure :

If $\overrightarrow{A}$ and $\overrightarrow{B}$ represent two adjacent sides of parallelogram , then $(\overrightarrow{A} + \overrightarrow{B})$, $(\overrightarrow{A} - \overrightarrow{B})$ represent the diagonals of the parallelogram and hence $\|\overrightarrow{A} + \overrightarrow{B}\| = \|\overrightarrow{A} - \overrightarrow{B}\|$ if the figure is a rectangle

i.e. $\overrightarrow{A} \perp \overrightarrow{B}$



Lesson Four Applications on vectors

First Geometric applications

We know that if $\overrightarrow{AB} = k\,\overrightarrow{DC}$, $k \neq 0$, then $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are :

- carried by the same straight line

i.e. A , B , C , D are collinear.

or

- carried by two parallel straight lines

i.e. $\overleftrightarrow{AB} \mathbin{//} \overleftrightarrow{DC}$

Remark

If ABCD is a quadrilateral in which $\overrightarrow{AB} = k\,\overrightarrow{DC}$, $k \neq 0$, then $\overleftrightarrow{AB} \mathbin{//} \overleftrightarrow{DC}$, $\|\overrightarrow{AB}\| = |k|\,\|\overrightarrow{DC}\|$ and vise versa.

For example :

If ABCD is a quadrilateral in which $\overrightarrow{AB} = -3\,\overrightarrow{CD}$, then $\overleftrightarrow{AB} \mathbin{//} \overleftrightarrow{CD}$, AB = 3 CD

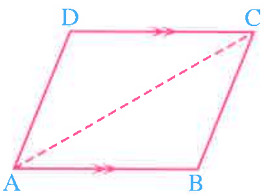
Thus we can use vectors to prove some theorems and geometric relations as follows :

Example 1

Using vectors , prove that if two opposite sides in any quadrilateral are parallel and equal in length , then the other two opposite sides are parallel and equal in length also.

i.e. The quadrilateral is a parallelogram.

Solution



Given In the figure ABCD :

$\overline{AB} \,//\, \overline{DC}$, AB = DC

R.T.P. $\overline{BC} \,//\, \overline{AD}$, BC = AD

Const. Draw $\overline{AC}$

Proof

$\because$ AB = DC , $\overline{AB} \,//\, \overline{DC}$ $\quad \therefore \overrightarrow{AB} \,//\, \overrightarrow{DC}$

In Δ ABC : $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ (addition definition)

In Δ ADC : $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ (addition definition)

$\because \overrightarrow{AB} = \overrightarrow{DC}$ $\quad \therefore \overrightarrow{BC} = \overrightarrow{AD}$

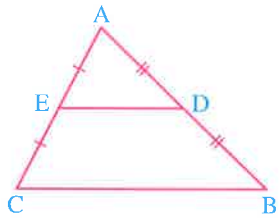
Thus $\overline{BC} \,//\, \overline{AD}$, BC = AD

$\therefore$ The figure ABCD is a parallelogram. (Q.E.D.)

Example 2

Using vectors , prove that the line segment drawn between the two midpoints of two sides in a triangle is parallel to the third side and its length is equal to half of this side length.

Solution



Given In Δ ABC : D is the midpoint of $\overline{AB}$

, E is the midpoint of $\overline{AC}$

R.T.P. $\overline{DE} \,//\, \overline{BC}$, DE = $\frac{1}{2}$ BC

Proof

$\because$ D is the midpoint of $\overline{AB}$

$\therefore$ AD = $\frac{1}{2}$ AB , $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$

$\because$ E is the midpoint of $\overline{AC}$

$\therefore$ AE = $\frac{1}{2}$ AC , $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$

In Δ ABC : $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$ (addition definition) (1)

In Δ ADE : $\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$ (addition definition)

$= \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$ (2)

From (1) and (2) we deduce that : $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

$\therefore \overline{DE} \,//\, \overline{BC}$, $\| \overrightarrow{DE} \| = \frac{1}{2} \| \overrightarrow{BC} \|$

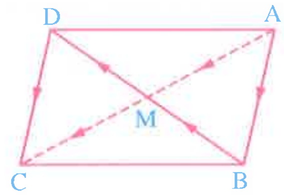
$\therefore$ The length of $\overline{DE} = \frac{1}{2}$ the length of $\overline{BC}$

$\therefore$ DE = $\frac{1}{2}$ BC (Q.E.D.)

Example 3

Using vectors , prove that the two diagonals of the parallelogram bisect each other.

Solution



Given ABCD is a parallelogram

R.T.P. The two diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other

Const. Let M be the midpoint of $\overline{BD}$, then draw the two vectors $\overrightarrow{AM}$, $\overrightarrow{MC}$

Proof In Δ ABM : $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$

In Δ CDM : $\overrightarrow{MC} = \overrightarrow{MD} + \overrightarrow{DC}$

$\because$ $\overrightarrow{BM} = \overrightarrow{MD}$ (by construction)

$\overrightarrow{AB} = \overrightarrow{DC}$ (Properties of parallelogram)

$\therefore$ $\overrightarrow{AM} = \overrightarrow{MC}$

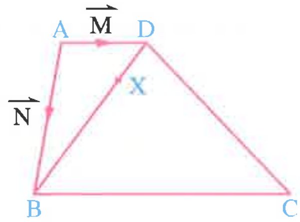
$\therefore$ $\overrightarrow{AM}$, $\overrightarrow{MC}$ have the same direction and they have the common point (M)

$\therefore$ A , M , C are collinear , $\| \overrightarrow{AM} \| = \| \overrightarrow{MC} \|$

$\therefore$ M is the midpoint of $\overline{AC}$, M is the midpoint of $\overline{BD}$ (by construction)

$\therefore$ The two diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. (Q.E.D.)

Example 4



In the opposite figure :

ABCD is a trapezium in which : $\overline{AD} \parallel \overline{BC}$,

BC = 3 AD , $\overrightarrow{AD} = \overrightarrow{M}$, $\overrightarrow{AB} = \overrightarrow{N}$

1 Express , in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$, each of $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$, $\overrightarrow{DC}$

2 If X $\in \overline{DB}$ where DX = $\frac{1}{3}$ XB

, **prove that :** The points A , X , C are collinear.

Solution

1 $\because$ $\overline{AD} \parallel \overline{BC}$, BC = 3 AD

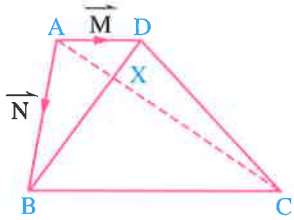
$\therefore$ $\overrightarrow{BC} = 3\,\overrightarrow{AD} = 3\,\overrightarrow{M}$

, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{N} + 3\,\overrightarrow{M}$, $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\overrightarrow{N} + \overrightarrow{M} = \overrightarrow{M} - \overrightarrow{N}$

, $\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{M} + \overrightarrow{N} + 3\,\overrightarrow{M} = \overrightarrow{N} + 2\,\overrightarrow{M}$

2 $\because DX = \frac{1}{3} XB$ $\therefore XB = 3\,XD$

$\because X \in \overline{BD}$ $\therefore \overrightarrow{XB} = 3\,\overrightarrow{DX}$



In Δ BCX :

$\because \overrightarrow{XC} = \overrightarrow{XB} + \overrightarrow{BC} = 3\,\overrightarrow{DX} + 3\,\overrightarrow{AD} = 3\,(\overrightarrow{DX} + \overrightarrow{AD})$

$= 3\,(\overrightarrow{AD} + \overrightarrow{DX}) = 3\,\overrightarrow{AX}$

$\therefore \overrightarrow{XC}$, $\overrightarrow{AX}$ have the same direction and have the common point X

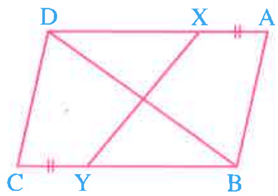
$\therefore$ A , X , C are collinear

TRY TO SOLVE

In the opposite figure :

ABCD is a parallelogram , $X \in \overline{AD}$, $Y \in \overline{BC}$ such that AX = CY

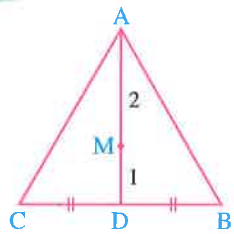
Using vectors , prove that : $\overline{XY}$, $\overline{BD}$ bisect each other.



Example 5

If A = (2 , 1) , B = (1 , 5) , C = (6 , –3) are the vertices of a triangle , find by using vectors the coordinates of the point of intersection of its medians.

Solution



- Draw the median $\overline{AD}$ in Δ ABC , M is the point of intersection of its medians.

$\because$ The point of intersection of the medians of the triangle divides each of them by the ratio 2 : 1 from the vertex.

$\therefore \overrightarrow{AM} = \frac{2}{3}\,\overrightarrow{AD}$ $\therefore \overrightarrow{AD} = \frac{3}{2}\,\overrightarrow{AM}$ (1)

In Δ ABC : $\because \overline{AD}$ is a median $\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$ (2)

From (1) and (2) : $\overrightarrow{AB} + \overrightarrow{AC} = 2\left(\frac{3}{2}\,\overrightarrow{AM}\right)$

$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 3\,\overrightarrow{AM}$ $\therefore \vec{B} - \vec{A} + \vec{C} - \vec{A} = 3\,(\vec{M} - \vec{A})$

$\therefore \vec{B} + \vec{C} - 2\,\vec{A} = 3\,\vec{M} - 3\,\vec{A}$ $\therefore \vec{B} + \vec{C} - 2\,\vec{A} + 3\,\vec{A} = 3\,\vec{M}$

$\therefore 3\,\vec{M} = \vec{A} + \vec{B} + \vec{C}$ $\therefore \vec{M} = \frac{1}{3}\,(\vec{A} + \vec{B} + \vec{C})$

$\therefore \vec{M} = \frac{1}{3}\,(2 + 1 + 6 , 1 + 5 - 3) = \frac{1}{3}\,(9 , 3) = (3 , 1)$

$\therefore$ The point of intersection of the medians of the triangle is (3 , 1)

Important remarks to solve the problems of the quadrilateral shapes

* To prove that the figure is a parallelogram , we prove one of the following properties :

1. Each two opposite sides are parallel.
2. Each two opposite sides are equal in length.
3. Two opposite sides are parallel and equal in length.
4. The two diagonals bisect each other.

For example :

To prove that the quadrilateral ABCD is a parallelogram , we prove that $\overline{BC} \parallel \overline{AD}$, BC = AD   i.e. we prove that $\overrightarrow{BC} = \overrightarrow{AD}$

* **To prove that the quadrilateral is a rectangle, rhombus or square, then we should prove first that the quadrilateral is a parallelogram as previous, then :**

• To prove that the parallelogram is a rectangle we prove one of the following properties:

1. Two adjacent sides are perpendicular.

   For example : $\overrightarrow{AB} \perp \overrightarrow{BC}$

2. The two diagonals are equal in length.

   For example : $\|\overrightarrow{AC}\| = \|\overrightarrow{BD}\|$

• To prove that the parallelogram is a rhombus we prove one of the following properties:

1. Two adjacent sides are equal in length.

   For example : $\|\overrightarrow{AB}\| = \|\overrightarrow{BC}\|$

2. The two diagonals are perpendicular.

   For example : $\overrightarrow{AC} \perp \overrightarrow{BD}$

• To prove that the parallelogram is a square we prove one of the properties of the rectangle and one of the properties of the rhombus together.

Example 6

ABCD is a quadrilateral in which A = (0 , 1) , B = (4 , 5) , C = (1 , 8) , D = (– 3 , 4)
Using vectors , prove that : The quadrilateral ABCD is a rectangle , then find its perimeter and its area.

Solution

$\because \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (4 , 5) - (0 , 1) = (4 , 4)$

$, \overrightarrow{DC} = \overrightarrow{C} - \overrightarrow{D} = (1 , 8) - (-3 , 4) = (4 , 4)$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$ $\therefore$ The figure is a parallelogram (1)

$, \because \overrightarrow{BC} = \overrightarrow{C} - \overrightarrow{B} = (1 , 8) - (4 , 5) = (-3 , 3)$

$\therefore \overrightarrow{AB} \perp \overrightarrow{BC}$ because $[4 \times -3 + 4 \times 3 = 0]$ (2)

From (1) and (2) we deduce that the figure ABCD is a rectangle

$\because \| \overrightarrow{AB} \| = \sqrt{(4)^2 + (4)^2} = 4\sqrt{2}$

$, \| \overrightarrow{BC} \| = \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2}$

$\therefore$ The perimeter of the rectangle $= 2\left(4\sqrt{2} + 3\sqrt{2}\right) = 14\sqrt{2}$ length unit.

, the area of the rectangle $= 4\sqrt{2} \times 3\sqrt{2} = 24$ square unit.

Example 7

ABCD is a quadrilateral in which $A = (5 , 3)$, $B = (3 , -2)$, $C = (-2 , -4)$, $D = (0 , 1)$

Using vectors , prove that : The figure is a rhombus , then find its area.

Solution

$\because \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (3 , -2) - (5 , 3) = (-2 , -5)$

$, \overrightarrow{DC} = \overrightarrow{C} - \overrightarrow{D} = (-2 , -4) - (0 , 1) = (-2 , -5)$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ The figure is a parallelogram (1)

$\because \overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (-2 , -4) - (5 , 3) = (-7 , -7)$

$, \overrightarrow{BD} = \overrightarrow{D} - \overrightarrow{B} = (0 , 1) - (3 , -2) = (-3 , 3)$

$\therefore \overrightarrow{AC} \perp \overrightarrow{BD}$ because $[-7 \times -3 + (-7) \times 3 = 0]$ (2)

From (1) and (2) : $\therefore$ The figure is a rhombus

$\because \| \overrightarrow{AC} \| = \sqrt{(-7)^2 + (-7)^2} = 7\sqrt{2}$ $, \| \overrightarrow{BD} \| = \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2}$

$\therefore$ The area of the rhombus $= \frac{1}{2}$ the product of lengths of the two diagonals

$= \frac{1}{2} \times 7\sqrt{2} \times 3\sqrt{2} = 21$ square unit.

Unit 4

TRY TO SOLVE

ABCD is a quadrilateral in which :

A = (– 1 , 4) , B = (1 , 1) , C = (– 1 , – 2) , D = (– 3 , 1)

Using vectors , prove that : The figure is a rhombus , then find its perimeter and its area.

Second Physical applications

1 The resultant force

- **The force :** is a vector passes through a given point and acts along a straight line.
- **The force :** is represented by a directed line segment and it is drawn by a suitable drawing scale.

Remember that

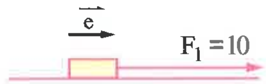
1. Consider $\vec{e}$ a unit vector in the East direction.
2. Choose a suitable drawing scale "Each 5 Newton is represented on drawing by 1 cm".

For example :

1. A force of magnitude $F_1 = 10$ Newton acts in the East direction.

   $\vec{F_1} = 10\,\vec{e}$

   $\vec{F_1}$ is represented by a directed line segment of length 2 cm.



2. A force of magnitude 15 Newtons towards West.

   $\vec{F_2} = -\,15\,\vec{e}$

   $\vec{F_2}$ is represented by a directed line segment of length 3 cm.

Friction force :

It is a mysterious force , its effect appears only when we try to move a body on a rough plane.

This force acts always in the opposite direction of motion.

- If the impetus force is greater than the friction force , then the body will move.
- If the impetus force is less than the friction force , then the body will still at rest.

The resultant force $(\vec{F})$

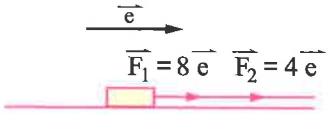
The forces acting on an object are subjected to the processes of adding vectors.
The result of this operation is known as the resultant force acting on the object where

$\vec{F} = \vec{F_1} + \vec{F_2} + \cdots\cdots$

For example :

1 If the force $\overrightarrow{F_1}$ of magnitude 8 newton acted towards East , then another force $\overrightarrow{F_2}$ of magnitude 4 newtons acted towards East also.

- Let $\vec{e}$ be a unit vector in the East direction



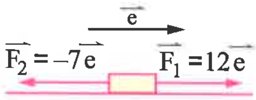
$\therefore \overrightarrow{F_1} = 8\vec{e}$, $\overrightarrow{F_2} = 4\vec{e}$

$\therefore$ The resultant force $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = 8\vec{e} + 4\vec{e} = 12\vec{e}$

i.e. F = 12 Newton acting towards East.

2 As trying to move a body under the effect of a force $\overrightarrow{F_1}$ of magnitude 12 newtons and the magnitude of the force of friction is 7 Newton.

- Let $\vec{e}$ be a unit vector in the direction of motion



$\therefore$ The impetus force of the body $\overrightarrow{F_1} = 12\vec{e}$

, the friction force $\overrightarrow{F_2} = -7\vec{e}$

$\therefore$ The resultant force $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = 12\vec{e} - 7\vec{e} = 5\vec{e}$

i.e. F = 5 newton and acts in the direction of motion.

Remark

Forces are measured in dyne , newton , gm.wt. , kg.wt.

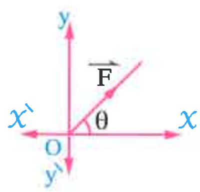
Example 8

The forces $\overrightarrow{F_1} = 5\vec{i} + 2\vec{j}$, $\overrightarrow{F_2} = -2\vec{i} + 7\vec{j}$, $\overrightarrow{F_3} = 3\vec{i} - \vec{j}$ act on a particle

Calculate the magnitude and the direction of the resultant of these forces

(given that the forces are measured in newton)

Solution



$\because$ The resultant of the forces $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$

$\therefore \vec{F} = (5 - 2 + 3)\vec{i} + (2 + 7 - 1)\vec{j} = 6\vec{i} + 8\vec{j}$

$\therefore$ The magnitude of the resultant $= \|\vec{F}\| = \sqrt{(6)^2 + (8)^2} = 10$ Newton.

, the direction of the resultant $\theta = \tan^{-1}\left(\frac{8}{6}\right) \approx 53°$

Example 9

Write in terms of the unit vector $\vec{e}$ the resultant of the forces shown in each of the following :

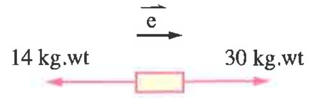


Fig. (1)

Fig. (2)

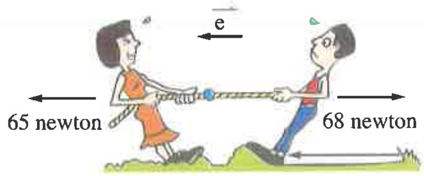


Fig. (3)

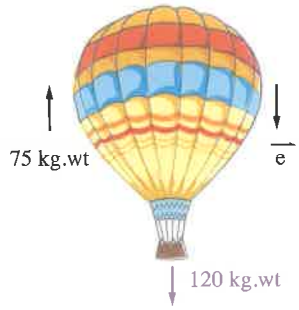


Fig. (4)

Solution

Fig. (1) : $30\,\vec{e} - 14\,\vec{e} = 16\,\vec{e}$

Fig. (2) : $470\,\vec{e} + 400\,\vec{e} = 870\,\vec{e}$

Fig. (3) : $65\,\vec{e} - 68\,\vec{e} = -3\,\vec{e}$

Fig. (4) : $120\,\vec{e} - 75\,\vec{e} = 45\,\vec{e}$

Remarks

1 If the two forces are equal in magnitude and act along the same line of action in two opposite directions , then the resultant force $\vec{F} = \vec{0}$

2 If the resultant of a system of forces meeting in one point = $\vec{0}$, this means that this system of forces are in equilibrium.

Example 10

If $\vec{F_1} = (5 , -3)$, $\vec{F_2} = a\,\vec{i} - 2\,\vec{j}$, $\vec{F_3} = (-7 , b)$ act on a particle.

, find the values of a and b if :

1 The resultant of these forces $= \vec{i} - 4\,\vec{j}$

2 The system of these forces is in equilibrium.

Solution

$\because \vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (5\,\vec{i} - 3\,\vec{j}) + (a\,\vec{i} - 2\,\vec{j}) + (-7\,\vec{i} + b\,\vec{j}) = (5 + a - 7)\,\vec{i} + (-3 - 2 + b)\,\vec{j}$

1 $\because \vec{F} = \vec{i} - 4\,\vec{j}$

$\therefore (5 + a - 7)\,\vec{i} + (-3 - 2 + b)\,\vec{j} = \vec{i} - 4\,\vec{j}$

$\therefore 5 + a - 7 = 1$ $\qquad \therefore a = 3$

$, -3 - 2 + b = -4$ $\qquad \therefore b = 1$

2 $\because$ The system is in equilibrium $\quad\therefore \vec{F} = \vec{0}$

$\therefore (5 + a - 7)\,\vec{i} + (-3 - 2 + b)\,\vec{j} = \vec{0}$

$\therefore 5 + a - 7 = 0 \quad \therefore a = 2$

$, -3 - 2 + b = 0 \quad \therefore b = 5$

TRY TO SOLVE

If the forces $\vec{F_1} = 5\,\vec{i} + 19\,\vec{j}$, $\vec{F_2} = 6\,\vec{i} - 3\,\vec{j}$, $\vec{F_3} = -4\,\vec{i} + 8\,\vec{j}$ act on a particle , **calculate the magnitude and the direction of the resultant of these forces** (given that forces are measure in dyne).

2 The relative velocity

- The train passenger may imagine that his train moves backwards when he sees from the window another train start motion in the same direction , but at last he discovers that his train still at rest when he sees again from the other side to the station building which is fixed at its position.

- When the driver of a car sees another car in front of him moving with speed less in magnitude than the speed of his car. It seems to him that the other car is moving back.

- When the driver of a car sees another car moves in his direction , it seems to him that it moves slowly while when he sees another car moves in the opposite direction , it seems to him that it moves with great speed.

The relative velocity vector

If $\vec{V_A}$ is the actual velocity vector of the body A and $\vec{V_B}$ is the actual velocity vector of the body B , then

1 $\vec{V}_{BA}$ is the relative velocity vector of the body B with respect to the body A

$$\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$$

It is the velocity that seems the body B moves with it , if we consider that the body A is at rest.

2 $\overrightarrow{V}_{AB}$ is the relative velocity vector of the body A with respect to the body B

$$\overrightarrow{V}_{AB} = \overrightarrow{V}_A - \overrightarrow{V}_B$$

It is the velocity that seems the body A moves with it , if we consider that the body B is at rest.

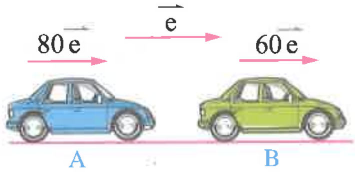
Example 11

A car A moves on a straight road with velocity 80 km./hr. and a car B moves on the same road with velocity 60 km./hr. **Find the velocity of the car A with respect to the car B if :**

1. The two cars move in the same direction.
2. The two cars move in opposite directions.

Solution

Let $\overrightarrow{e}$ be a unit vector in the direction of motion of the car A

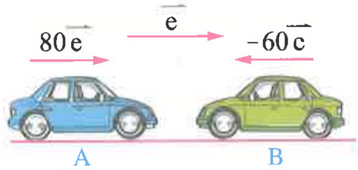


1 The two cars move in the same direction.

$\therefore \overrightarrow{V}_A = 80\,\overrightarrow{e}$, $\overrightarrow{V}_B = 60\,\overrightarrow{e}$

$\therefore \overrightarrow{V}_{AB} = \overrightarrow{V}_A - \overrightarrow{V}_B = 80\,\overrightarrow{e} - 60\,\overrightarrow{e} = 20\,\overrightarrow{e}$

i.e. The driver of the car B feels that the car A moves with velocity 20 km./h.



2 The two cars move in opposite directions.

$\therefore \overrightarrow{V}_A = 80\,\overrightarrow{e}$, $\overrightarrow{V}_B = -60\,\overrightarrow{e}$

$\therefore \overrightarrow{V}_{AB} = \overrightarrow{V}_A - \overrightarrow{V}_B = 80\,\overrightarrow{e} - \left(-60\,\overrightarrow{e}\right) = 140\,\overrightarrow{e}$

i.e. The driver of the car B feels that the car A moves with velocity 140 km./h.

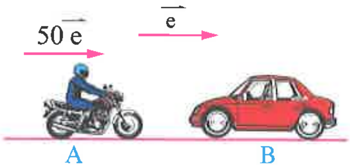
Example 12

A motorcycle (A) moves with velocity 50 km./h. its rider feels that a car B moving in the opposite direction with velocity 110 km./h. with respect to him.

Find the actual velocity of the car.

Solution

Assuming that $\vec{e}$ is a unit vector in the direction of the motion of the motorcycle (A)



$\therefore \vec{V}_A = 50\,\vec{e}$, $\vec{V}_{BA} = -110\,\vec{e}$

$\therefore \vec{V}_{BA} = \vec{V}_B - \vec{V}_A \qquad \therefore -110\,\vec{e} = \vec{V}_B - 50\,\vec{e}$

$\therefore \vec{V}_B = -110\,\vec{e} + 50\,\vec{e} = -60\,\vec{e}$

i.e. The car B moves with velocity 60 km./h. in the opposite direction of the motion of the motorcycle (A)

TRY TO SOLVE

A car moves on a straigh road with velocity 80 km./h. If a motorcycle moves on the same road with velocity 30 km./h. **Find the relative velocity of the motorcycle with respect to the car in each of the following cases :**

1 The motorcycle moves in the same direction of the car.

2 The motorcycle moves in the opposite direction of the car.

Unit 5

STRAIGHT LINE

Unit Lessons

Lesson One	:	Division of a line segment.
Lesson Two	:	Equation of the straight line.
Lesson Three	:	Measure of the angle between two straight lines.
Lesson Four	:	The length of the perpendicular from a point to a straight line.
Lesson Five	:	General equation of the straight line passing through the point of intersection of two lines

Learning outcomes

By the end of this unit, the student should be able to :

- Find the coordinates of the division point of a line segment internally or externally if the ratio of the division is known.
- Find the ratio by which the line segment is divided internally or externally if the coordinates of the division point are known.
- Recognize the different forms of the equation of the straight line.
- Find the vector equation, parametric equations and cartesian equation of the straight line.
- Find the general form of the equation of the straight line.
- Find the equation of the straight line in terms of the intercepted parts of the two axes.
- Find the measure of the acute angle between two straight lines.
- Find the length of the perpendicular drawn from a point to a straight line.
- Find the general equation of a straight line passing through the point of intersection of two straight lines.

Lesson One — Division of a line segment

- If $\overrightarrow{AB}$ is a directed line segment $\subset \overleftrightarrow{AB}$, then any point $C \in \overleftrightarrow{AB}$ divides $\overrightarrow{AB}$ into two directed line segments $\overrightarrow{AC}$, $\overrightarrow{CB}$, where $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

  If the point C divides $\overrightarrow{AB}$ by a given ratio $m_2 : m_1$ and $\vec{r}_1$, $\vec{r}_2$, $\vec{r}$ are the vectors which are represented by the directed line segments $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ where O is the origin point.

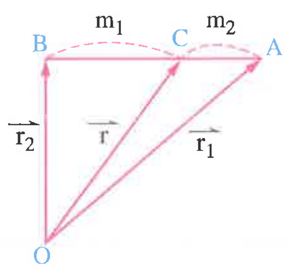


Fig. (1)

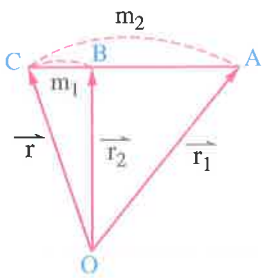


Fig. (2)

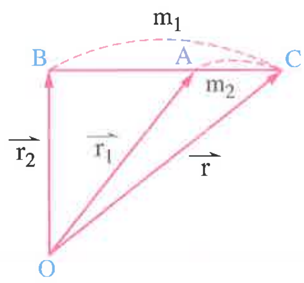


Fig. (3)

Then $\dfrac{AC}{CB} = \dfrac{m_2}{m_1}$ $\quad\therefore m_1 \times \overrightarrow{AC} = m_2 \times \overrightarrow{CB}$

$\therefore m_1 (\vec{C} - \vec{A}) = m_2 (\vec{B} - \vec{C})$ $\quad\therefore m_1 (\vec{r} - \vec{r}_1) = m_2 (\vec{r}_2 - \vec{r})$

$\therefore m_1 \vec{r} - m_1 \vec{r}_1 = m_2 \vec{r}_2 - m_2 \vec{r}$ $\quad\therefore m_1 \vec{r} + m_2 \vec{r} = m_1 \vec{r}_1 + m_2 \vec{r}_2$

$\therefore \vec{r} (m_1 + m_2) = m_1 \vec{r}_1 + m_2 \vec{r}_2$

$\therefore \vec{r} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ which is called the vector form

Remarks

1. If $C \in \overline{AB}$, then C divides $\overline{AB}$ internally , then $\overrightarrow{AC}$ and $\overrightarrow{CB}$ have the same direction and the two values m_1 and m_2 are positive.

   i.e. $\frac{m_2}{m_1} > 0$ [Fig. (1)]

2. If $C \in \overleftrightarrow{AB}$, $C \notin \overline{AB}$, then C divides $\overline{AB}$ externally , then $\overrightarrow{AC}$ and $\overrightarrow{CB}$ have two opposite directions and one of the two values m_1 and m_2 is positive and the other is negative.

   i.e. $\frac{m_2}{m_1} < 0$, in this case we have two cases :

   First : $|m_2| > |m_1|$, then $C \in \overrightarrow{AB}$, $C \notin \overline{AB}$ [Fig. (2)]

   Second : $|m_2| < |m_1|$, then $C \in \overrightarrow{BA}$, $C \notin \overline{AB}$ [Fig. (3)]

3. $\frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{CB}\|} = \left|\frac{m_2}{m_1}\right|$ **i.e.** $\frac{AC}{CB} = \left|\frac{m_2}{m_1}\right|$

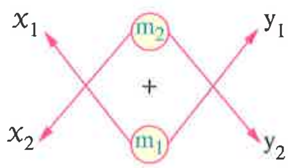
- If we assume that $A = (x_1 , y_1)$, $B = (x_2 , y_2)$, $C = (x , y)$

$$\because \vec{r} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

$$\therefore (x , y) = \frac{m_1 (x_1 , y_1) + m_2 (x_2 , y_2)}{m_1 + m_2} = \frac{(m_1 x_1 + m_2 x_2 , m_1 y_1 + m_2 y_2)}{m_1 + m_2}$$

$$\therefore (x , y) = \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} , \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \right)$$ which is called the cartesian form

- We can use the opposite figure to facilitate finding the cartesian form.



Example 1

If A = (1 , – 4) , B = (6 , 6) , find the coordinates of the point C which divides $\overline{AB}$ internally by the ratio 3 : 2

Solution

Notice that

C divides $\overline{AB}$

$\therefore \vec{r_1} = A (1 , -4)$, $\vec{r_2} = B (6 , 6)$

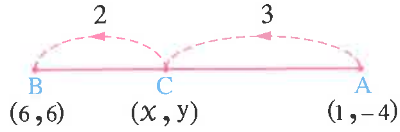
$\because$ C divides $\overline{AB}$ internally $\therefore \frac{m_2}{m_1} = \frac{3}{2}$

$$\because \vec{r} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

$$\therefore \vec{r} = \frac{2 (1 , -4) + 3 (6 , 6)}{2 + 3} = \left(\frac{2 \times 1 + 3 \times 6}{5} , \frac{2 \times -4 + 3 \times 6}{5} \right) = (4 , 2)$$

$\therefore C = (4 , 2)$

Another Solution : (Using vectors)



$\because$ C divides $\overrightarrow{AB}$ internally by the ratio 3 : 2

$\therefore \frac{AC}{CB} = \frac{3}{2}$ $\qquad \therefore 2\,\overrightarrow{AC} = 3\,\overrightarrow{CB}$

$\therefore 2\,(x - 1 , y + 4) = 3\,(6 - x , 6 - y)$

$\therefore (2\,x - 2 , 2\,y + 8) = (18 - 3\,x , 18 - 3\,y)$

$\therefore 2\,x - 2 = 18 - 3\,x$, then $x = 4$

, $2\,y + 8 = 18 - 3\,y$, then $y = 2$

$\therefore C = (4 , 2)$

Example 2

If A = (2 , – 3) , B = (1 , – 1) , find the coordinates of the point C which divides $\overrightarrow{BA}$ externally by the ratio 4 : 3

Solution

$\because$ C divides $\overrightarrow{BA}$ externally $\qquad \therefore \frac{m_2}{m_1} = \frac{-4}{3}$

$\because \vec{r} = \frac{m_1\,\vec{r_1} + m_2\,\vec{r_2}}{m_1 + m_2}$

Notice that

C divides $\overrightarrow{BA}$

$\therefore \vec{r_1} = B\,(1 , -1)$, $\vec{r_2} = A\,(2 , -3)$

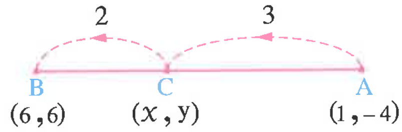
$\therefore \vec{r} = \frac{3\,(1 , -1) + (-4)\,(2 , -3)}{3 + (-4)} = \left(\frac{3 \times 1 - 4 \times 2}{-1} , \frac{3 \times -1 - 4 \times -3}{-1}\right) = (5 , -9)$

$\therefore C = (5 , -9)$

Notice that : We consider the ratio of division $m_2 : m_1 = -4 : 3$ and if we consider it $4 : -3$, we will get the same result.

$\vec{r} = \frac{-3\,(1 , -1) + 4\,(2 , -3)}{-3 + 4} = (5 , -9)$

Another Solution : (Using vectors)



$\because$ C divides $\overrightarrow{BA}$ externally by the ratio 4 : 3

$\therefore \frac{BC}{AC} = \frac{4}{3}$ $\qquad \therefore 3\,\overrightarrow{BC} = 4\,\overrightarrow{AC}$

$\therefore 3\,(x - 1 , y + 1) = 4\,(x - 2 , y + 3)$

$\therefore (3\,x - 3 , 3\,y + 3) = (4\,x - 8 , 4\,y + 12)$

$\therefore 3\,x - 3 = 4\,x - 8$, then $x = 5$

, $3\,y + 3 = 4\,y + 12$, then $y = -9$

$\therefore C = (5 , -9)$

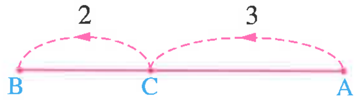
Example 3

If $A = (3 , -1)$, $B = (5 , 2)$ and $C \in \overleftrightarrow{AB}$ such that $2\,AC = 3\,CB$, find the coordinates of C if :

1 The division is internally. **2** The division is externally.

Solution

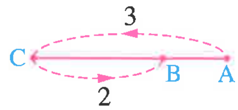
$\because 2\,AC = 3\,CB \quad \therefore \dfrac{AC}{CB} = \dfrac{3}{2} \quad \therefore \left|\dfrac{m_2}{m_1}\right| = \dfrac{3}{2}$



1 If the division is internally , then $\dfrac{m_2}{m_1} = \dfrac{3}{2}$

$\therefore \vec{r} = \dfrac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} = \dfrac{2\,(3 , -1) + 3\,(5 , 2)}{2 + 3} = \left(\dfrac{21}{5} , \dfrac{4}{5}\right)$

$\therefore C = \left(\dfrac{21}{5} , \dfrac{4}{5}\right)$



2 If the division is externally , then $\dfrac{m_2}{m_1} = -\dfrac{3}{2} = \dfrac{3}{-2}$

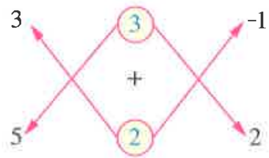
$\therefore \vec{r} = \dfrac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} = \dfrac{-2\,(3 , -1) + 3\,(5 , 2)}{-2 + 3} = (9 , 8)$

$\therefore C = (9 , 8)$

Notice that : $|m_2| > |m_1|$, thus $C \in \overrightarrow{AB}$, $C \notin \overline{AB}$

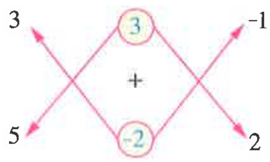
Another Solution : (Using the cartesian form of the point of division)

$C = \left(\dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} , \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2}\right)$



1 $A = (3 , -1)$, $B = (5 , 2)$, $m_2 : m_1 = 3 : 2$

$\therefore C = \left(\dfrac{2 \times 3 + 3 \times 5}{2 + 3} , \dfrac{2 \times -1 + 3 \times 2}{2 + 3}\right) = \left(\dfrac{21}{5} , \dfrac{4}{5}\right)$



2 $A = (3 , -1)$, $B = (5 , 2)$, $m_2 : m_1 = 3 : -2$

$\therefore C = \left(\dfrac{-2 \times 3 + 3 \times 5}{-2 + 3} , \dfrac{-2 \times -1 + 3 \times 2}{-2 + 3}\right) = (9 , 8)$

TRY TO SOLVE

$A = (1 , 2)$, $B = (8 , -5)$ Find the coordinates of C which divides $\overrightarrow{AB}$ by the ratio $4 : 3$ if :

1 The division is internally. **2** The division is externally.

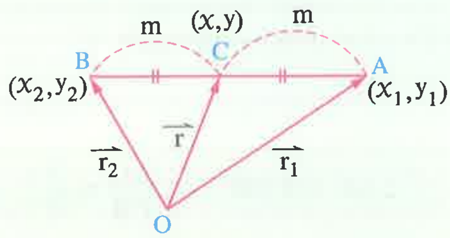
Notice that

If C is the midpoint of $\overrightarrow{AB}$ where $A = (x_1, y_1)$, $B = (x_2, y_2)$, then $m_1 = m_2 = m$

$$\therefore \vec{r} = \frac{m\,\vec{r_1} + m\,\vec{r_2}}{m+m} = \frac{m(\vec{r_1}+\vec{r_2})}{2m}$$

$$\therefore \vec{r} = \frac{\vec{r_1}+\vec{r_2}}{2} \quad \text{the vector form}$$

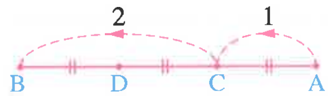
$$, (x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \quad \text{the cartesian form.}$$



Example 4

If A = (– 1 , 4) , B = (5 , – 2) , find the coordinates of the two points C and D which divide $\overline{AB}$ into three equal parts in length.

Solution



$\because$ C divides $\overrightarrow{AB}$ internally by the ratio $\dfrac{m_2}{m_1} = \dfrac{1}{2}$

$$\therefore \vec{r} = \frac{2(-1, 4) + (5, -2)}{2+1} = (1, 2)$$

$\therefore C = (1, 2)$

, $\because$ D is the midpoint of $\overrightarrow{CB}$

$$\therefore \vec{r} = \frac{\vec{r_1}+\vec{r_2}}{2} = \frac{(1, 2) + (5, -2)}{2} = (3, 0)$$

$\therefore D = (3, 0)$

We can get the coordinates of D also regarding that it divides $\overrightarrow{AB}$ internally by the ratio $m_2 : m_1 = 2 : 1$

Example 5

ABCD is a parallelogram in which A = (5 , 2) , B = (0 , 3) , C = (– 2 , – 1)
Find the coordinates of the vertex D

Solution

Let $D = (x, y)$

$\because$ The two diagonals in the parallelogram bisect each other

$\therefore$ The midpoint of $\overline{AC}$ = the midpoint of $\overline{BD}$

$\therefore \dfrac{5-2}{2} = \dfrac{0+x}{2}$ $\qquad \therefore x = 3$

, $\dfrac{2-1}{2} = \dfrac{3+y}{2}$ $\qquad \therefore y = -2$ $\qquad \therefore D = (3, -2)$

Remarks

- To prove that the points A , B and C are collinear , then we prove that :
  either $\overrightarrow{AB} = k\ \overrightarrow{AC}$, $k \neq 0$ (using vectors)
  or the slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{AC}$ (using the slope).
  or AB = BC + AC (using the distance between two points where AB is the longest length).
- If C divides $\overline{AB}$ by the ratio $m_2 : m_1$ then the division is :

  1 internally if $\frac{m_2}{m_1}$ is positive. 2 externally if $\frac{m_2}{m_1}$ is negative.

Example 6

Prove that the points A = (1 , – 3) , B = (– 2 , – 9) , C = (5 , 5) are collinear , then find :

1 The ratio by which the point C divides $\overrightarrow{AB}$

2 The ratio by which the point A divides $\overrightarrow{BC}$

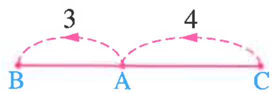
Solution

$\because \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (-2 , -9) - (1 , -3) = (-3 , -6) = -3\ (1 , 2)$

, $\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (5 , 5) - (1 , -3) = (4 , 8) = 4\ (1 , 2)$ $\therefore \overrightarrow{AC} = -\frac{4}{3}\ \overrightarrow{AB}$

$\therefore$ A , B , C are collinear , B , C are in different sides of the point A

$\frac{\| \overrightarrow{AC} \|}{\| \overrightarrow{AB} \|} = \frac{4}{3}$, and we deduce that :



1 C divides $\overrightarrow{AB}$ by the ratio 4 : 7 externally.

2 A divides $\overrightarrow{BC}$ by the ratio 3 : 4 internally.

Another Solution : (Using the slope)

$\because$ The slope of $\overleftrightarrow{AB} = \frac{-9+3}{-2-1} = 2$

, the slope of $\overleftrightarrow{AC} = \frac{5+3}{5-1} = 2$

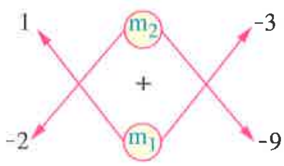
$\therefore$ A , B , C are collinear

1 Let C = (5 , 5) divide $\overrightarrow{AB}$ by the ratio $m_2 : m_1$

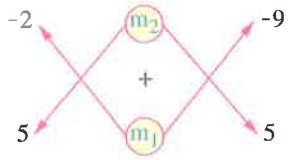
$\therefore \frac{m_1 - 2\ m_2}{m_1 + m_2} = 5$ $\therefore m_1 - 2\ m_2 = 5\ m_1 + 5\ m_2$

$\therefore 4\ m_1 = -7\ m_2$ $\therefore \frac{m_2}{m_1} = -\frac{4}{7}$ (negative)

$\therefore$ C divides $\overrightarrow{AB}$ by the ratio 4 : 7 externally



2 Let A = (1 , – 3) divide $\overrightarrow{BC}$ by the ratio $m_2 : m_1$



$\therefore \frac{-2\,m_1 + 5\,m_2}{m_1 + m_2} = 1$ $\quad\therefore -2\,m_1 + 5\,m_2 = m_1 + m_2$

$\therefore 3\,m_1 = 4\,m_2$ $\quad\therefore \frac{m_2}{m_1} = \frac{3}{4}$ (positive)

$\therefore$ A divides $\overrightarrow{BC}$ by the ratio 3 : 4 internally.

Third Solution : (Using the distance between two points)

$\because AB = \sqrt{(1+2)^2 + (-3+9)^2} = 3\sqrt{5}$ length unit.

, $BC = \sqrt{(-2-5)^2 + (-9-5)^2} = 7\sqrt{5}$ length unit.

, $CA = \sqrt{(1-5)^2 + (-3-5)^2} = 4\sqrt{5}$ length unit.

$\therefore BC = AB + CA$

$\therefore$ A , B , C are collinear , $C \notin \overline{AB}$, $\frac{AB}{AC} = \frac{3\sqrt{5}}{4\sqrt{5}} = \frac{3}{4}$

$\therefore$ C divides $\overrightarrow{AB}$ by the ratio 4 : 7 externally and A divides $\overrightarrow{BC}$ by the ratio 3 : 4 internally.

Example 7

Find the ratio by which $\overrightarrow{AB}$ is divided at the two points of intersection with the coordinate axes if A = (4 , – 3) , B = (– 3 , 5) , then find the coordinates of the points of division.

Solution

First : Let C (x , 0) be the point of intersection of $\overline{AB}$ with the x-axis

$\therefore 0 = \frac{m_2 \times 5 + m_1 \times (-3)}{m_2 + m_1}$ $\quad\therefore 0 = 5\,m_2 - 3\,m_1$

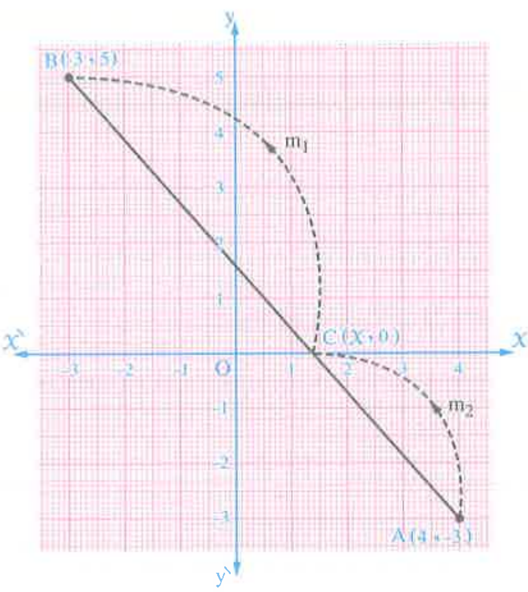
$\therefore 5\,m_2 = 3\,m_1$ $\quad\therefore \frac{m_2}{m_1} = \frac{3}{5}$

$\therefore \overline{AB}$ is divided at the point of its intersection with the x-axis by the ratio 3 : 5 internally.

$\therefore x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$= \frac{5 \times 4 + 3 \times -3}{3 + 5} = \frac{11}{8}$

$\therefore$ C (the point of division) = $(\frac{11}{8}, 0)$



Second : Let D $(0 , y)$ be the point of intersection of $\overline{AB}$ with the y-axis

$\therefore 0 = \frac{m_2 \times -3 + m_1 \times 4}{m_2 + m_1}$ $\therefore 0 = -3\, m_2 + 4\, m_1$

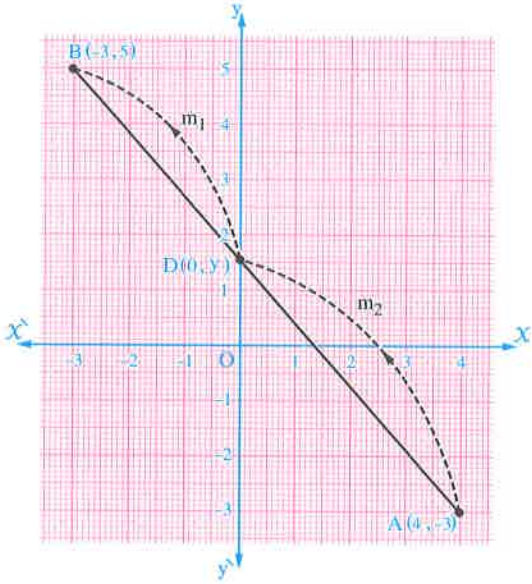
$\therefore 3\, m_2 = 4\, m_1$ $\therefore \frac{m_2}{m_1} = \frac{4}{3}$

$\therefore \overline{AB}$ is divided at the point of its intersection with the y-axis by the ratio 4 : 3 internally

$\therefore y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$

$= \frac{3 \times -3 + 4 \times 5}{3 + 4} = \frac{11}{7}$

$\therefore$ D (the point of division) $= \left(0 , \frac{11}{7}\right)$



TRY TO SOLVE

If A = (2 , 3) , B = (– 2 , 1) find the ratio by which $\overrightarrow{AB}$ is divided at its intersection point with the X-axis , **then find the coordinates of the point of division.**

Example 8

If A = (2 , 1) , B = (1 , 5) , C = (6 , – 3) are the vertices of a triangle , find the coordinates of the point of intersection of its medians.

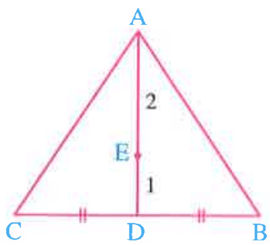
Solution

The point of intersection of the medians of the triangle divides each of them internally by the ratio 2 : 1 from the vertex.

Let D be the midpoint of $\overline{BC}$

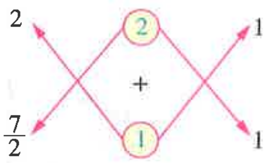
$\therefore D = \left(\frac{1+6}{2} , \frac{5-3}{2}\right) = \left(\frac{7}{2} , 1\right)$

, E (the point of intersection of the medians) divides $\overline{AD}$ internally by the ratio 2 : 1



$\therefore X = \frac{2 \times \frac{7}{2} + 1 \times 2}{2 + 1} = 3$

$, y = \frac{2 \times 1 + 1 \times 1}{2 + 1} = 1$



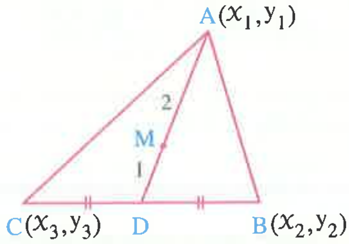
$\therefore E = (3 , 1)$

Unit 5

Remark

If ABC is a triangle whose vertices are $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ and the point M is the point of intersection of its medians, then

$$M = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$



The previous example can be solved as follows :

$$M = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{2+1+6}{3}, \frac{1+5-3}{3}\right) = (3, 1)$$

Notice the difference

If $C \in \overleftrightarrow{AB}$ and :

1. $\overrightarrow{AC} = 2\,\overrightarrow{CB}$, then C divides $\overline{AB}$ internally.
2. $\overrightarrow{AC} = -2\,\overrightarrow{CB}$, then C divides $\overline{AB}$ externally.
3. $AC = 2\,CB$, then C divides $\overline{AB}$ internally or externally.

Lesson TWO — Equation of the straight line

You studied in the previous years that :

- **The general form of the equation of the straight line is** $a x + b y + c = 0$

where a , b , c are real numbers , a and b are not equal to zero together. This equation is represented by a straight line.

For example :

The relations : $x + \sqrt{3}\, y = 6$, $y = 3$, $x - 4 = 0$

represent straight lines , but the relations : $y + \sqrt{x} = 4$, $x + \frac{1}{y} = 5$

do not represent straight lines.

The slope of the straight line

1 If the straight line L passes through the two points (x_1 , y_1) , (x_2 , y_2)

, then m (the slope of the straight line) $= \dfrac{\text{Difference between y coordinates}}{\text{Difference between } x \text{ coordinates}} = \dfrac{y_2 - y_1}{x_2 - x_1}$

For example :

The straight line which passes through the two points (1 , 3) , (4 , 2)

, its slope equals $\frac{2-3}{4-1} = -\frac{1}{3}$

2 If the equation of the straight line is in the form $a x + b y + c = 0$

, then the slope of the straight line $= \dfrac{-\text{The coefficient of } x}{\text{The coefficient of y}}$

For example :

The straight line whose equation is : $5 x + 2 y + 7 = 0$, its slope $= \frac{-5}{2}$

3 If the equation of the straight line is in the form $y = m\,x + c$, then its slope $= m$ and intercepts from y-axis a part of length = the absolute value of the number c , and it passes through the point $(0 , c)$

For example :

The straight line whose equation is : $y = 3\,x - 5$, its slope $= 3$ and intercepts from the negative part of y-axis 5 length units and it passes through the point $(0 , -5)$

4 If θ is the measure of the positive angle , which the straight line makes with the positive direction of x-axis , then the slope of the straight line $= \tan\theta$

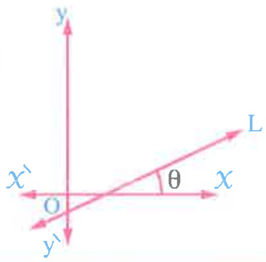
For example :

If the measure of the positive angle which the straight line makes with the positive direction of x-axis $= 45°$, then the slope of the straight line $= \tan 45° = 1$

So , we notice that the slope of the straight line changes as the measure of the angle θ changes as follows :

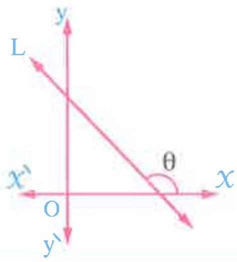
(1) acute

If the slope is positive



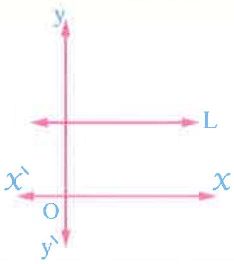
(2) obtuse

If the slope is negative



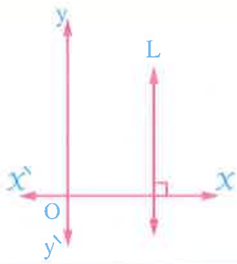
(3) zero

If the slope $= 0$



(4) right

If the slope is undefined



5 The slope of x-axis and the slope of any horizontal straight line (parallel to x-axis) are equal to zero.

6 The slope of y-axis and the slope of any vertical straight line (parallel to y-axis) are undefined.

- If the slope of $\overleftrightarrow{AB}$ = The slope of $\overleftrightarrow{BC}$, then the points A , B and C are collinear.

The relation between the two parallel straight lines and the perpendicular straight lines

If L_1 and L_2 are two straight lines of slopes m_1 and m_2 respectively , then :

1 $L_1 // L_2 \Leftrightarrow m_1 = m_2$

i.e. The two parallel straight lines have equal slopes and vice versa.

2 $L_1 \perp L_2 \Leftrightarrow m_1 \times m_2 = -1$

(unless one of them is parallel to one of the two coordinate axes)

i.e. The product of the slopes of any two perpendicular straight lines = – 1 and vice versa.

For example :

If the straight line L_1 passes through the two points (3 , 5) , (– 3 , – 1)

, then its slope $m_1 = \frac{5+1}{3+3} = 1$

and the straight line L_2 , whose equation is $3X - 3y + 5 = 0$ its slope $m_2 = \frac{-3}{-3} = 1$, where the straight line L_3 makes with the positive direction of X-axis a positive angle of measure 135°

$\therefore$ Its slope $m_3 = \tan 135° = -1$

$\because m_1 = m_2$ $\qquad \therefore L_1 // L_2$

$\because m_1 \times m_3 = 1 \times -1 = -1$ $\qquad \therefore L_1 \perp L_3$

$\because m_2 \times m_3 = 1 \times -1 = -1$ $\qquad \therefore L_2 \perp L_3$

- Any two different points in the plane one and only one straight line passes through them , and from any point outside this straight line we can draw another unique straight line parallel to it.
- To determine the equation of any straight line , then we should know two information about this straight line.

i.e. We know two given points on it or a point on it and its slope or something like this as will be shown in the following explaination.

Definition of the direction vector of any straight line :

Every non-zero vector can be represented by a directed line segment on a straight line is called a direction vector of this straight line.

In the opposite figure :

Each of $\overrightarrow{XY}$, $\overrightarrow{YZ}$, $\overrightarrow{ZX}$, $\overrightarrow{YX}$ is a direction vector to the straight line L

- If $\vec{u} \neq \vec{O}$, $\vec{u}$ // the straight line L , then $\vec{u}$ is a direction vector to the straight line L

- If $\vec{u} = (a , b)$ is a direction vector to a straight line , then $k\vec{u}$ is a direction vector to the same straight line , where $k \in \mathbb{R}^*$

For example :

If $\vec{u} = (3 , 4)$ is a direction vector of a straight line , then each of the vectors $(6 , 8)$, $(-3 , -4)$, $(1.5 , 2)$, $(15 , 20)$, ... is a direction vector of this straight line.

Notice :

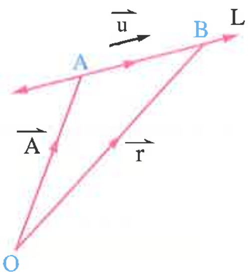
If $\vec{u} = (a , b)$ is a direction vector of a straight line , then the slope of this straight line $= \frac{b}{a}$ and vice versa.

For example :

If $(2 , -3)$ is a direction vector of a straight line , then the slope of this straight line $= \frac{-3}{2}$ and the straight line whose slope $= \frac{-4}{7}$, then the vector $\vec{u} = (7 , -4)$ is a direction vector to it.

The different forms of the equation of the straight line

- If L is a straight line passing through the point A , $\vec{u}$ is a direction vector to it and assuming that a point B lies on the straight line L , and $\vec{A}$ and $\vec{r}$ are the two vectors represented by the two directed line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.



- There exist a number $k \in \mathbb{R}^*$ such that $\overrightarrow{AB} = \vec{r} - \vec{A} = k\,\vec{u}$

$\therefore \vec{r} = \vec{A} + k\,\vec{u}$ « This form is called the vector equation of the straight line »

where k is a real number called a parameter and at each value of the parameter k we can find a point on the straight line.

- Assuming that $B = (x , y)$, $A = (x_1 , y_1)$, $\vec{u} = (a , b)$

$\therefore$ The equation of the straight line is $(x , y) = (x_1 , y_1) + k\,(a , b)$

$\therefore x = x_1 + k\,a \quad , \quad y = y_1 + k\,b$ « This form is called the parametric equations of the straight line »

$\therefore \frac{x - x_1}{a} = k \quad , \quad \frac{y - y_1}{b} = k$ eliminating k from the two equations :

$\therefore \frac{x - x_1}{a} = \frac{y - y_1}{b} \qquad \therefore \frac{y - y_1}{x - x_1} = \frac{b}{a}$

, $\because$ The slope $(m) = \frac{b}{a}$

$\therefore \frac{y - y_1}{x - x_1} = m$ « This form is called the cartesian equation of the straight line »

We can summarize the previous in the following :

The straight line L which passes through the point $A = (x_1 , y_1)$ and the vector $\overrightarrow{u} = (a , b)$ is a direction vector to it , then :

- The vector equation is $\boxed{\overrightarrow{r} = \overrightarrow{A} + k \overrightarrow{u}}$

i.e. $(x , y) = (x_1 , y_1) + k (a , b)$

- The two parametric equations are $\boxed{x = x_1 + k a \quad , \quad y = y_1 + k b}$
- The cartesian equation is $\boxed{\frac{y - y_1}{x - x_1} = m}$

Example 1

Find the different forms of the equation of the straight line which passes through the point $A = (3 , -2)$ and $\overrightarrow{u} = (-2 , 1)$ is a direction vector of it.

Solution

- The vector equation of the straight line is : $\overrightarrow{r} = \overrightarrow{A} + k \overrightarrow{u}$

$\therefore \overrightarrow{r} = (3 , -2) + k (-2 , 1)$

i.e. $(x , y) = (3 , -2) + k (-2 , 1)$

- The two parametric equations of the straight line are :

$x = 3 - 2k \quad , \quad y = -2 + k$

- The cartesian equation is :

$\frac{y + 2}{x - 3} = \frac{-1}{2}$

$\therefore x - 3 = -2y - 4$

$\therefore$ The general form is : $x + 2y + 1 = 0$

Remember that

$\because (-2 , 1)$ is a direction vector of the straight line.

$\therefore$ The slope of the straight line $= \frac{1}{-2}$

Another solution to get the cartesian equation

Eliminating k from the two parametric equations

$\therefore \frac{x - 3}{-2} = \frac{y + 2}{1}$

i.e. $x + 2y + 1 = 0$

Example 2

Find the different forms of the equation of the straight line which passes through the point $(-2, 1)$ and its slope $= -\frac{4}{5}$

Solution

$\because$ The slope $= -\frac{4}{5}$

$\therefore$ The vector $\vec{u} = (5, -4)$ is a direction vector of this straight line.

- The vector equation is : $\vec{r} = (-2, 1) + k(5, -4)$

  i.e. $(x, y) = (-2, 1) + k(5, -4)$

- The two parametric equations are : $x = -2 + 5k \quad , \quad y = 1 - 4k$

- The cartesian equation is : $\frac{y-1}{x+2} = -\frac{4}{5}$

  $\therefore$ The general form is : $4x + 5y + 3 = 0$

TRY TO SOLVE

Find the equation of the straight line which passes through the point $(1, 4)$ and makes with the positive direction of x-axis an angle of measure $\frac{3\pi}{4}$

The equation of the straight line given two points lying on it $P = (x_1, y_1)$, $N = (x_2, y_2)$

Let the vector $\vec{u} = \overrightarrow{PN} = \vec{N} - \vec{P}$ be a direction vector of the straight line

$\therefore$ The vector equation is $\vec{r} = \vec{P} + k(\vec{N} - \vec{P})$

, $\because$ the slope $(m) = \frac{y_2 - y_1}{x_2 - x_1}$

Substituting by the slope in the cartesian form

$\therefore$ The cartesian form is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

Example 3

Find the different forms of the equation of the straight line which passes through the two points : $P = (3, -1)$, $N = (-2, 4)$

Solution

$\vec{u} = \vec{N} - \vec{P} = (-2, 4) - (3, -1) = (-5, 5)$ is a direction vector of the required straight line.

$\therefore \vec{u}' = \frac{1}{5}\vec{u} = \frac{1}{5}(-5, 5) = (-1, 1)$ is a direction vector also of the required straight line.

$\therefore$ The slope of the straight line $= \frac{1}{-1} = -1$

$\therefore$ The vector equation is $\vec{r} = \vec{P} + k\,\vec{u}$ $\therefore$ $\vec{r} = (3\,,-1) + k\,(-1\,,1)$

i.e. $(x\,,y) = (3\,,-1) + k\,(-1\,,1)$

$\therefore$ The two parametric equations are : $x = 3 - k$, $y = -1 + k$

$\therefore$ The cartesian equation is : $\frac{y+1}{x-3} = -1$

i.e. $y + 1 = -x + 3$ $\therefore$ The general form is : $y + x - 2 = 0$

Remarks

1. The equation of the straight line which passes through the origin point O $(0\,,0)$ is :
   - The vector equation is $\vec{r} = k\,\vec{u}$, where $\vec{u}$ is the direction vector of the straight line.
   - The cartesian equation is $y = m\,x$, where m is the slope of the straight line.
2. The direction vector of the straight line passing through the origin point and the point $(x_1\,,y_1)$ is $\vec{u} = (x_1\,,y_1)$
3. The straight line which is parallel to x-axis and passes through the point $(x_1\,,y_1)$
   The vector $\vec{i} = (1\,,0)$ is a direction vector to it.
   its vector equation is : $\vec{r} = (x_1\,,y_1) + k\,(1\,,0)$
   its cartesian equation is : $\frac{y-y_1}{x-x_1} = \frac{0}{1}$ **i.e.** $y = y_1$
4. The straight line which is parallel to y-axis and passes through the point $(x_1\,,y_1)$
   The vector $\vec{j} = (0\,,1)$ is a direction vector to it.
   its vector equation is : $\vec{r} = (x_1\,,y_1) + k\,(0\,,1)$
   its cartesian equation is : $\frac{y-y_1}{x-x_1} = \frac{1}{0}$ (undefined) **i.e.** $x = x_1$
5. The equation of x-axis is $y = 0$ or $\vec{r} = k\,(1\,,0)$
6. The equation of y-axis is $x = 0$ or $\vec{r} = k\,(0\,,1)$

Example 4

Find the vector form and the cartesian form of the equation of the straight line which passes through the origin point and the point $P = (-3\,,5)$

Solution

$\because$ The straight line passes through the origin point

$\therefore$ The vector $\vec{u} = (-3\,,5)$ is a direction vector to this straight line.

, the slope of the straight line $= \frac{-5}{3}$

$\therefore$ The vector equation is : $\vec{r} = k\,\vec{u}$ $\therefore$ $\vec{r} = k\,(-3\,,5)$

, the cartesian equation is : $y = m\,x$

$\therefore$ $y = -\frac{5}{3}\,x$ $\therefore$ $3\,y + 5\,x = 0$

The perpendicular direction vector of a straight line

- If $\vec{u} = (a , b)$ is a direction vector of a straight line , then any one of the family of vectors which are in the form $k (b , -a)$ where $k \in \mathbb{R}^*$ is the perpendicular direction vector to the vector $\vec{u}$
- If $\vec{N} = (a , b)$ is a perpendicular to the straight line , then any one of the family of vectors which are in the form $k (b , -a)$ where $k \in \mathbb{R}^*$ is a direction vector to this straight line.

For example :

If $\vec{u} = (4 , 5)$ is a direction vector of a straight line , then the perpendicular direction vector to it is $(5 , -4)$, $(-5 , 4)$, $(-10 , 8)$, ...

Example 5

Find the different forms of the equation of the straight line L , which passes through the point P $(-3 , 2)$ and is perpendicular to the vector $\vec{N} = (1 , 4)$

Solution

$\because$ $\vec{N} = (1 , 4)$ is perpendicular to the straight line L

$\therefore$ $\vec{u} = (4 , -1)$ is a direction vector to the straight line L

$\therefore$ The vector equation is $\vec{r} = \vec{P} + k \vec{u}$

$\therefore$ $\vec{r} = (-3 , 2) + k (4 , -1)$ i.e. $(x , y) = (-3 , 2) + k (4 , -1)$

$\therefore$ The two parametric equations are : $x = -3 + 4k$, $y = 2 - k$

$\therefore$ The cartesian equation is : $\dfrac{y-2}{x+3} = \dfrac{-1}{4}$ i.e. $4y - 8 = -x - 3$

$\therefore$ The general form is : $x + 4y - 5 = 0$

Remark

If the general equation of the straight line is : $a x + b y + c = 0$, then :
- The vector $\vec{N} = (a , b) =$ (coefficient of x , coefficient of y) is the perpendicular direction vector to the straight line.
- The vector $\vec{u} = (b , -a)$ is the direction vector to this straight line.

For example :

With respect to the straight line whose equation is : $2x + 3y + 7 = 0$, we get the vector $\vec{N} = (2 , 3)$ is the perpendicular direction vector to it.

the vector $\vec{u} = (3 , -2)$ is the direction vector to the straight line.

Example 6

Find the vector form of the equation of the straight line : $3X - 2y + 12 = 0$

Solution

$\because$ The straight line : $3X - 2y + 12 = 0$

$\therefore$ The vector $\overrightarrow{N} = (3, -2)$ is the perpendicular direction vector of this straight line.

$\therefore$ The vector $\overrightarrow{u} = (2, 3)$ is the direction vector of this straight line.

To get the vector form of the equation of this straight line we should search for getting a point on the straight line , this can be carried out by taking X or y any value , then we get the corresponding value of y or X

taking $X = 0$ we find that : $-2y + 12 = 0$

$\therefore y = 6$ $\qquad$ $\therefore$ The straight line passes through the point $(0, 6)$

$\therefore$ Its vector equation is $\overrightarrow{r} = (0, 6) + k(2, 3)$

TRY TO SOLVE

Find the vector form and the cartesian form of the equation of the straight line L which passes through the point $(-4, 1)$ and the vector $(-3, 6)$ is a perpendicular vector to it.

The equation of the straight line given its slope and the intercepted part from y-axis

$\because$ The straight line has a slope m and intersects the y-axis at the point $(0, c)$

i.e. intercepts from y-axis a part of length = the absolute value of the number c , then by substituting in the cartesian form we find that $\frac{y - c}{X - 0} = m$

i.e. $y = mX + c$

The equation of the straight line given the two intercepted parts from the two coordinate axes

Let the straight line intersect X-axis at the point $(a, 0)$, and y-axis at the point $(0, b)$

$\therefore$ The slope of the straight line $m = \frac{b - 0}{0 - a} = \frac{-b}{a}$

Substituting in the cartesian form

$\therefore \frac{y - 0}{X - a} = \frac{-b}{a}$ $\qquad$ $\therefore ay = -bX + ab$

$\therefore bX + ay = ab$ «dividing by a b»

$\therefore \frac{X}{a} + \frac{y}{b} = 1$

Example 7

Find the general form of the equation of each of the following :

1 The straight line L_1 whose slope = 3 , and intercepts from the negative part of y-axis a part = 7 length units.

2 The straight line L_2 which intercepts from the positive part of x-axis 4 length units , and from the negative part of y-axis 3 length units.

Solution

1 The equation of the straight line L_1 is : $y = m x + c$ $\therefore y = 3x - 7$

2 The equation of the straight line L_2 is : $\frac{x}{a} + \frac{y}{b} = 1$ $\therefore \frac{x}{4} + \frac{y}{-3} = 1$

i.e. $3x - 4y - 12 = 0$

TRY TO SOLVE

Find the lengths of the two intercepted parts from the two coordinate axes by the straight line : $3x + 8y - 24 = 0$

Remarks

- The equation $ax + by + c = 0$, where a and b not equal to zero together is called the general form of the straight line.

  1 If $a = 0$, $b \neq 0$, then $by + c = 0$

  i.e. $y = \frac{-c}{b}$, which is the equation of a straight line parallel to x-axis and passes through the point $\left(0 , \frac{-c}{b}\right)$

  2 If $a \neq 0$, $b = 0$, then $ax + c = 0$

  i.e. $x = \frac{-c}{a}$, which is the equation of a straight line parallel to y-axis and passes through the point $\left(\frac{-c}{a} , 0\right)$

  3 If $c = 0$, then $ax + by = 0$, which is the equation of a straight line passing through the origin point.

- To find the point of intersection of the straight line with x-axis , put $y = 0$
- To find the point of intersection of the straight line with y-axis , put $x = 0$

Example 8

Find the measure of the positive angle which the straight line : $3\,X + 2\,y + 6 = 0$ makes with the positive direction of X-axis , then find the points of its intersection with the coordinate axes.

Solution

$\because$ The slope of the straight line $= \dfrac{-\text{The coefficient of } X}{\text{The coefficient of y}}$

$\therefore \tan\theta = \dfrac{-3}{2}$

$\because$ The slope is negative. $\therefore$ The angle is obtuse.

$\because$ The measure of the acute angle whose tangent $= \dfrac{3}{2}$ is $56^\circ\ 19'$

$\therefore \theta = 180^\circ - 56^\circ\ 19' = 123^\circ\ 41'$

To find the point of intersection with y-axis , put $X = 0$

$\therefore 3\,(0) + 2\,y + 6 = 0$ $\therefore y = -3$

$\therefore$ The point of intersection is $(0 , -3)$

To find the point of intersection with X-axis , put $y = 0$

$\therefore 3\,X + 2 \times 0 + 6 = 0$ $\therefore X = -2$

$\therefore$ The point of intersection is $(-2 , 0)$

Another solution to find the points of intersection with coordinate axes :

$\because 3\,X + 2\,y + 6 = 0$ $\therefore 3\,X + 2\,y = -6$

dividing by -6 $\therefore \dfrac{X}{-2} + \dfrac{y}{-3} = 1$

$\therefore$ The straight line cuts X-axis at the point $(-2 , 0)$ and cuts y-axis at the point $(0 , -3)$

Example 9

Prove that : The points $A = (4 , -3)$, $B = (-6 , 7)$ and $C = (5 , -4)$ are collinear.

Solution

$\because$ The slope of $\overleftrightarrow{AB} = \dfrac{7+3}{-6-4} = -1$

, the slope of $\overleftrightarrow{BC} = \dfrac{-4-7}{5+6} = -1$

$\therefore$ The points A , B and C are collinear.

Another solution :

The equation of $\overleftrightarrow{AC}$ is : $\frac{y+3}{x-4} = \frac{-4+3}{5-4} = -1$ i.e. $y + x - 1 = 0$

$\because$ The point B = (– 6 , 7) satisfies the equation

$\therefore B \in \overleftrightarrow{AC}$ $\therefore$ A , B and C are collinear.

Example 10

Find the general equation of each of the following straight lines :

1. The straight line L_1 which passes through the point (3 , – 1) , and its slope $= -\frac{3}{4}$
2. The straight line L_2 which passes through the point $\left(4 , \sqrt{3}\right)$, and makes with the positive direction of x-axis a positive angle of measure 120°
3. The straight line L_3 which passes through the point (– 2 , – 5) , and the vector $\vec{u} = (3 , 1)$ is the direction vector of it.
4. The straight line L_4 which passes through the point (4 , – 2) , and is perpendicular to the vector $\vec{N} = (-1 , 5)$
5. The straight line L_5 which passes through the point (– 3 , 7) , and is parallel to x-axis.
6. The straight line L_6 which passes through the two points (4 , – 2) , (5 , 3)
7. The straight line L_7 which passes through the point (1 , 2) , and is parallel to the straight line $2x + 3y - 6 = 0$
8. The straight line L_8 which passes through the point (2 , 3) , and is perpendicular to the straight line whose slope $= \frac{5}{2}$

Solution

1. The equation of the straight line L_1 is :

   $\frac{y+1}{x-3} = -\frac{3}{4}$ i.e. $3x + 4y - 5 = 0$

2. $\because$ The slope of the straight line $L_2 = \tan 120° = -\sqrt{3}$

   $\therefore$ The equation of the straight line L_2 is : $\frac{y-\sqrt{3}}{x-4} = -\sqrt{3}$ i.e. $y + \sqrt{3}\,x = 5\sqrt{3}$

3. $\because$ The slope of the straight line $L_3 = \frac{1}{3}$

   $\therefore$ The equation of the straight line L_3 is : $\frac{y+5}{x+2} = \frac{1}{3}$ i.e. $x - 3y - 13 = 0$

4 $\because$ The vector $\overrightarrow{u} = (5, 1)$ is a direction vector of the straight line L_4

$\therefore$ its slope $= \frac{1}{5}$

$\therefore$ The equation of the straight line L_4 is : $\frac{y+2}{x-4} = \frac{1}{5}$ **i.e.** $x - 5y - 14 = 0$

5 The equation of the straight line L_5 is $y = 7$ **i.e.** $y - 7 = 0$

6 $\because$ The slope of the straight line $L_6 = \frac{3+2}{5-4} = 5$

$\therefore$ The equation of the straight line L_6 is : $\frac{y+2}{x-4} = 5$ **i.e.** $5x - y - 22 = 0$

7 $\because$ The slope of the given straight line $= \frac{-2}{3}$

$\therefore$ The slope of the required straight line $= \frac{-2}{3}$

$\therefore$ Its equation is : $\frac{y-2}{x-1} = \frac{-2}{3}$ **i.e.** $3y + 2x - 8 = 0$

8 $\because$ The slope of the given straight line $= \frac{5}{2}$

$\therefore$ The slope of the required straight line $= \frac{-2}{5}$

$\therefore$ Its equation is : $\frac{y-3}{x-2} = -\frac{2}{5}$ **i.e.** $2x + 5y - 19 = 0$

Example 11

ABC is a triangle , its vertices are A = (– 1 , 5) , B = (4 , – 2) , C = (– 3 , 0)
Find the equation of the straight line passing through the vertex A perpendicular to $\overrightarrow{BC}$

Solution

$\because$ $\overrightarrow{BC} = \overrightarrow{C} - \overrightarrow{B} = (-3, 0) - (4, -2) = (-7, 2)$

$\because$ The vector $\overrightarrow{BC} = (-7, 2)$ is perpendicular to the straight line L

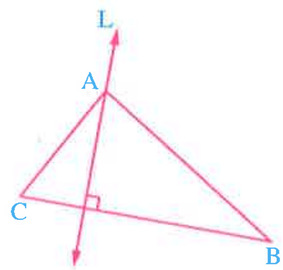
$\therefore$ The vector $\overrightarrow{u} = (2, 7)$ is a direction vector of the straight line L

$\therefore$ The slope of the straight line $L = \frac{7}{2}$

$\because$ The straight line passes through the point $A = (-1, 5)$

$\therefore$ The equation of the straight line is : $\frac{y-5}{x+1} = \frac{7}{2}$

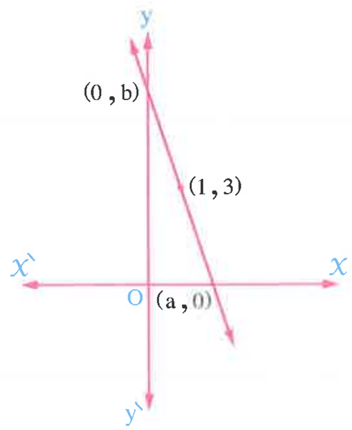
$\therefore$ $7x + 7 = 2y - 10$ **i.e.** $7x - 2y + 17 = 0$



Example 12

Find the equation of the straight line passing through the point (1 , 3) and its slope is negative and it makes with the coordinate axes a triangle of area 6 square units.

Solution



Assume that the straight line cuts X-axis at (a , 0) and y-axis at (0 , b)

$\therefore$ Its equation is on the form : $\frac{X}{a} + \frac{y}{b} = 1$

$\because$ (1 , 3) on the straight line

$\therefore \frac{1}{a} + \frac{3}{b} = 1$ $\qquad \therefore b + 3a = ab \qquad (1)$

, $\because$ the area of the triangle = 6 square units

$\therefore \frac{1}{2} ab = 6$ $\qquad \therefore ab = 12 \qquad (2)$

Substituting from (2) in (1) :

$\therefore b + 3a = 12$ $\qquad \therefore b = 12 - 3a$

Substituting in (2) : $\qquad \therefore a(12 - 3a) = 12$

$\therefore 12a - 3a^2 = 12$ $\qquad \therefore 3a^2 - 12a + 12 = 0$

$\therefore a^2 - 4a + 4 = 0$ $\qquad \therefore (a-2)^2 = 0$

$\therefore a = 2$ $\qquad$, then $b = 6$

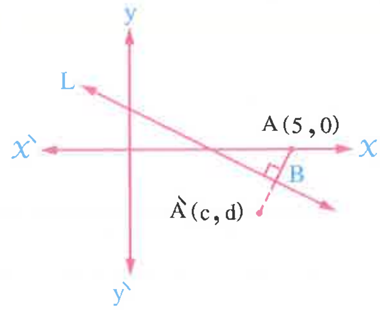
$\therefore$ The equation of the straight line is :

$\frac{X}{2} + \frac{y}{6} = 1$ $\qquad$ i.e. $3X + y = 6$

Example 13

Find the projection of the point A (5 , 0) on the straight line L : 2 X + y = 5 , then find the image of the point A by reflection in the same straight line.

Solution



Let B is the projection of the point A on L

, $\because$ the equation of the straight line L is $2X + y = 5 \quad (1)$

$\therefore$ The slope of L = – 2 $\qquad \therefore$ The slope of $\overleftrightarrow{AB} = \frac{1}{2}$

$\therefore$ The equation of $\overleftrightarrow{AB}$ is $\frac{y-0}{X-5} = \frac{1}{2}$

i.e. $X - 2y = 5 \qquad (2)$

By solving the two equations (1) and (2)

$\therefore X = 3 , y = -1$ $\qquad \therefore$ B (3 , – 1)

i.e. The projection of the point A on the line $2X + y = 5$ is the point B (3 , – 1)

• To find $\grave{A}$ (c , d) is the image of A (5 , 0) by reflection in line L

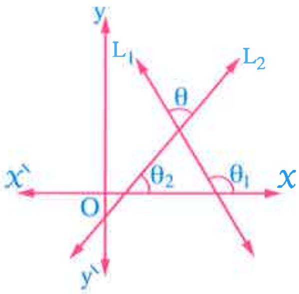
$\therefore$ B is midpoint of $\overline{A\grave{A}}$ $\qquad \therefore \left(\frac{5+c}{2} , \frac{0+d}{2}\right) = (3 , -1)$

$\therefore c = 1 , d = -2$ $\qquad \therefore \grave{A} = (1 , -2)$

Lesson Three — Measure of the angle between two straight lines

- Generally , if two straight lines intersect , then there will be two angles (each of them supplements the other) , they are either two right angles or one of them is an acute angle and the other is an obtuse angle.

- If θ is the measure of the included angle between the two straight lines L_1 and L_2 whose slopes are m_1 and m_2 , then



$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad \text{where } \theta \in \left[0 , \frac{\pi}{2}\right]$$

, $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

With noticing the following :

1. If the tangent is positive , then we obtain an acute angle.
2. If the tangent is zero , then the measure of the included angle is zero , then $m_1 = m_2$ and the two straight lines are parallel or coincident.
3. If the tangent is undefined , then the measure of the included angle is 90° , then $m_1 m_2 = -1$ and the two straight lines are orthogonal (perpendicular).
4. The measure of the obtuse angle = the measure of the supplementary angle of the acute angle.

Example 1

Find the measure of the acute angle between the two straight lines :

$L_1 : x - 2y + 5 = 0$, $L_2 : 2x + 4y - 7 = 0$

Solution

$\because m_1 = \frac{1}{2}$, $m_2 = \frac{-2}{4} = \frac{-1}{2}$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} + \frac{1}{2}}{1 + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}\right| = \frac{4}{3}$

$\therefore \theta \approx 53° \, 8'$

Remember that

The slope of the straight line : $a x + b y + c = 0$ equals $\frac{-a}{b}$

Example 2

Find the measure of the acute angle between the two straight lines :

$L_1 : \vec{r} = (2, 3) + k(4, 3)$, $L_2 : \vec{r} = (1, 6) + k'(7, -1)$

Solution

$\because m_1 = \frac{3}{4}$, $m_2 = \frac{-1}{7}$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{3}{4} + \frac{1}{7}}{1 + \left(\frac{3}{4}\right)\left(\frac{-1}{7}\right)}\right| = 1$

$\therefore \theta = 45°$

TRY TO SOLVE

Find the measure of the acute angle between the two straight lines :

$L_1 : x + 5y = 3$, $L_2 : \vec{r} = (2, 3) + k(4, 1)$

Example 3

If the measure of the angle between the two straight lines

$L_1 : x - 2y + 1 = 0$, $L_2 : x + ky + 2 = 0$ equals 45° , find the value of k

Solution

$\because m_1 = \frac{1}{2}$, $m_2 = \frac{-1}{k}$, $\theta = 45°$

$\therefore \tan 45° = \left|\frac{\frac{1}{2} + \frac{1}{k}}{1 - \frac{1}{2k}}\right|$ $\qquad \therefore 1 = \left|\frac{\frac{1}{2} + \frac{1}{k}}{1 - \frac{1}{2k}}\right|$

$\therefore \dfrac{\frac{1}{2} + \frac{1}{k}}{1 - \frac{1}{2k}} = \pm 1$ $\therefore$ either $\frac{1}{2} + \frac{1}{k} = 1 - \frac{1}{2k}$

$\therefore \frac{3}{2k} = \frac{1}{2}$ $\therefore k = 3$

or $\frac{1}{2} + \frac{1}{k} = \frac{1}{2k} - 1$

$\therefore \frac{1}{2k} = \frac{-3}{2}$ $\therefore k = -\frac{1}{3}$

Example 4

Find the measures of the angles of Δ ABC whose vertices are A = (6 , 5) , B = (6 , 1) and C = (3 , 1)

Solution

$\because$ The slope of $\overleftrightarrow{AB} = \frac{5-1}{6-6} = \frac{4}{0}$ (undefined)

$\because \overleftrightarrow{AB}$ // y-axis (1)

$\because$ The slope of $\overleftrightarrow{BC} = \frac{1-1}{6-3} = \frac{0}{3}$ = zero

$\therefore \overleftrightarrow{BC}$ // X-axis (2)

$\because$ The slope of $\overleftrightarrow{AC} = \frac{5-1}{6-3} = \frac{4}{3}$ (3)

From (1) and (2) : $\therefore m(\angle B) = 90^\circ$

$\therefore \angle$ A and $\angle$ C are acute angles

From (2) and (3) : $\therefore \tan C = \left| \dfrac{\frac{4}{3} - \text{zero}}{1 + \text{zero} \times \frac{4}{3}} \right| = \frac{4}{3}$

$\therefore m(\angle C) \approx 53^\circ\ 8'$

$\therefore m(\angle A) = 180^\circ - (53^\circ\ 8' + 90^\circ) = 36^\circ\ 52'$

Remark

To determine the type of the triangle ABC according to the measures of its angles (where AC represents the length of the greatest side in the triangle) :

1. If $(AC)^2 > (AB)^2 + (BC)^2$, then the triangle is an obtuse-angled triangle at B
2. If $(AC)^2 = (AB)^2 + (BC)^2$, then the triangle is a right-angled triangle at B
3. If $(AC)^2 < (AB)^2 + (BC)^2$, then the triangle is an acute-angled triangle.

Example 5

Find the measures of the angles of the triangle whose vertices are A = (4 , 3) , B = (– 1 , 1) and C = (– 6 , 4) , then find its area.

Solution

$\because$ AB $= \sqrt{(4+1)^2 + (3-1)^2} = \sqrt{29}$ length units.

, BC $= \sqrt{(-1+6)^2 + (1-4)^2} = \sqrt{34}$ length units.

, AC $= \sqrt{(4+6)^2 + (3-4)^2} = \sqrt{101}$ length units.

$\therefore (AC)^2 > (AB)^2 + (BC)^2$

$\therefore \Delta$ ABC is an obtuse-angled triangle at B

$\therefore \angle$ A and $\angle$ C are acute angles.

$\because$ The slope of $\overleftrightarrow{AB} = \frac{3-1}{4+1} = \frac{2}{5}$, the slope of $\overleftrightarrow{BC} = \frac{1-4}{-1+6} = -\frac{3}{5}$

and the slope of $\overleftrightarrow{AC} = \frac{3-4}{4+6} = \frac{-1}{10}$

$\therefore \tan A = \left| \frac{\frac{2}{5} + \frac{1}{10}}{1 - \frac{2}{50}} \right| = \frac{25}{48}$ $\qquad \therefore m(\angle A) \approx 28°$

, $\tan C = \left| \frac{\frac{-3}{5} + \frac{1}{10}}{1 + \frac{3}{50}} \right| = \frac{25}{53}$ $\qquad \therefore m(\angle C) \approx 25°$

$\therefore m(\angle B) = 180° - (28° + 25°) = 127°$

, the area of the triangle $= \frac{1}{2} \times$ the product of two side lengths

$\times$ sine of the included angle between them

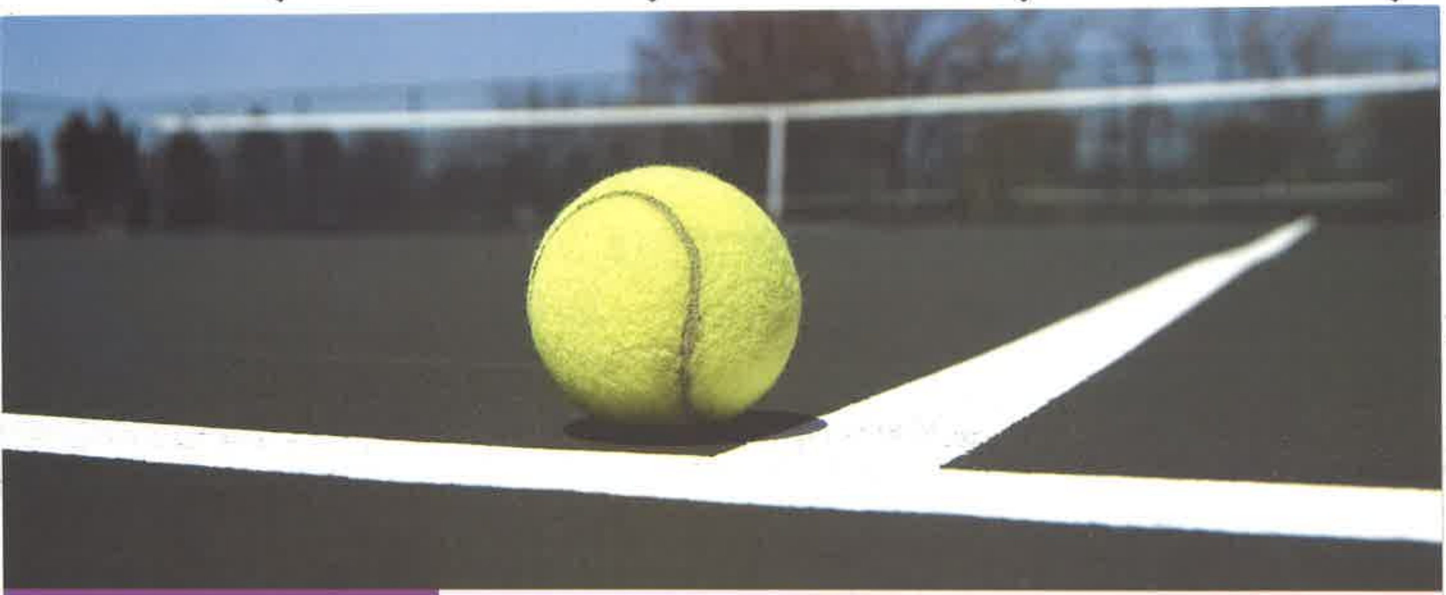
$= \frac{1}{2} \times AB \times AC \times \sin A = \frac{1}{2} \times \sqrt{29} \times \sqrt{101} \times \sin 28°$

≈ 12.7 square units.

TRY TO SOLVE

Find the measures of the angles of the triangle whose vertices are :

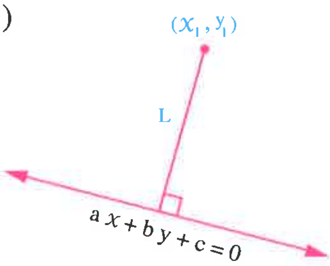
A = (2 , 3) , B = (– 1 , 3) , C = (2 , 5)



Lesson Four

The length of the perpendicular from a point to a straight line

- The length of the perpendicular (L) drawn from the point (X_1 , y_1) to the straight line whose equation is : $a X + b y + c = 0$ is determined by the relation :



$$\text{The length of the perpendicular (L)} = \frac{|a X_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

Important Remarks

1. If the length of the perpendicular drawn from the point (X_1 , y_1) to the straight line $a X + b y + c = 0$ equals zero , then the point lies on the straight line.
2. The length of the perpendicular drawn from the origin point $(0 , 0)$ to the straight line : $a X + b y + c = 0$ equals $\frac{|c|}{\sqrt{a^2 + b^2}}$
3. The length of the perpendicular drawn from the point (X_1 , y_1) to X-axis $= |y_1|$
4. The length of the perpendicular drawn from the point (X_1 , y_1) to y-axis $= |X_1|$
5. If (X_1 , y_1) and (X_2 , y_2) are two points in the Cartesian plane which contains the straight line : $a X + b y + c = 0$ and the two expressions $a X_1 + b y_1 + c$ and $a X_2 + b y_2 + c$ have the same sign , then the two points (X_1 , y_1) and (X_2 , y_2) are on the same side of the straight line , and if they have different signs , then the two points are in two different sides of the straight line.

Unit 5

Example 1

Find the length of the perpendicular from the point (3 , 5) to the straight line : $\vec{r} = (-1 , 2) + k\,(4 , -3)$

Solution

$\because$ The straight line $\vec{r} = (-1 , 2) + k\,(4 , -3)$ passes through the point $(-1 , 2)$ and its slope $= \frac{-3}{4}$

$\therefore$ The cartesian form is : $\frac{y-2}{x+1} = \frac{-3}{4}$ $\qquad \therefore 4y - 8 = -3x - 3$

$\therefore$ The general form is : $3x + 4y - 5 = 0$

$\therefore$ The length of the perpendicular $= \frac{|3\,(3) + 4\,(5) - 5|}{\sqrt{(3)^2 + (4)^2}} = 4.8$ length units.

TRY TO SOLVE

Find the length of the perpendicular from the point $(-2 , 3)$ to the straight line : $\vec{r} = (1 , 3) + k\,(4 , 3)$

Example 2

Find the length of the perpendicular drawn from the point A = (2 , 4) to the straight line passing through the point B = (– 2 , 0) and its slope = $\frac{5}{6}$

Solution

$\because$ The equation of the straight line passing through the point $B = (-2 , 0)$ and its slope $= \frac{5}{6}$ is $\frac{y-0}{x+2} = \frac{5}{6}$ $\qquad$ i.e. $5x - 6y + 10 = 0$

$\therefore$ The length of the perpendicular drawn from A to the straight line

$= \frac{|5 \times 2 - 6 \times 4 + 10|}{\sqrt{25 + 36}} = \frac{4}{\sqrt{61}}$ length units.

Example 3

If the length of the perpendicular drawn from the point (7 , c) to the straight line : $6x - 8y + 17 = 0$ equals 3.5 length units , find the value of c

Solution

$\because 3.5 = \frac{|6 \times 7 - 8 \times c + 17|}{\sqrt{36 + 64}}$ $\qquad \therefore 3.5 = \frac{|59 - 8c|}{10}$

$\therefore 35 = |59 - 8c|$ $\qquad \therefore 59 - 8c = 35$ or $59 - 8c = -35$

$\therefore -8c = -24$ or $-8c = -94$ $\qquad \therefore c = 3$ or $c = \frac{47}{4}$

Example 4

The length of the perpendicular from the point (2 , 5) to a straight line equals 3 units and the vector (3 , 4) is the direction vector of it. Find the equation of this straight line.

Solution

$\because$ The vector (3 , 4) is the direction vector of the straight line.

$\therefore$ The vector (4 , – 3) is perpendicular vector to the straight line.

$\therefore$ The equation of the straight line is : $4x - 3y + c = 0$

, $\because$ the length of the perpendicular on it from the point (2 , 5) = 3 length units.

$\therefore \frac{|4 \times 2 - 3 \times 5 + c|}{\sqrt{16+9}} = 3$ $\qquad \therefore |8 - 15 + c| = 3 \times 5$

$\therefore |c - 7| = 15$ $\qquad \therefore c - 7 = \pm 15$

$\therefore c = 7 + 15 = 22$ or $c = 7 - 15 = -8$

$\therefore$ The equation of the straight line is : $4x - 3y + 22 = 0$ or $4x - 3y - 8 = 0$

Example 5

ABC is a triangle whose vertices are A = (1 , 5) , B = (5 , – 3) and C = (1 , 0) Find its area.

Solution

Let $\overline{BC}$ (one of the sides of the triangle) be the base of the triangle , then we find the height which is the length of the perpendicular drawn from A to the straight line $\overleftrightarrow{BC}$ and we find also the length of $\overline{BC}$, then we calculate the area of the triangle as follows :

$\because BC = \sqrt{(1-5)^2 + (0+3)^2} = \sqrt{16+9} = 5$ length units.

, the equation of $\overleftrightarrow{BC}$ is : $\frac{y+3}{x-5} = \frac{0+3}{1-5} = \frac{3}{-4}$ i.e. $3x + 4y - 3 = 0$

$\therefore$ The length of the perpendicular from A to $\overleftrightarrow{BC}$

$= \frac{|3 \times 1 + 4 \times 5 - 3|}{\sqrt{9+16}} = \frac{|3 + 20 - 3|}{5} = \frac{20}{5} = 4$ length units.

$\therefore$ The area of the triangle ABC $= \frac{1}{2} \times 5 \times 4 = 10$ square units.

TRY TO SOLVE

If the points A = (– 3 , 0) , B = (3 , 2) , C = (– 1 , 5) represent vertices of a triangle , **find :**

1. The length of $\overline{BC}$
2. The equation of the straight line $\overleftrightarrow{BC}$
3. The length of the perpendicular from A to $\overleftrightarrow{BC}$
4. The area of Δ ABC

Example 6

Find the area of the circle whose centre is M (1 , 2) and the straight line whose equation L : $6x + 8y - 2 = 0$ is a tangent to it. ($\pi = 3.14$)

Solution

The length of the perpendicular drawn from the centre M (1 , 2) to the tangent L

$$= \frac{|6 \times 1 + 8 \times 2 - 2|}{\sqrt{6^2 + 8^2}} = \frac{20}{10} = 2 \text{ length units.}$$

$\because$ The radius length = the length of the perpendicular drawn from the centre to the tangent L

$\therefore$ r = 2 length units.

$\therefore$ The area of the circle = $\pi r^2 = 3.14 \times 4 = 12.56$ sqaure units.

Example 7

Prove that the two points : A = (3 , 1) and B = (– 3 , 2) lie on two different sides of the straight line L : $3x - 4y + 6 = 0$ and at equal distances from it.

Solution

$\because$ The length of the perpendicular from A to the straight line L

$$= \frac{|3 \times 3 - 4 \times 1 + 6|}{\sqrt{9 + 16}} = \frac{|11|}{5} = \frac{11}{5} = 2.2 \text{ length units.}$$

, the length of the perpendicular from B to the straight line L

$$= \frac{|3 \times (-3) - 4 \times 2 + 6|}{\sqrt{9 + 16}} = \frac{|-11|}{5} = \frac{11}{5} = 2.2 \text{ length units.}$$

$\therefore$ A and B are equidistant from the straight line L

$\because$ The expression $3x - 4y + 6$ has two different signs as substituting by the coordinates of each of A and B

$\therefore$ The two points A and B lie on different sides of the straight line L

Example 8

Prove that the two straight lines L_1 and L_2 are parallel and find the distance between them in each of the following :

1 $L_1 : x - 2y + 11 = 0$, $L_2 : 2x - 4y + 7 = 0$

2 $L_1 : \vec{r} = (2 , -5) + k (3 , -4)$, $L_2 : \vec{r} = (1 , 4) + \grave{k} (-6 , 8)$

Solution

1 $\because$ The slope of $L_1 = \frac{-1}{-2} = \frac{1}{2}$ and the slope of $L_2 = \frac{-2}{-4} = \frac{1}{2}$

$\therefore$ The two slopes are equal. $\therefore$ The two straight lines are parallel.

- Putting $X = 1$ (for example) in the equation of L_1

$\therefore y = 6$ $\qquad \therefore (1 , 6) \in L_1$

$\therefore$ The distance between the two straight lines

= the length of the perpendicular from the

point $(1 , 6)$ to $L_2 = \frac{|2 \times 1 - 4 \times 6 + 7|}{\sqrt{4 + 16}}$

$= \frac{|-15|}{\sqrt{20}} = \frac{15}{2\sqrt{5}} = \frac{3\sqrt{5}}{2}$ length units.

Remark

To find the distance between L_1 and L_2 , determine a point lying on one of them , then find the length of the perpendicular drawn from it to the other straight line.

2 The vector $\vec{u} = (3 , -4)$ is the direction vector of the straight line L_1

, the vector $\vec{u}' = (-6 , 8)$ is the direction vector of the straight line L_2

, $\because \vec{u}' = (-6 , 8) = -2 (3 , -4) = -2 \vec{u}$

$\therefore L_1 // L_2$ $\qquad$, $\because$ the point $N = (2 , -5) \in L_1$

, $\because L_2 : X = 1 - 6 k' , y = 4 + 8 k'$ $\qquad \therefore L_2 : \frac{X - 1}{-6} = \frac{y - 4}{8}$

$\therefore L_2 : 4 X + 3 y - 16 = 0$

$\therefore$ The distance between the two straight lines = the length of the perpendicular from

the point $(2 , -5)$ to $L_2 = \frac{|4 \times 2 + 3 \times (-5) - 16|}{\sqrt{16 + 9}} = 4.6$ length units.

Example 9

Prove that the point (4 , 6) lies on one of the two angle bisectors of the angle between the two straight lines : $L : 9 X - 13 y - 8 = 0$, $L' : X = 5 + 3 k$, $y = k + 3$

Solution

The point lies on one of the two angle bisector of the angle between the two straight lines L and L' if it on equal distance from the two straight lines.

The distance of the point $(4 , 6)$ from the straight line $L = \frac{|9 \times 4 - 13 \times 6 - 8|}{\sqrt{81 + 169}}$

$= \frac{|36 - 78 - 8|}{\sqrt{250}} = \frac{|-50|}{5\sqrt{10}} = \sqrt{10}$ length unit. $\qquad$ (1)

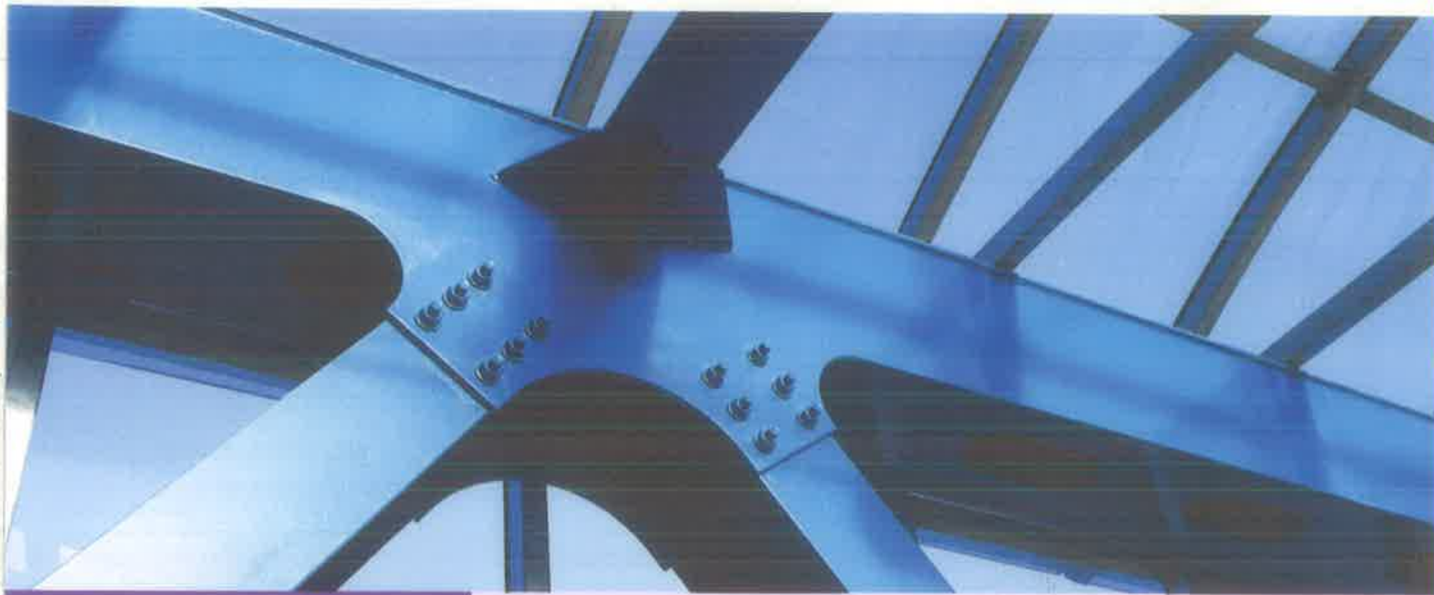
, the equation of L' is : $\frac{X - 5}{3} = \frac{y - 3}{1}$ $\qquad$ i.e. $X - 3 y + 4 = 0$

$\therefore$ The distance of the point $(4 , 6)$ from the straight line $L' = \frac{|4 - 3 \times 6 + 4|}{\sqrt{1 + 9}}$

$= \frac{|-10|}{\sqrt{10}} = \sqrt{10}$ length unit $\qquad$ (2)

from (1) and (2) :

$\therefore$ The point $(4 , 6)$ lies on one of the two angle bisector of the angle between the two straight lines L and L'



Lesson Five — General equation of the straight line passing through the point of intersection of two lines

If the two straight lines $L_1 : a_1 X + b_1 y + c_1 = 0$ and $L_2 : a_2 X + b_2 y + c_2 = 0$ intersect at a point , then the general equation of all straight lines which pass through the point of their intersection is :

$m (a_1 X + b_1 y + c_1) + \ell (a_2 X + b_2 y + c_2) = 0 \qquad (1)$

where $m \in \mathbb{R}$, $\ell \in \mathbb{R}$

- When $m = 0$, we get the equation of L_2
- When $\ell = 0$, we get the equation of L_1
- When $m \neq 0$, $\ell \neq 0$, we get the general equation of any line passing through the point of intersection of L_1 and L_2 other than L_1 and L_2
  , in this case equation (1) can be written as :
  $a_1 X + b_1 y + c_1 + k (a_2 X + b_2 y + c_2) = 0$, where k is a non-zero constant.

Example 1

Find the equation of the straight line which passes through the point of intersection of the two straight lines :

$2 X + 3 y = 18$ and $5 X - 2 y - 7 = 0$ and passes through the point (5 , 3)

Solution

The general equation of the straight line which passes through the point of intersection of the two given straight lines except them is : $2 X + 3 y - 18 + k (5 X - 2 y - 7) = 0 \qquad (1)$

$\because$ The point (5 , 3) lies on this straight line.

$\therefore$ It satisfies its equation.

$\therefore 10 + 9 - 18 + k (25 - 6 - 7) = 0$

$\therefore 1 + 12\,k = 0 \qquad \therefore k = \frac{-1}{12}$

Substituting in (1) :

$\therefore$ The equation of the required straight line is : $2\,x + 3\,y - 18 + \frac{-1}{12}(5\,x - 2\,y - 7) = 0$

«multiplying by 12»

$\therefore 24\,x + 36\,y - 216 - 5\,x + 2\,y + 7 = 0$

i.e. $19\,x + 38\,y - 209 = 0$ i.e. $x + 2\,y - 11 = 0$

Another Solution :

We find the intersection point of the two straight lines : $2\,x + 3\,y = 18$ and $5\,x - 2\,y = 7$ by solving the two equations algebraically that by multiplying the first equation by 2 and the second equation by 3

$\therefore$ The two equations will be : $4\,x + 6\,y = 36$ and $15\,x - 6\,y = 21$

Then by adding $\therefore 19\,x = 57$

$\therefore x = 3$ and substituting in any of the two equations.

$\therefore y = 4$ $\qquad \therefore$ The point of intersection is $(3 , 4)$

Then we find the equation of the straight line passing through the two points $(3 , 4)$ and $(5 , 3)$ as we studied before.

TRY TO SOLVE

Find the equation of the straight line which passes through the point of intersection of the two straight lines :

$2\,x + 3\,y = 9$, $4\,x + 5\,y = 15$ and passes through the point $(5 , -4)$

Example 2

Find the equation of the straight line passing through the point of intersection of the two straight lines : $3\,x + 2\,y = 10$ and $5\,x - 3\,y - 4 = 0$ and it is perpendicular to the straight line : $2\,x + 7\,y - 4 = 0$

Solution

The general equation of the straight line passing through the point of intersection of the two straight lines except them is :

$3\,x + 2\,y - 10 + k\,(5\,x - 3\,y - 4) = 0 \qquad (1)$

$\because$ The slope of the straight line : $2\,x + 7\,y - 4 = 0$ is $\frac{-2}{7}$

$\therefore$ The slope of the required straight line is $\frac{7}{2}$

And from (1) : $\therefore 3\,x + 2\,y - 10 + 5\,k\,x - 3\,k\,y - 4\,k = 0$

$\therefore x(3+5k)+y(2-3k)-10-4k=0$

$\therefore$ Its slope $= -\dfrac{3+5k}{2-3k}$ $\qquad \therefore \dfrac{7}{2} = -\dfrac{3+5k}{2-3k}$

$\therefore 6+10k=-14+21k$, $\qquad \therefore k=\dfrac{20}{11}$

Substituting in (1) :

$\therefore$ The equation of the required straight line is : $3x+2y-10+\dfrac{20}{11}(5x-3y-4)=0$

i.e. $33x+22y-110+100x-60y-80=0$

i.e. $133x-38y-190=0$ $\qquad$ **i.e.** $7x-2y-10=0$

Example 3

Prove that the two straight lines :
$4x-3y+7=0$, $\vec{r}=(2,5)+k(-4,3)$ are intersecting orthogonally ,
then find their point of intersection.

Solution

$\because m_1=\dfrac{-4}{-3}=\dfrac{4}{3}$, $m_2=-\dfrac{3}{4}$

$\therefore m_1\times m_2=\dfrac{4}{3}\times -\dfrac{3}{4}=-1$

$\therefore$ The two straight lines are intersecting orthogonally.

- To find the point of intersection of the two straight lines , we find the general equation of the second straight line at first.

  $\because$ The second straight line passes through the point (2 , 5) and its slope $=-\dfrac{3}{4}$

  $\therefore$ The cartesian equation is : $\dfrac{y-5}{x-2}=\dfrac{-3}{4}$

  $\therefore$ The general equation is : $3x+4y-26=0$

- Solving the two equations simultaneously :

  $4x-3y+7=0$ $\qquad$ (1)

  , $3x+4y-26=0$ $\qquad$ (2)

  Multiplying the equation (1) × 4 :

  $\therefore 16x-12y+28=0$ $\qquad$ (3)

  Multiplying the equation (2) × 3 :

  $\therefore 9x+12y-78=0$ $\qquad$ (4)

  Adding (3) and (4) :

  $\therefore 25x-50=0$ $\qquad \therefore x=2$

  Substituting in (1) : $\qquad \therefore y=5$

  $\therefore$ The point of intersection is (2 , 5)

Mathematics

By a group of supervisors

Exercises

SECOND TERM

1 SEC.

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First

Algebra and Trigonometry

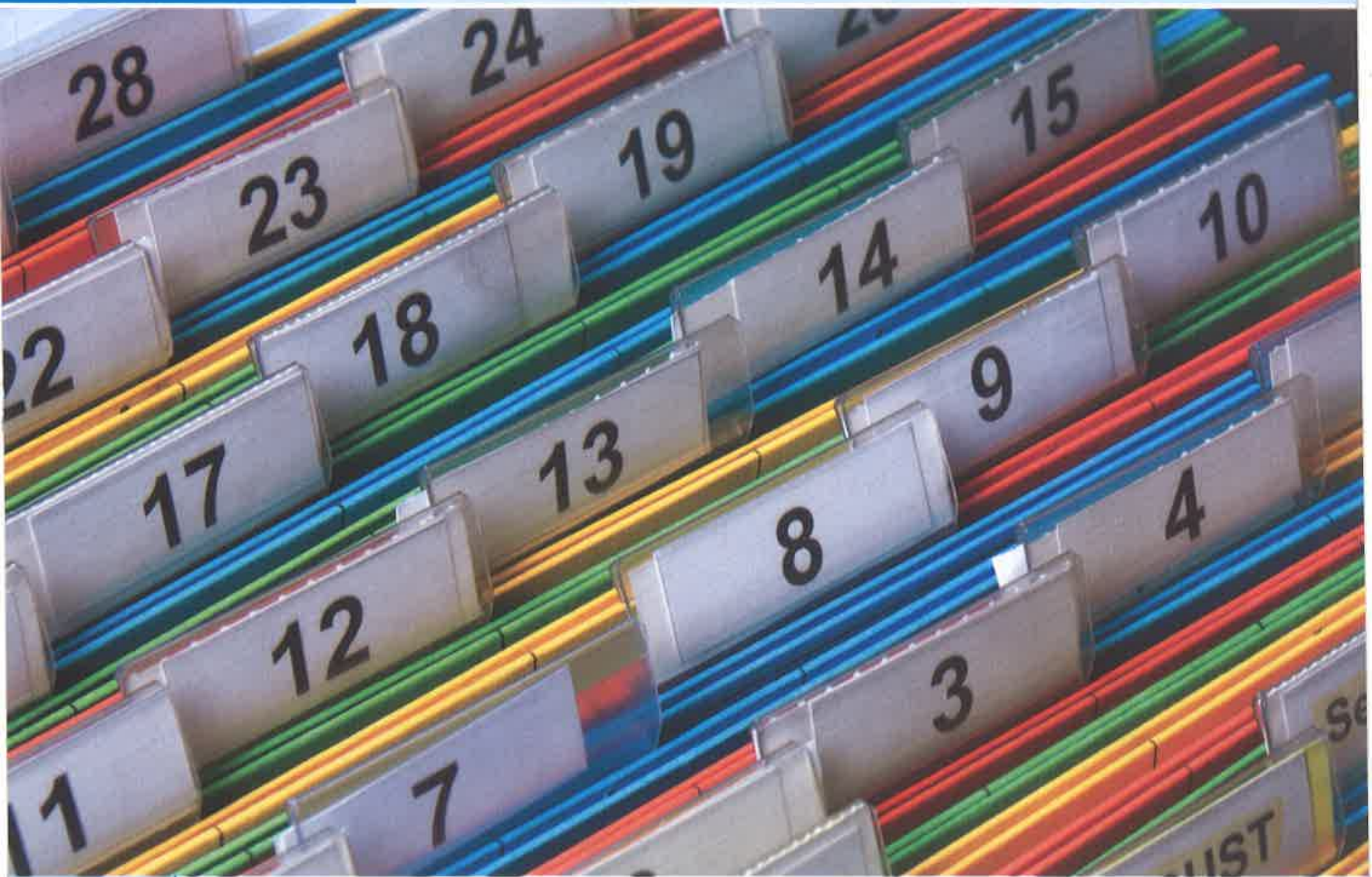
Unit One Matrices.

Unit Two Linear programming.

Unit Three Trigonometry.

Unit 1

MATRICES.



Exercise One	:	Organizing data in matrices.
Exercise Two	:	Adding and subtracting matrices.
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Exercise Four	:	Determinants.
Exercise Five	:	Multiplicative inverse of a matrix.

Exercise One

Organizing data in matrices

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) The matrix $(3 \quad 2 \quad 1)$ of order ....

(a) 2×1 (b) 1×3 (c) 3×2 (d) 3×1

(2) If : $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \\ -1 & 7 \end{pmatrix}$, then $a_{12} + a_{32} =$ ....

(a) 8 (b) 12 (c) zero (d) 10

(3) If : $A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 7 \\ 8 & 9 \end{pmatrix}$, then $a_{21} + c_{12} =$ ....

(a) 5 (b) 4 (c) -5 (d) 3

(4) If A is a matrix of order 2×3 , then the number of elements in matrix A is ....

(a) 4 (b) 9 (c) 6 (d) 5

(5) If B is a matrix of order 3×1 , then B^t is a matrix of order ....

(a) 3×1 (b) 3×3 (c) 1×1 (d) 1×3

(6) If O is a zero matrix of order 2×2 , then the number of its elements = ...

(a) zero (b) $\varnothing$ (c) 2 (d) 4

(7) If A is a matrix of order 3×4 , then the row contains .... elements.

(a) 3 (b) 4 (c) 7 (d) 12

(8) If A is a matrix of order 3×2 , then the matrix 2 A is of order ....

(a) 6×4 (b) 3×4 (c) 6×2 (d) 3×2

(9) If : $A=\begin{pmatrix} 3 & -2 & 7 \\ 5 & -4 & 2 \end{pmatrix}$, $B=A^t$, then $a_{13}+b_{31}=$ ....

(a) 4 (b) 9 (c) 14 (d) 10

(10) If : $A=\begin{pmatrix} 2 & 3 & -1 \\ 4 & -7 & 6 \end{pmatrix}$, then $2A^t=$ ....

(a) $\begin{pmatrix} 4 & 6 & -2 \\ 8 & -14 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 4 \\ 6 & -7 \\ 2 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 8 \\ 6 & -14 \\ -2 & 12 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 8 \\ 2 & -14 \\ -1 & 11 \end{pmatrix}$

(11) If $\begin{pmatrix} 1 & x & 2 \\ -1 & 6 & 5 \end{pmatrix}^t=\begin{pmatrix} 1 & -1 \\ -3 & 6 \\ 2 & y \end{pmatrix}$, then : $xy=$ ....

(a) -15 (b) -2 (c) 2 (d) 15

(12) If $\begin{pmatrix} 3 & 5 \\ x & -2 \end{pmatrix}=\begin{pmatrix} 3 & 5 \\ 7 & y+1 \end{pmatrix}$, then $x+y=$ ....

(a) 7 (b) -3 (c) 4 (d) 10

(13) If $\begin{pmatrix} 4 & x \\ -1 & y \end{pmatrix}=\begin{pmatrix} 4 & z \\ 0 & 3 \end{pmatrix}^t$, then $\begin{pmatrix} y & x \\ 0 & z \end{pmatrix}$ is a .... matrix.

(a) unit (b) zero

(c) diagonal (d) skew symmetric

(14) If $A=\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 2d & -1 \\ 3e & 4 \end{pmatrix}$, $A=B^t$, then : $d+e=$ ....

(a) zero (b) 12 (c) -2 (d) -1

(15) If $\begin{pmatrix} 12 & -2x \\ 16 & -8 \end{pmatrix}=-4\begin{pmatrix} -3 & 4 \\ 2y-x & 2 \end{pmatrix}$, then : $\sqrt{xy}=$ ....

(a) 1 (b) 2 (c) 3 (d) 4

(16) If the matrix : $A=\begin{pmatrix} 1 & 2x-4 \\ -2 & 3 \end{pmatrix}$ is symmetric , then $x=$ ....

(a) 1 (b) 3 (c) 2 (d) -2

(17) If $A = \begin{pmatrix} 1 & 1 & x-1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$ is a symmetric matrix , then : $x =$ ....

(a) -1 (b) zero (c) 4 (d) 6

(18) If $\begin{pmatrix} x^2 & 3 \\ 1 & x+4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, then $x =$ ....

(a) ± 2 (b) 2 (c) -2 (d) zero

(19) If $\begin{pmatrix} x & y \\ z & \ell \end{pmatrix} = 3\,I$ where I is a unit matrix , then $x + y + z + \ell =$ ....

(a) 2 (b) 3 (c) 6 (d) 12

(20) If the matrix : $A = \begin{pmatrix} 4 & m & 2 \\ y-x & 5 & 7 \\ y & x-1 & 1 \end{pmatrix}$ is symmetric , then m = ...

(a) 4 (b) -6 (c) 10 (d) -8

(21) If $A = \begin{pmatrix} 0 & y-5 & 3x \\ x & 0 & 2z \\ y-7 & -4 & 0 \end{pmatrix}$ is a skew symmetric , then $x + y + z =$ ....

(a) 3 (b) 2 (c) -2 (d) 7

(22) If A is a matrix of order 2×2 where $a_{ij} = j - 2i$, then A = ...

(a) $\begin{pmatrix} -1 & 0 \\ -3 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -3 \\ 0 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(23) If A is a matrix and $a_{xy} = x\,y$ for each $x \in \{1 , 2\}$, $y \in \{1 , 2 , 3\}$
, then matrix A = ...

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$

(24) If A is a matrix of order 2×2 and $a_{xy} = \frac{x}{y}$, then $a_{11} \times a_{12} \times a_{21} \times a_{22} =$ ....

(a) 4 (b) 2 (c) 1 (d) $\frac{1}{2}$

(25) If the order of the matrix A is 3×2 and : $a_{11} = 2$, $a_{21} = 3$, $a_{32} = \frac{1}{2}\,a_{11}$,
$a_{22} = a_{21} + 3$, $a_{31} = -9$, $a_{12} = \frac{1}{3}\,a_{31}$, then the matrix A = ...

(a) $\begin{pmatrix} 2 & -9 \\ 3 & 6 \\ 1 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 3 & 6 \\ -9 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -3 \\ 3 & 6 \\ -9 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 3 & -9 \\ -3 & 6 & 1 \end{pmatrix}$

(26) If A is a row matrix and $a_{xz} = 5$, then $x =$ ....

(a) 5 (b) 5 z (c) $\frac{5}{z}$ (d) 1

(27) If A is a skew symmetric matrix of order 3×3 , then $a_{11} + a_{22} + a_{33} =$ ....

(a) 3 (b) 2 (c) 1 (d) zero

(28) If A is a matrix of order $m \times n$ where $m < n$ and the number of its elements equals 3 and the matrix B of order $n \times 2$, then the number of elements of matrix B equals ....

(a) 2 (b) 3 (c) 6 (d) 9

(29) If A is a matrix in the order 3×2 and the sum of elements of matrix A equals x , then sum of elements of matrix 2 A equals ....

(a) x (b) $2x$ (c) $6x$ (d) $12x$

(30) If A is a symmetric matrix which of the following could be a rule to identify the elements of A ?

(a) $a_{ij} = 2i - j$ (b) $a_{ij} = i + j$ (c) $a_{ij} = i^j$ (d) $a_{ij} = 3i + 2j$

(31) If the matrix : $A = \begin{pmatrix} 0 & x-1 \\ x^2 - 1 & 0 \end{pmatrix}$ is a skew symmetric , then $x \in$ ....

(a) $\{-1 , 2\}$ (b) $\{0 , -1\}$ (c) $\{1 , -2\}$ (d) $\{0 , 1\}$

(32) If A is a diagonal matrix of order 3×3 and $a_{xy} = 5$ for every $x = y$, then ....

(a) A = I (b) A = 5 I (c) A = 5 O (d) A = O

(33) If A is a symmetric matrix and skew symmetric in the same time, then ....

(a) A = I (b) A = O

(c) A is a diagonal matrix. (d) A is a row matrix.

(34) If A is a skew symmetric matrix of order 3×3 and $a_{13} = 4$ which of the following statements is true ?

(1) $a_{31} = -4$ **(2)** $a_{11} = 0$ **(3)** $a_{23} + a_{32} = 0$

(a) (1) only. (b) (1) , (2) only.

(c) (2) , (3) only. (d) (1) , (2) and (3) are true.

(35) If : $X = \begin{pmatrix} a-b & 1 \\ 3 & b-2 \end{pmatrix}$, $Y = \begin{pmatrix} a-6 & 6 \\ 2 & 4-a \end{pmatrix}$

and : $2X = Y^t$, then $a + 2b =$ ....

(a) 4 (b) 8 (c) 10 (d) 12

(36) If : $A = \begin{pmatrix} \sin 30^\circ & 1 \\ \tan^2 60^\circ & \cot 45^\circ \end{pmatrix}$, $B = \begin{pmatrix} m - \cos 60^\circ & 1 \\ p + \sec^2 45^\circ & n + \cos^2 90^\circ \end{pmatrix}$ and A = B

, then $m + p - n =$ ....

(a) 2 (b) 10 (c) – 2 (d) 1

Second Essay questions

1 If $A = \begin{pmatrix} 4 & -1 & 2 \\ -3 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 & -1 \\ 4 & 3 & 2 \\ 6 & -7 & 8 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{3} \\ -\frac{1}{5} \end{pmatrix}$

(1) Mention the order of each matrix.

(2) Write each of the following elements : a_{23} , b_{11} , c_{31} , a_{31} , b_{33} , c_{21}

2 Write the type and the order of each of the following matrices :

(1) $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix}$ (3) $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (5) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ (6) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

3 If $A = \begin{pmatrix} 15 & -12 & 10 \\ 20 & -10 & 7 \\ -2 & 1 & 3 \end{pmatrix}$, **find :** $-5A$

4 **Find the transpose of each of the following matrices showing the order of the resulted matrix:**

(1) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ (3) $\begin{pmatrix} -1 & \sqrt{5} \\ 2 & 3 \\ -4 & 7 \end{pmatrix}$

(4) $\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$ (5) $\begin{pmatrix} 1 & -2 & 6 \end{pmatrix}$ (6) $\begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}$

5 **Write all the following elements of matrices :**

(1) $A = (a_{ij})$, $i = 1 , 2$, $j = 1 , 2 , 3$

(2) $B = (b_{ij})$, $i = 1 , 2 , 3$, $j = 1$

(3) $C = (c_{ij})$, $i = 1 , 2$, $j = 1 , 2$

6 If $A = (a_{xy})$ for each x , $y \in \{1 , 2 , 3\}$, **write the matrix A , given that :**

$a_{xy} = y - x$, then find A^t

7 Form the matrix $A = (a_{ij})$ of order 3×2 where $a_{ij} = i - j + 2$

, then find the matrix C where $C = A^t$, mention its order and find c_{ji} if $i = 3j$

---

8 **Find the value of each of a and b if :** $\begin{pmatrix} 4 & -1 \\ 2a-1 & 3b+1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 7 \end{pmatrix}$ « 2 , 2 »

---

9 If $\begin{pmatrix} 2x-5 & 4 \\ 3 & 3y+12 \end{pmatrix} = \begin{pmatrix} 25 & 4 \\ 3 & y+18 \end{pmatrix}$ **, find the value of each of :** x and y

« 15 , 3 »

---

10 If $\begin{pmatrix} x+8 & -5 \\ 3 & -y \end{pmatrix} = \begin{pmatrix} 38 & -5 \\ 3 & 4y-10 \end{pmatrix}$ **, find the value of each of :** x and y

« 30 , 2 »

---

11 If $\begin{pmatrix} 8 & 15 \\ -1 & z+y \end{pmatrix} = \begin{pmatrix} \ell^3 & x^2-1 \\ y & 8 \end{pmatrix}$ **, find the value of each of :** x , y , z and ℓ

« ± 4 , − 1 , 9 , 2 »

---

12 If $\begin{pmatrix} 3x & x+y & x-z \end{pmatrix} = \begin{pmatrix} -9 & 4 & -10 \end{pmatrix}$

, find the values of : x , y and z « − 3 , 7 , 7 »

---

13 **Find the values of a , b , c and d if :**

(1) $\begin{pmatrix} 3 & -5 \\ a-3 & 3d-2 \end{pmatrix} = \begin{pmatrix} a-2 & 2b+1 \\ c & 16 \end{pmatrix}$ « 5 , − 3 , 2 , 6 »

(2) $\begin{pmatrix} 15 & 2b \\ 0 & 2a+c \end{pmatrix} = \begin{pmatrix} 3a & 10 \\ 2b-d & 10 \end{pmatrix}$ « 5 , 5 , 0 , 10 »

(3) $\begin{pmatrix} a+b & a-b \\ a+b+c & a-b+2d \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 7 & 5 \end{pmatrix}$ « 3 , 6 , − 2 , 4 »

---

14 If $\begin{pmatrix} \cot 60° & \cos 30° & \tan 60° \\ \sec 60° & \csc 60° & \sin 30° \end{pmatrix} = \sqrt{3} \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{3}}{3} \\ \frac{1}{2}x & 2y \\ 1 & \sqrt{3}z \end{pmatrix}^t$

, then find the value of each of : x , y and z « 1 , $\frac{1}{3}$, $\frac{1}{6}$ »

---

15 **Show which of the following matrices is symmetric and which is skew symmetric :**

(1) $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & -1 & 4 \\ -1 & 2 & 6 \\ 4 & 6 & 5 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & -\frac{5}{2} & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$

16 If A is a symmetric matrix where $A = \begin{pmatrix} 5 & 2y & 3y + x \\ z + 1 & -1 & z + 3 \\ 2 + 3x & 8 & 0 \end{pmatrix}$

, then find the value of each of : x , y and z « 3.5 , 3 , 5 »

17 If B is a skew symmetric matrix where $B = \begin{pmatrix} 0 & -2x & -5 \\ z + 6 & 0 & 3z \\ 3y + x & 6 & 0 \end{pmatrix}$

, then find the value of : x y z « – 4 »

Third Problems that measure high standard levels of thinking

- **Choose the correct answer from those given :**

(1) If $\begin{pmatrix} 2x & 1 \\ x & x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t$ and if ad – bc = 1 , then $x \in$ ……………

(a) $\{1 , -\frac{1}{2}\}$ (b) $\{-1 , -\frac{1}{2}\}$ (c) $\{1 , \frac{1}{2}\}$ (d) $\{-1 , \frac{1}{2}\}$

(2) If ℓ and m are the two roots of the equation $x^2 - 3x + 1 = 0$ and if

$\begin{pmatrix} \ell^2 m & -\ell^2 \\ m^2 & m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then ad – bc = ……………

(a) 18 (b) 1 (c) 2 (d) 4

(3) If $\begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$ is a skew symmetric , then $\frac{a + b + c + f}{d + x + y + z} =$ ……………

(a) 1 (b) zero (c) – 1 (d) e

(4) If the matrix (a_{xy}) of order 3 × 2 where $a_{xy} = x + 2y$ and the sum of the first row elements = k^2 , then k = ……………

(a) 2 (b) – 2 (c) $2\sqrt{2}$ (d) $\pm 2\sqrt{2}$

(5) If A is a matrix of order m × n and A^t of order (2 m – 1) × (n – 1) , then m + n = ……………

(a) 3 (b) 4 (c) 5 (d) 6

(6) If $\begin{pmatrix} 1+i & 1-i \\ \sin x & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a b c d = 1 , then x = ...

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

(7) If $\begin{pmatrix} x^3-y^3 & 6 \\ x^2+xy+y^2 & 12 \end{pmatrix} = 3\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, then xy = ...

(a) 3 (b) – 3 (c) – 2 (d) 2

(8) If the matrix $A = \begin{pmatrix} \sin\left(\frac{\pi}{2}\right) & \sin(\text{zero}) \\ \cos\left(\frac{\pi}{2}\right) & \cos(\text{zero}) \end{pmatrix}$

which of the following statements is true ?

(1) A is a unit matrix. **(2)** A is a symmetric matrix.

(3) A is a square matrix.

(a) (1) only. (b) (1) , (2) only.

(c) (2) , (3) only. (d) (1) , (2) and (3) are true.

(9) If A is a matrix of order 3 × 3 where $a_{xy} = \begin{cases} x+y & \text{for} \quad x \neq y \\ 6 & \text{for} \quad x = y \end{cases}$

, then the sum of the elements of the main diagonal equals ....

(a) 1 (b) 6 (c) 12 (d) 18

(10) If $A = \begin{pmatrix} x & e \\ f & y \end{pmatrix}$ is a skew symmetric , then 2 e + 3 f = ...

(a) e (b) $x + y$ (c) f + y (d) x

(11) If $\begin{pmatrix} 3^x & 2x+y \\ x+y & 3^y \end{pmatrix} = \begin{pmatrix} 25 & b \\ a & 5 \end{pmatrix}$, then the value of : $\frac{b}{a}$ = ...

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) 15 (d) 5

Life applications

1 **Sports :** The coach of a team of the basketball in the school recorded the scores of three players in the classes league , as follows :

Samir : played 10 games , 20 shots , 5 scores.

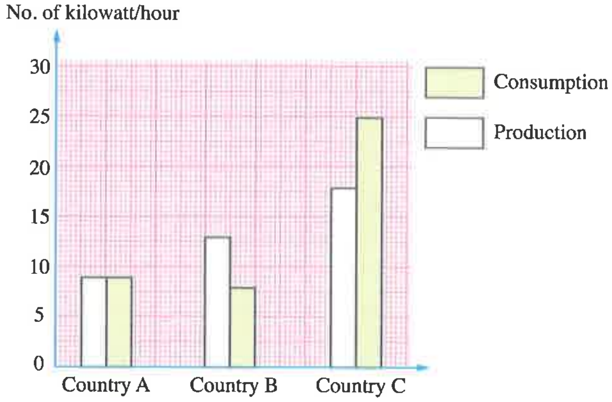
Hazem : played 16 games , 35 shots , 8 scores.

Karim : played 18 games , 41 shots , 10 scores.

(1) Arrange these data in a matrix such that the players are arranged ascendingly according to their number of scores.

(2) Determine the order of the matrix. What is the value of a_{23} ?

2 **Energy :** It is possible to measure the energy in kilowatt/hour, the opposite graph shows the production and consumption of energy of some countries. Write a matrix representing the data in the opposite graph.



3 **Industry :**

The opposite table shows the number of general national factories of food and leather industry in three different cities in Egypt.

City	Food industry	Leather industry
6 October	44	68
Sadat	28	52
10 Ramadan	37	14

(1) Organize the data in a matrix.

(2) Add the elements in each column, what's your interpretation of the results obtained ?

(3) Add the elements in each row, are the results you obtained possible to help us with an extra data ? Explain your answer.

Exercise Two

Adding and subtracting matrices

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) If O is the zero matrix of order 2×3 , then $\begin{pmatrix} 4 & 0 & -3 \\ -1 & 5 & 0 \end{pmatrix} + O =$ ....

(a) I (b) O (c) $\begin{pmatrix} 4 & 0 & -3 \\ -1 & 5 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} -4 & 0 & 3 \\ 1 & -5 & 0 \end{pmatrix}$

(2) If $A = \begin{pmatrix} -4 \\ 8 \\ -5 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$, then $A + B =$ ....

(a) $\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} -6 \\ 12 \\ -10 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(3) $\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} =$ ....

(a) O (b) $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

(4) $\begin{pmatrix} -5 & -2 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 6 & -4 \\ 2 & -6 \end{pmatrix}^t =$ ....

(a) O (b) $\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$ (d) I

(5) If $X + \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} = O$, then $X =$ ....

(a) O (b) I (c) $\begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -4 \\ -5 & 3 \end{pmatrix}$

(6) If $\begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix} + B = I$, then B = ...

(a) $\begin{pmatrix} -5 & -1 \\ -4 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & -1 \\ -4 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -5 & 0 \\ -3 & -1 \end{pmatrix}$

(7) $\begin{pmatrix} 5 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 1 & -2 \end{pmatrix}^t =$ ....

(a) $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 8 \\ -9 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

(8) If $2X + \begin{pmatrix} -2 & -2 \\ 4 & \text{zero} \end{pmatrix} = 0$, then the matrix X = ...

(a) $\begin{pmatrix} -2 & -2 \\ 4 & \text{zero} \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -1 \\ 2 & \text{zero} \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 2 \\ -4 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ -2 & \text{zero} \end{pmatrix}$

(9) $\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix} + 2\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(a) $\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

(10) If $\begin{pmatrix} x \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, then $x + y =$ ....

(a) 1 (b) 4 (c) 5 (d) 3

(11) If $\begin{pmatrix} 2 & 4 \\ -3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 2 & 6 \end{pmatrix}^t = \begin{pmatrix} 3k & -y \\ 2x & 11 \end{pmatrix}$, then $x + y + k =$ ....

(a) – 3 (b) – 6 (c) 1 (d) – 8

(12) If $\begin{pmatrix} 5 & 1 \\ 2 & -3 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t =$ ....

(a) $\begin{pmatrix} 1 & -2 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$

(13) If $\begin{pmatrix} 6 & 1 \\ 5 & -3 \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} - y\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, then $x + y =$ ....

(a) 4 (b) 3 (c) 2 (d) – 2

(14) If x , y and z are real numbers and $x\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$

, then $x\,y\,z =$ ....

(a) 12 (b) 4 (c) – 12 (d) 3

(15) If $\begin{pmatrix} 3 & 0 \\ -2-x & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ x+y & y+2 \end{pmatrix} = O$, then $x + y =$ ....

(a) -2 (b) -1 (c) 0 (d) -4

(16) For any matrix A , A + (– A) = ...

(a) A (b) – A (c) O (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(17) If $A + A^t = O$, then A is a .... matrix.

(a) row (b) column

(c) symmetric (d) skew symmetric

(18) If A is a skew symmetric matrix , then $A + A^t =$ ....

(a) 2A (b) $2A^t$ (c) O (d) zero

(19) $\left(X^t\right)^t - X =$ ....

(a) O (b) X (c) 2 X (d) zero

(20) If the matrices A and B are of the same order $m \times n$, then the matrix $A - 2B$ is of order ....

(a) $m \times 1$ (b) $1 \times n$ (c) $n \times m$ (d) $m \times n$

(21) If A and B are two matrices of the order 3×2 , then the matrix $(5A + 3B)^t$ is of order ....

(a) 3×2 (b) 5×3 (c) 2×3 (d) 3×3

(22) If $A = \begin{pmatrix} x & 3 \\ -y & y+1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 3 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$

and $A - 2B = 3C$, then $xy =$ ....

(a) -6 (b) -2 (c) 6 (d) 9

(23) If $A^t + B^t = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$, then $2(A+B)^t =$ ....

(a) $\begin{pmatrix} 2 & 6 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 \\ 6 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$

(24) If $A = \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}$, $(A+B)^t = \begin{pmatrix} 3 & -2 \\ 1 & -4 \end{pmatrix}$, then the matrix B = ...

(a) $\begin{pmatrix} 2 & -1 \\ 4 & -6 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 \\ -1 & -6 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 \\ 0 & -6 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 1 & -6 \end{pmatrix}$

Unit 1

Second Essay questions

1 If $A = \begin{pmatrix} -4 & -1 \\ -3 & -7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -7 \\ 8 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

, find each of the following "if possible" : (1) $A + B$ (2) $A + C$

---

2 If $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 5 \\ -2 & 4 \end{pmatrix}$

, find each of the following : (1) $B - C$ (2) $A + 2B - C$

---

3 If $A = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 \\ 6 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}$

, find the matrix : $2A - 3B + 4C$

---

4 If $X = \begin{pmatrix} -4 & -2 \\ 3 & 6 \\ 0 & 4 \end{pmatrix}$, $Y = \begin{pmatrix} 5 & 1 \\ 0 & -2 \\ 4 & -3 \end{pmatrix}$ and $Z = \begin{pmatrix} -2 & -4 \\ -3 & 2 \\ 6 & 0 \end{pmatrix}$

, then find the matrix : $3X - Y + Z$

---

5 If $A = \begin{pmatrix} 3 & 1 & -1 \\ -2 & 5 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 4 & 0 \\ 1 & -1 & 3 \end{pmatrix}$ **, check that :** $4(A + B) = 4A + 4B$

---

6 If $A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$ **, check that :**

(1) $(A + B)^t = A^t + B^t$ (2) $(A - B)^t = A^t - B^t$

(3) $A - B \neq B - A$ (4) $-(A + B) = (-A) + (-B)$

---

7 If $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & 1 \\ 3 & 4 \end{pmatrix}$ **, find :**

(1) $\frac{1}{3}(A + C)$ (2) $2A - B + 2C$

(3) $A - 2B + 2(B + C)$ (4) $A + 2B^t - C^t$

---

8 If $A = \begin{pmatrix} -7 & 0 & -5 \\ 4 & 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & 4 \\ 0 & 7 \\ -6 & -5 \end{pmatrix}$, then find the result of each of the following operations "if possible" , give reasons in case of impossible solution :

(1) $A + B$ (2) $A + B^t$ (3) $A^t + B$

---

9 If $A = \begin{pmatrix} -4 \\ 6 \\ -12 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, prove that the matrix A is the additive inverse of 2 B

10 If $\begin{pmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \\ 0 & -6 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -3 \\ 4 & -6 & -7 \\ -1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} x & 5 & a \\ 8 & y & -2 \\ b & -3 & z \end{pmatrix}$,

find the value of each of : x , y , z , a and b « 3 , 1 , 13 , – 4 , – 1 »

---

11 If $3\begin{pmatrix} 1 & 3 \\ x & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & y \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 6 \\ 8 & 2 \end{pmatrix}^t + \begin{pmatrix} 6 & -5 \\ -4 & 2 \end{pmatrix}$,

find the value of each of : x and y « 4 , 3 »

---

12 **Find the value of each of a , b , c and d which satisfies the equation :**

$2\begin{pmatrix} a & 3 \\ 6 & b \end{pmatrix} = 3\begin{pmatrix} a & d \\ c & -2 \end{pmatrix} - 4\begin{pmatrix} c & 3 \\ 0 & a \end{pmatrix}$ « 16 , – 35 , 4 , 6 »

---

13 **Find the value of each of x , y , z and ℓ that satisfies that :**

$x\begin{pmatrix} 1 & 3 \\ 5 & y \end{pmatrix} + z\begin{pmatrix} 2 & \ell \\ 0 & 4 \end{pmatrix} + 5\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = O_{2\times 2}$ « – 1 , 2 , – 2 , 1 »

---

14 If $\begin{pmatrix} 2x-1 & 8 \\ z & 6 \end{pmatrix} - \begin{pmatrix} x+3 & 5 \\ -2z & 4 \end{pmatrix} = \begin{pmatrix} 4x & -3y \\ 3 & k \end{pmatrix} + 2\begin{pmatrix} -x & 6y \\ -6 & 2k \end{pmatrix}$,

find the value of each of : x , y , z and k « – 4 , $\frac{1}{3}$, – 3 , $\frac{2}{5}$ »

---

15 If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -3 \\ 4 & 5 & -1 \end{pmatrix}$ and $(A+B)^t = \begin{pmatrix} 5 & 3 & 6 \\ -1 & 5 & 8 \\ 4 & 0 & 3 \end{pmatrix}$, **find the matrix B**

---

16 If $A = \begin{pmatrix} 4 & 8 & -6 \\ 2 & -4 & 8 \\ 6 & 12 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -6 & 2 \\ 4 & -10 & 0 \\ -1 & 8 & -4 \end{pmatrix}$,

then find the matrix X where : X = 2 A – 3 B

---

17 If $A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & 3 \\ 4 & 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & -1 & 3 \\ 4 & 12 & -5 \\ 10 & 21 & 3 \end{pmatrix}$,

find the matrix X such that : 2 A + B – X = O

---

18 If $A = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 \\ 6 & -2 \end{pmatrix}$, **find the matrix X such that :** $2A - X = B^t$

---

19 **Solve the matrix equation :** $4\left[X + \begin{pmatrix} 5 & -1 \\ 0 & -3 \end{pmatrix}\right] = 2X + \begin{pmatrix} 12 & -4 \\ 5 & -6 \end{pmatrix}$

20 If $A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 & 1 \\ -2 & -6 & 4 \end{pmatrix}$, **find the matrix X such that :**

(1) $X^t = A^t + B^t$ (2) $2(X^t + A) = 4B$ (3) $5X^t - 2(A - B) = O$

---

21 If $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 \\ 1 & -2 \\ -1 & 0 \end{pmatrix}$,

find the matrix X such that : $2B + X^t = 3X^t - A$

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) If A is a square matrix, then the matrix $(A + A^t)$ is .... matrix.

(a) symmetric (b) skew symmetric (c) zero (d) diagonal

(2) If A is a square matrix, then the matrix $(A - A^t)$ is ....

(a) symmetric. (b) skew symmetric.

(c) zero matrix. (d) diagonal matrix.

(3) If $A^t + B^t = A + B$, then ....

(a) A is symmetric. (b) B is symmetric.

(c) $(A + B)$ is symmetric. (d) $(A + B)$ is skew symmetric.

(4) If A is a matrix of order 3×3 where $a_{ij} = 2i - j$, B is a matrix of order 3×3 where $b_{ij} = j - i$, then $A + B =$ ....

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -2 \\ 4 & -3 & 0 \end{pmatrix}$

(5) If A is a matrix of order 2×2 and $A + A^t = I$, then the sum of the elements of A is ....

(a) 4 (b) 2 (c) 1 (d) zero

(6) If A and B are two matrices of order 2×2 and $(A + B)$ is a symmetric, then $\frac{a_{21} - a_{12}}{b_{21} - b_{12}} =$ ....

(a) zero (b) -1 (c) 1 (d) 2

2 **Check the following identity :** $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

, then write the matrix $\begin{pmatrix} 3 & -2 \\ 7 & 5 \end{pmatrix}$ in the form of sum of 4 matrices where each of them (one of its elements is one and all other elements are zeroes) multiplied by a real number.

---

3 If $X + 2Y = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}$ and $X + Y = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$, **find the two matrices X and Y**

---

4 If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $(X - X^t)^t = \begin{pmatrix} e & f \\ m & n \end{pmatrix}$, **then find the value of :** $e + f + m + n$

« zero »

---

5 If $A = \begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}$ and $B^t = \begin{pmatrix} -3 & 2 \\ 0 & 1 \end{pmatrix}$,

then find the matrix X that satisfies the relation : $3A - 2B^t + 2X = B - X^t$

Exercise Three

Multiplying matrices

Test yourself

From the school book

First Multiple choice questions

Choose the correct answer from those given :

(1) If A is a matrix of order $m \times n$ and B is a matrix of order $r \times \ell$, then AB is defined if ....

(a) $m = r$ (b) $n = r$ (c) $n = \ell$ (d) $m = \ell$

(2) If A is a matrix of order 3×1 and B is a matrix of order 1×3 , then AB is a matrix of order ....

(a) 3×1 (b) 1×1 (c) 3×3 (d) 1×3

(3) If A is a matrix of order 2×3 and AB is defined as a matrix of order 2×1 , then B is a matrix of order ....

(a) 3×2 (b) 2×1 (c) 3×1 (d) 2×2

(4) If A is a matrix of order 2×3 and B^t is a matrix of order 1×3 , then AB is a matrix of order ....

(a) 3×1 (b) 2×1 (c) 3×2 (d) 1×3

(5) If A is a matrix of order 1×3 and B^t is a matrix of order 1×3 , then which of the following operation we can do it ?

(a) $A + B$ (b) $B^t + A^t$ (c) AB^t (d) AB

(6) $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} =$ ....

(a) $\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(7) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \dots\dots$

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (d) not possible

(8) $\begin{pmatrix} 4 & 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix} = \dots\dots$

(a) $\begin{pmatrix} 20 & -12 & -8 \end{pmatrix}$ (b) $\begin{pmatrix} 20 \\ -12 \\ -8 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \end{pmatrix}$ (d) not possible

(9) If $X = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} -3 & 5 \end{pmatrix}$, then $Y\,X = \dots\dots$

(a) $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} -9 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} -9 & 15 \\ 6 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \end{pmatrix}$

(10) If A and B are two matrices, where $AB = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$, then $B^t\,A^t = \dots\dots$

(a) $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -1 \\ 4 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$

(11) If $A = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \end{pmatrix}$, then $(BA)^t = \dots\dots$

(a) $\begin{pmatrix} 6 & 15 \\ -4 & -10 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & -4 \\ 15 & -10 \end{pmatrix}$ (c) $\begin{pmatrix} -4 \end{pmatrix}$ (d) $\begin{pmatrix} 4 \end{pmatrix}$

(12) If I is the identity matrix, then $I^n = \dots\dots$ where n is a positive integer.

(a) I^t (b) I^2

(c) I^3 (d) All the previous

(13) If $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, then $A^2 = \dots\dots$

(a) $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$ (c) $O_{1\times 1}$ (d) $O_{2\times 2}$

(14) If $A = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$, then $A^4 = \dots\dots$

(a) A (b) 2 A (c) 4 A (d) 8 A

(15) If $A = \begin{pmatrix} 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} x & 0 \\ 1 & y \end{pmatrix}$, $A \times B = \begin{pmatrix} 9 & 6 \end{pmatrix}$, then : $x + y =$ ....

(a) 3 (b) 5 (c) 7 (d) 9

(16) If $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 17 & k \end{pmatrix}$, then $k =$ ....

(a) 7 (b) -7 (c) 9 (d) -9

(17) If $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x & 7 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 11 & 18 \end{pmatrix}$, then : $x + y =$ ....

(a) -3 (b) 3 (c) -2 (d) -1

(18) If $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$, then $A^2 - A =$ ....

(a) I (b) O (c) 3 I (d) 5 I

(19) If $X = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then $X^2 + 3X =$ ....

(a) 12 I (b) 18 I (c) 9 X (d) 12 X

(20) If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then $x =$ ....

(a) 3 (b) 4 (c) -3 (d) -4

(21) If X is a matrix such that $X\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then X = ...

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \end{pmatrix}$

(22) If $\begin{pmatrix} 3 & 2 & x \\ 4 & 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ y & 2 \end{pmatrix} = \begin{pmatrix} 14 & -1 \\ 19 & -2 \end{pmatrix}$, then $x - y =$ ....

(a) 5 (b) 3 (c) -5 (d) -3

(23) If $A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 0 \\ 0 & -5 & 4 \end{pmatrix}$

and $AB^t + X = \begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}$, then X = ...

(a) $\begin{pmatrix} 6 & 4 \\ 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 12 & 7 \\ 6 & 13 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 4 \\ -1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix}$

(24) If : A = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, B = $\begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}$, C = $\begin{pmatrix} 3 & 1 \end{pmatrix}$, then ACB = ...

(a) $\begin{pmatrix} 10 & -18 \\ 15 & -27 \end{pmatrix}$ (b) $\begin{pmatrix} -21 & -7 \\ 12 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & -6 \\ 18 & 0 \end{pmatrix}$ (d) not possible

(25) If $\begin{pmatrix} i^3 & 0 \\ -i & i \end{pmatrix}\begin{pmatrix} i & -i \\ 0 & i \end{pmatrix}$ = ... (where $i^2 = -1$)

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ (d) not possible

(26) $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$ = ...

(a) I (b) – I (c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Second Essay questions

1 **Find the matrix of the product in each of the following (if it is possible) , showing the order of the resultant matrix :**

(1) $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$

(2) $\begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 4 & 0 & 2 \\ -1 & 1 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

(5) $\begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(6) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(7) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$

(8) $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(9) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -3 \end{pmatrix}$

(10) $\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}\begin{pmatrix} 4 & 2 & -1 \end{pmatrix}$

(11) $\begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix}\begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}$

(12) $\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} -2 & 9 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}$

---

2 If A = $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 3 \\ 3 & 9 \end{pmatrix}$ **, then find each of the following :**

(1) AB (2) BA (3) $(A + B)A$

---

3 If A = $\begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix}$ and B = $\begin{pmatrix} -3 & 0 \\ 7 & 6 \end{pmatrix}$ **, find :** AB , BA , A^2 , B^2 , A^tB , AB^t

4 If $X = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$, **find :** $X^2 - Y^2$

5 If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 5 & 6 \\ -1 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 5 & -7 \\ 10 & -5 & 5 \\ -8 & 5 & -3 \end{pmatrix}$, **prove that :** $AB = 10\,I$

6 If $A = \begin{pmatrix} 4 & 1 \\ -1 & 0 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 5 \\ 2 & 0 \end{pmatrix}$, **check that :** $(AB)^t = B^t A^t$

7 If $A = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, **prove that :** $A(B + C) = AB + AC$

8 If $A = \begin{pmatrix} 3 & -2 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ 5 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 2 & -1 \end{pmatrix}$,

check each of the following :

(1) $(AB)\,C = A\,(BC)$ (2) $BI = IB = B$

9 If $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $(A + B)^t = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, **find :** $(A^t B)^t$

10 If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$, **show that :** $AB = O$, while $BA \neq O$

11 If $A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -3 \\ 4 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 5 \\ 2 & 3 \end{pmatrix}$, **check that :** $(ABC)^t = C^t B^t A^t$

12 If $A^t = \begin{pmatrix} 2 & -4 \\ 4 & 3 \end{pmatrix}$, **then prove that :** $A^2 - 5A + 22\,I = O$

13 **Find the value of each of x, y and z that satisfies :**

$$\begin{pmatrix} x & -1 \\ 1 & y \end{pmatrix}\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & z & -13 \\ 5 & 0 & 5 \end{pmatrix}$$

« 3 , 2 , 7 »

14 If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ x & y \end{pmatrix}$,

find the value of each of x and y that satisfies : $AB = BA$ « 0 , 2 »

15 If $M = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, **find the value of each of x and y that satisfies :** $M^2 - x\,M + y\,I = O$

« 4 , 3 »

16 If $A = \begin{pmatrix} 13 & 7 \\ -1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 4 & 5 \\ 9 & 0 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & 5 \end{pmatrix}$,

find the matrix X that satisfies the relation : $2X^t = A^2 + (BC)^t$

Third Problems that measure high standard levels of thinking

1 Choose the correct answer from those given :

(1) If A is a matrix and $B = AA^t$, then B is ....

(a) symmetric. (b) skew symmetric. (c) unit matrix I (d) zero matrix O

(2) If each of A , B is a symmetric matrix , then the matrix (ABA) is .... matrix.

(a) symmetric (b) skew symmetric (c) diagonal (d) triangular

(3) If $X = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $Y = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$, then $XX^t + YY^t =$ ....

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(4) If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $A^{27} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $a + b + c + d =$ ....

(a) 19 (b) 27 (c) 29 (d) 36

(5) If ℓ , m are two roots of the equation $x^2 - 3x + 1 = 0$

, then $\begin{pmatrix} 3 & \ell \\ \ell & 2 \end{pmatrix}\begin{pmatrix} 2 & m \\ m & 3 \end{pmatrix} =$ ....

(a) $\begin{pmatrix} 6 & 3 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 9 \\ 6 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 6 \\ 3 & -2 \end{pmatrix}$

(6) If neither of the two matrices A nor B is a row or a column matrix and the number of elements of the matrix A = 10 and the number of elements of the matrix B = 6 and AB exists , then the number of the elements of (AB) equals ....

(a) 60 (b) 16 (c) 15 (d) 6

(7) If A and B are two square matrices of the same order

, then $(A + B)^2 = A^2 + 2AB + B^2$ if ....

(a) AB = BA (b) A = – B (c) $A = B^t$ (d) 2 A = – B

(8) If $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, then $A^n =$ ....

(a) $\begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} n & nk \\ 0 & n \end{pmatrix}$ (c) $\begin{pmatrix} n & k^n \\ 0 & n \end{pmatrix}$ (d) $\begin{pmatrix} 1 & k^n \\ 0 & 1 \end{pmatrix}$

(9) If $A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $n \in \mathbb{N}$, then $A^{4n} =$ ....

(a) I (b) $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ (c) $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ (d) O

(10) If AB = O, then ....

(a) A = O or B = O (b) A = O and B = O

(c) it is not necessary that A = O or B = O (d) no thing of the previous.

(11) If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A^{x} = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix}$, then $x =$ ....

(a) 3 (b) 4 (c) 5 (d) 6

(12) If : A is a matrix of order 2 × 2 and $A^2 = 3A + 2I$, $A^3 = mA + \ell I$

, then the value of $m + \ell =$ ....

(a) 20 (b) 17 (c) 11 (d) 6

(13) If A is a square matrix of the order 2 × 2, $A \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

, $A \times \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, then : $A \times \begin{pmatrix} 3 \\ -2 \end{pmatrix} =$ ....

(a) $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$

(14) If $A = \begin{pmatrix} 4 & 0 \\ 3 & -4 \end{pmatrix}$, then $A^{60} =$ ....

(a) $2^{30}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $2^{60}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $2^{90}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $2^{120}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(15) If : $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, then $A^{2019} =$ ....

(a) A (b) A^2 (c) $2019\,A^3$ (d) 2019 I

---

2 If $B^t + C^t = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$,

find the matrix X that satisfies : $X = (AB + AC)^t$

---

3 If $X = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, **find a and b if :** $X^2 + X = O$

---

4 If X, Y and Z are three square non-zero matrices and $Z = Y^t X^t + XY$,

prove that : Z is a symmetric matrix.

5 If $X = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, **prove that :** $X^{2014} = I$

Life applications

1 **Tourism :** A hotel in the tourist town of Hurghada consumes the following quantities of meat , vegetables and fruits in kilograms for lunch and dinner meals as in the following table :

	Meat	Vegetables	Fruits
Lunch meal	200	100	150
Dinner meal	120	80	100

If the average of the price of one kilogram of meat was 65 pounds , the average of the price of one kilogram of vegetables was 4 pounds and the average of the price of one kilogram of fruits was 5 pounds. Find using multiplication of matrices the total cost of the two meals. « 14150 , 8620 »

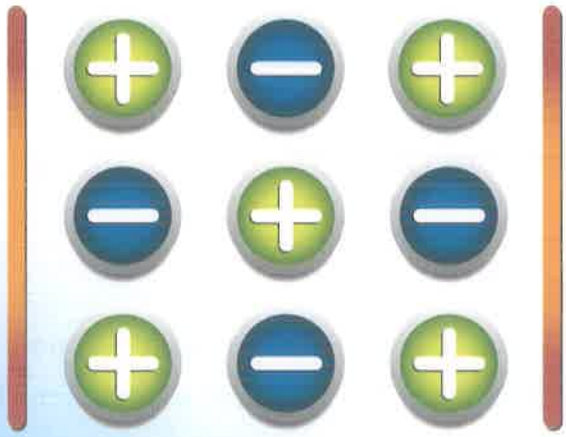
2 **Tourism :** A tourist company has 3 hotels in Hurghada, the following table shows the number of different rooms in each hotel. If the daily fare of 1 - bedroom is 250 pounds, 2-bedroom is 450 pounds and the suite is 600 pounds.

Hotel	1-bedroom	2-bedroom	Suite
Venus	28	64	8
Pearl	35	95	20
Diamond	20	80	15

(1) Write a matrix representing the number of different rooms in the three hotels , then write a matrix of prices of rooms.

(2) Write a matrix representing the daily income for the company. Assuming that all the rooms have been occupied.

(3) What is the daily income for the company , assuming that all the rooms have been occupied ? « 154 100 »



Exercise Four

Determinants

From the school book

Test yourself

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) The value of the determinant : $\begin{vmatrix} 7 & 5 \\ 3 & 2 \end{vmatrix} = \cdots\cdots\cdots$

(a) 29 (b) 1 (c) -1 (d) 11

(2) The value of the determinant : $\begin{vmatrix} -2 & -2 \\ 4 & 0 \end{vmatrix} = \cdots\cdots\cdots$

(a) -8 (b) 8 (c) zero (d) 10

(3) The value of the determinant : $\begin{vmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 4 & 2 & 5 \end{vmatrix}$ equals $\cdots\cdots\cdots$

(a) 10 (b) 30 (c) 15 (d) 5

(4) If the matrix $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 4 & 2 \\ 2 & 3 & -5 \end{pmatrix}$, then $|A| = \cdots\cdots\cdots$

(a) 8 (b) -8 (c) 20 (d) -20

(5) The value of the determinant : $\begin{vmatrix} 0 & 42 & 3 \\ 2 & 18 & 7 \\ 0 & 28 & 3 \end{vmatrix} = \cdots\cdots\cdots$

(a) zero (b) 84 (c) 48 (d) -84

(6) $\begin{vmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 3 & -7 \\ -2 & 5 \end{vmatrix} = \cdots\cdots\cdots$

(a) zero (b) 1 (c) -1 (d) -29

(7) If $A = \begin{pmatrix} 2 & -2 \\ -1 & 4 \end{pmatrix}$, then : $\frac{|2A|}{2|A|} =$ ....

(a) 1 (b) 2 (c) 4 (d) 8

(8) If $\begin{vmatrix} 2 & 1 \\ 4 & x \end{vmatrix} = 0$, then : $x =$ ....

(a) 4 (b) 3 (c) 2 (d) 1

(9) If $\begin{vmatrix} x^2 & 5 \\ 3 & x \end{vmatrix} = 12$, then : $x =$ ....

(a) 15 (b) 3 (c) 12 (d) 27

(10) If $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 2x & 5 \\ 6 & 1 & 4 \end{vmatrix} = 11$, then : $x =$ ....

(a) 3 (b) 2 (c) 4 (d) 5

(11) If $\begin{vmatrix} x & 12 \\ 3 & x \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 10 & 5 \end{vmatrix}$, then : $x =$ ....

(a) 2 (b) 5 (c) 6 (d) ± 6

(12) If $\begin{vmatrix} x & -1 \\ 2 & x \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 2$, then : $x =$ ....

(a) 3 or -2 (b) -3 or 2 (c) 3 or 2 (d) -3 or -2

(13) The solution set of the equation :

$\begin{vmatrix} x & 0 & 0 \\ 5 & 3 & 0 \\ 4 & -1 & 2x \end{vmatrix} = 6$ is ....

(a) $\{3\}$ (b) $\{6\}$ (c) $\{1, -1\}$ (d) $\{6, -6\}$

(14) If $\begin{vmatrix} 2x & 0 & 0 \\ 1 & 3x & 0 \\ 2 & 4 & -x \end{vmatrix} = 48$, then the value of $x =$ ....

(a) 2 (b) -2 (c) ± 2 (d) zero

(15) The solution set of the equation :

$\begin{vmatrix} x^2 & -2 \\ 2 & 1 \end{vmatrix} =$ zero in C is ....

(a) $\varnothing$ (b) $\{-2, 2\}$ (c) $\{-2i, 2i\}$ (d) $\{-i, i\}$

(16) The solution set of the equation : $\begin{vmatrix} x+2 & 2 \\ x & x-3 \end{vmatrix} = 4$ is ....

(a) $\{3, -2\}$ (b) $\{2, -3\}$ (c) $\{5, -2\}$ (d) $\{2, -5\}$

(17) The possible value of a which makes the determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & a & 1 \\ -1 & 4 & -2 \end{vmatrix} = 0$ is ....

(a) 5 (b) -2 (c) 1 (d) 3

(18) If A $(3, 5)$, B $(2, 0)$, C $(-3, 3)$

, then the area of the triangle ABC equals .... square unit.

(a) 28 (b) 14 (c) 7 (d) 2

(19) If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 6$ and $\begin{vmatrix} ak & kc \\ b & d \end{vmatrix} = -24$, then : $k =$ ....

(a) 4 (b) 3 (c) -3 (d) -4

(20) If $\begin{vmatrix} x & y \\ z & \ell \end{vmatrix} = 4$, then : $\begin{vmatrix} x-y & 4y \\ z-\ell & 4\ell \end{vmatrix} =$ ....

(a) 1 (b) 10 (c) 12 (d) 16

(21) If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 5$ and $d - c = 7$, then : $\begin{vmatrix} a+2 & b+2 \\ c & d \end{vmatrix} =$ ....

(a) 5 (b) 14 (c) -9 (d) 19

(22) If $\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \times \begin{pmatrix} x & 2y \\ 0 & z \end{pmatrix} + I = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}^t$

, then $x \times y \times z =$ ....

(a) 1 (b) -1 (c) 2 (d) 4

(23) If $\begin{vmatrix} 2k-1 & 2 \\ 3 & k+1 \end{vmatrix} = 2k^2 + 1$, then the value of $k =$ ....

(a) 2 (b) 4 (c) 6 (d) 8

(24) If $\begin{vmatrix} \ell & 2 & -1 \\ x-2 & m & 3 \\ 0 & 0 & n \end{vmatrix} = \ell mn$ where ℓ, m, n are non – zero values, then : $x =$ ....

(a) 2 (b) -2 (c) 6 (d) ℓm

(25) If $\begin{vmatrix} a & 5 \\ 5 & b \end{vmatrix} = 0$, $\begin{vmatrix} b & 2 \\ 2 & c \end{vmatrix} = 0$, $\begin{vmatrix} a & 1 \\ 1 & c \end{vmatrix} = 0$

, then $\begin{vmatrix} a & 0 & 0 \\ -1 & b & 0 \\ 2 & 5 & c \end{vmatrix} =$ ....

(a) ± 10 (b) ± 50 (c) ± 100 (d) ± 20

(26) If $A = \begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$, $B = \begin{pmatrix} 1 & x \\ 2 & y \end{pmatrix}$, $|B| = 3$, $|A + B| = 5$

, then $x =$ ....

(a) 3 (b) 15 (c) 9 (d) 21

(27) If A is a square matrix such that $|A| = 2$, then $|A^t| =$ ....

(a) zero (b) -2 (c) $\frac{1}{2}$ (d) 2

(28) If A is a matrix of order 2×2 and $|A| = 7$, then $|A^2| =$ ....

(a) 14 (b) 28 (c) 49 (d) 56

(29) If A is a matrix of order 2×2 and $|A| = 15$, then $|2A| =$ ....

(a) 15 (b) 30 (c) 60 (d) 120

(30) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $|B| = 12$, then $|AB| =$ ....

(a) -24 (b) 24 (c) 48 (d) 3

(31) If $A^3 = \begin{pmatrix} 1 & 2 \\ 3 & 14 \end{pmatrix}$, then $|A| =$ ....

(a) 8 (b) -8 (c) 2 (d) -2

(32) If $A = -A^t$ where A is a matrix of order 3×3 , then $|A| =$ ....

(a) -1 (b) zero (c) 1 (d) 2

(33) If A is a square matrix of order 2×2 and $|2A| = 8$, then $|3A| =$ ....

(a) 9 (b) 12 (c) 18 (d) 24

(34) The system of equations $a_1 x + b_1 y = c_1$, $a_2 x + b_2 y = c_2$

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 5$, $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = -10$

, $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 15$, then : $(x , y) =$ ....

(a) $(-2 , 3)$ (b) $(3 , -2)$ (c) $(-50 , 75)$ (d) $(75 , -50)$

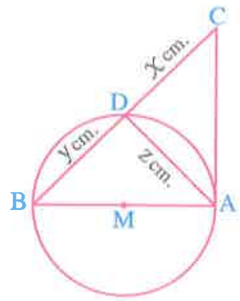
(35) When solving the system of equations $2x + 3y - z = 1$, $3x + 5y + 2z = 8$, $x - 2y - 3z = -1$, then $\frac{\Delta_x}{\Delta} =$ ....

(a) -1 (b) 2 (c) -2 (d) 3

(36) In the opposite figure $\overline{AC}$ is a tangent segment of the circle M , $\overline{AB}$ is diameter of the circle then :

The value of the determinant

$$\begin{vmatrix} 5 & 0 & 0 \\ 3 & z & y \\ 2 & x & z \end{vmatrix} = \text{..............}$$



(a) 5 (b) 3 (c) Zero (d) 2

(37) The solution set of the equation : $\begin{vmatrix} x-2 & 0 & 0 \\ 3 & x-3 & 0 \\ 4 & -1 & x \end{vmatrix} = 0$ is ....

(a) $\{0\}$ (b) $\{3 , 4 , -1\}$ (c) $\{2 , 3\}$ (d) $\{0 , 2 , 3\}$

(38) The solution set of the equation $\begin{vmatrix} x-2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & x+2 \end{vmatrix} = 5$ is ....

(a) $\{2 , -2\}$ (b) $\{3 , -3\}$ (c) $\{3 , 2\}$ (d) $\{1 , -1\}$

(39) If $\begin{vmatrix} \sin\theta & 0 & 0 \\ 2 & \csc\theta & 0 \\ 1 & 3 & x \end{vmatrix} + \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0$

, then $x =$ ....

(a) 1 (b) -1 (c) Zero (d) $\sin\theta$

(40) If $\begin{vmatrix} x & 1 & 3 \\ 0 & 2x & 5 \\ 0 & 0 & x \end{vmatrix} = 8x$, then the solution set of the equation is ....

(a) $\{2 , -2\}$ (b) $\{0 , -2 , 2\}$ (c) $\{\frac{1}{2} , \frac{-1}{2}\}$ (d) $\{0 , \frac{-1}{2} , \frac{1}{2}\}$

(41) If $\theta \in]0 , \frac{\pi}{2}[$, then the solution set of the equation : $\begin{vmatrix} \tan\theta & \cos\theta \\ \cos\theta & \cot\theta \end{vmatrix} = \frac{3}{4}$ is ....

(a) $\{\frac{\pi}{3}\}$ (b) $\{\frac{\pi}{4}\}$ (c) $\{\frac{\pi}{6}\}$ (d) $\{\frac{\pi}{12}\}$

(42) If ℓ , M are roots of the equations $x^2 - 4x - 10 = 0$, then the value of the determinant $\begin{vmatrix} 2\ell & -1 \\ 3 & m \end{vmatrix}$ equals ....

(a) – 17 (b) – 12 (c) – 8 (d) – 6

Second Essay questions

1 **Find the value of each of the following determinants :**

(1) $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$

(2) $\begin{vmatrix} 5 & 0 \\ 7 & -1 \end{vmatrix}$

(3) $\begin{vmatrix} -\frac{3}{4} & \frac{2}{3} \\ -1 & \frac{8}{9} \end{vmatrix}$

(4) $\begin{vmatrix} x & -2 \\ x & 1 \end{vmatrix}$

(5) $\begin{vmatrix} a+x & a \\ b+y & b \end{vmatrix}$

(6) $\begin{vmatrix} x+1 & x^2+1 \\ y+1 & y^2+1 \end{vmatrix}$

(7) $\begin{vmatrix} 1 & \sec\theta \\ \sec\theta & \tan^2\theta \end{vmatrix}$

(8) $\begin{vmatrix} \frac{1}{\cos\theta} & 1+\tan^2\theta \\ 1 & \frac{1}{\cos\theta} \end{vmatrix}$

2 **Prove that :**

(1) $\begin{vmatrix} 2x & -1 \\ 2 & 3x \end{vmatrix} + \begin{vmatrix} 3 & 6x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 13 \\ -2 & -7 \end{vmatrix}$

(2) $\begin{vmatrix} \csc\theta & \cot^2\theta \\ 1 & \csc\theta \end{vmatrix} \times \begin{vmatrix} 2 & -3 \\ 5 & -7 \end{vmatrix} = 1$

3 If $\begin{vmatrix} x & y \\ z & \ell \end{vmatrix} = 3$ **, find the value of each of :**

(1) $\begin{vmatrix} 2x & 5y \\ 2z & 5\ell \end{vmatrix}$

(2) $\begin{vmatrix} x-y & 4y \\ z-\ell & 4\ell \end{vmatrix}$

4 **Find the value of each of the following determinants :**

(1) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 4 \\ 0 & 7 & 8 \end{vmatrix}$

(2) $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 5 \\ 0 & 0 & 1 \end{vmatrix}$

(3) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

(4) $\begin{vmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 2 & -1 \end{vmatrix}$

(5) $\begin{vmatrix} 2 & 3 & -7 \\ 0 & -1 & 12 \\ 0 & 0 & 0 \end{vmatrix}$

(6) $\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix}$ (where $i^2 = -1$)

5 **Prove that :** $\begin{vmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 3 & -7 \\ -2 & 5 \end{vmatrix} = -1$

6 **Solve each of the following equations :**

(1) $\begin{vmatrix} 3x & -4 \\ 2 & 2-x \end{vmatrix} = -1$ « -1 or 3 »

(2) $\begin{vmatrix} x^2-1 & x+1 \\ x+1 & x^2-1 \end{vmatrix} = 0$ « -1 or 0 or 2 »

(3) $\begin{vmatrix} 0 & -1 & x \\ x & 4 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 10$ « 8 or -2 »

(4) $\begin{vmatrix} 0 & 3 & x \\ x & 1 & 0 \\ 1 & x+3 & 3 \end{vmatrix} = 0$ « -5 or 0 or 2 »

(5) $\begin{vmatrix} \sin 20° & \sin 70° \\ -\cos 20° & \cos 70° \end{vmatrix} = 2x - 3$ « 2 »

7 **Find the value of x satisfying :** $\begin{vmatrix} 3x & 1 \\ -3 & 2 \end{vmatrix}$ is equal to three times $\begin{vmatrix} -1 & -2 & 3 \\ 2 & 1 & 5 \\ -1 & x & 7 \end{vmatrix}$

« $-\frac{11}{3}$ »

8 If $x^3 + y^3 + z^3 = 20$ and $x\,y\,z = 4$,

find the numerical value of the determinant : $\begin{vmatrix} x & z & y \\ y & x & z \\ z & y & x \end{vmatrix}$ « 8 »

9 **Find using determinants the area of the triangle :**

(1) A (2 , 4) , B (– 2 , 4) , C (0 , – 2) « 12 square units »

(2) X (3 , 3) , Y (– 4 , 2) , Z (1 , – 4) « $23\frac{1}{2}$ square units »

(3) X (– 1 , – 3) , Y (2 , 4) , Z (– 3 , 5) « 19 square units »

10 **Use determinants to prove that each of the following points are collinear :**

(1) (3 , 5) , (4 , – 1) , (5 , – 7)

(2) (3 , 2) , (– 1 , 0) , (– 5 , – 2)

11 **Solve each of the following systems of linear equations by Cramer's rule :**

(1) $2x - 3y = 5$, $3x + 4y = -1$ « 1 , – 1 »

(2) $x + 3y = 5$, $2x + 5y = 8$ «-1 , 2»

(3) $x + 2y = 0$, $2x - 3y = 1$ «$\frac{2}{7}$, $\frac{-1}{7}$»

(4) $3x = 1 - 4y$, $5x + 12 = 7y$ «-1 , 1»

12 Solve each of the following systems of linear equations by Cramer's rule :

(1) $2x + y - 2z = 10$, $3x + 2y + 2z = 1$, $5x + 4y + 3z = 4$ «1 , 2 , -3»

(2) $x + 2y - 3z = 6$, $2x - y - 4z = 2$, $4x + 3y - 2z = 14$ «2 , 2 , 0»

(3) $y + 2x + 3z = 6$, $2x - y + z = -3$, $x - 2y + 2z = -11$ «$\frac{5}{3}$, $\frac{65}{12}$, $\frac{-11}{12}$»

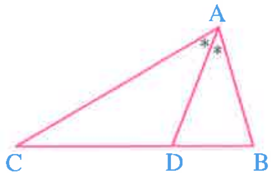
(4) $x + 2y + 3z = 4$, $2x + 5y = 3$, $3y + z = 4$ «-1 , 1 , 1»

(5) $x - y = 5$, $z - y = 8$, $x + y + z = 4$ «2 , -3 , 5»

(6) $3x + z = 7 - 2y$, $5y + 3z = 4 - x$, $x = 2y - z - 1$ «2 , 1 , -1»

(7) $x + y = 5 - 3z$, $y = z$, $2x - 4z = -2$ «1 , 1 , 1»

13 In the opposite figure :



ABC is a triangle , $\overrightarrow{AD}$ bisects $\angle BAC$

Find the value of : $\begin{vmatrix} 5 & 0 & 0 \\ 6 & AB & AC \\ 7 & BD & DC \end{vmatrix}$ «0»

Third Problems that measure high standard levels of thinking

1 Choose the correct answer from those given :

(1) If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \cos\theta & -1 \\ \sin\theta & 1 \end{pmatrix}$ where $\theta \in \left]0 , \frac{\pi}{2}\right[$, $|A \times B| = \frac{1}{2}$

, then $\theta = \dots\dots$

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

(2) If $\begin{vmatrix} 3 + \sin\theta & 1 \\ 4 & \sin\theta \end{vmatrix} = 0$, then $\theta = \dots\dots$

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

(3) The solution of the equation $\begin{vmatrix} \sin x & 0 & 0 \\ \cos x & \csc x & 0 \\ \sec x & \cot x & \tan x \end{vmatrix} = 1$

where $x \in [0 , 360°[$ is $\dots\dots$

(a) 45° or 135° (b) 135° or 225° (c) 45° or 225° (d) 45° or 315°

(4) The points A (– 1 , 5) , B (2 , 2) , C (3 , 1) are ....

(a) vertices of right-angled triangle whose area 5 square units.

(b) vertices of an isosceles triangle whose area 10 square units.

(c) vertices of an equilateral triangle whose area 9 square units.

(d) collinear.

(5) If $\begin{vmatrix} k+\frac{1}{k} & 1 \\ 1 & k+\frac{1}{k} \end{vmatrix} = 15$, then $k^2+\frac{1}{k^2} =$ ....

(a) 16 (b) 15 (c) 14 (d) 13

(6) If the area of the triangle whose vertices (k , 0) , (4 , 0) , (0 , 2) are 4 square units , then k = ...

(a) zero or – 8 (b) – 4 or 4 (c) zero or 8 (d) 8 or – 8

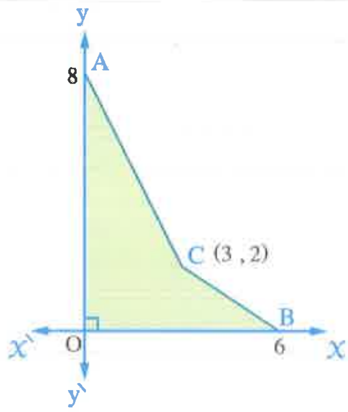
(7) The system of equations $a_1 X + b_1 y = c_1$, $a_2 X + b_2 y = c_2$ have a unique solution if ....

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0$

(c) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ (d) $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \neq 0$

2 **In the opposite figure :**

Find the area of the shaded region using determinants.



« 18 »

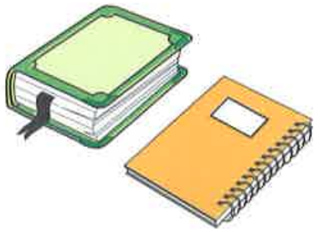
3 **Using Cramer's rule , solve the following equations :**

$\begin{vmatrix} 2 & y \\ -1 & X \end{vmatrix} = 1$, $\begin{vmatrix} 3 & z \\ -2 & y \end{vmatrix} = 1$, $\begin{vmatrix} 3 & X \\ 1 & z \end{vmatrix} = 2$

« 1 , 1 , 1 »

Life applications

1 **Consumer :** Fady bought 3 notebooks and 2 books for 85 pounds , Karim bought 2 notebooks and 4 books from the same kinds for 110 pounds.
Use Cramar's rule to find the price of each of the notebook and the book.



« 15 pounds , 20 pounds »

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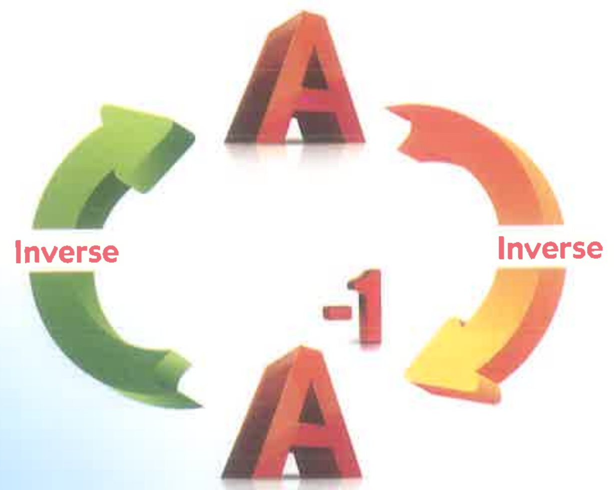
2 An agricultural piece of land is on a shape of a triangle. If the measure of one of its angles is double the measure of the second angle and more than the measure of the third by 20°
, find the measures of the three angles.

« 80° , 40° , 60° »



Exercise Five

Multiplicative inverse of a matrix

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) Which of the following matrices has no multiplicative inverse ?

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 4 \\ 2 & 8 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 6 \\ 3 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}$

(2) Which of the following matrices has a multiplicative inverse ?

(a) $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ -3 & 1 \end{pmatrix}$

(3) If the matrix $\begin{pmatrix} x & 1 \\ 6 & 3 \end{pmatrix}$ has no multiplicative inverse , then $x =$ ....

(a) -2 (b) zero (c) 2 (d) 3

(4) The value of x which makes the matrix $\begin{pmatrix} 6 & 2 \\ x-4 & -4 \end{pmatrix}$ has no multiplicative inverse is ....

(a) -8 (b) -10 (c) 8 (d) 10

(5) If the matrix $\begin{pmatrix} x & 8 \\ 2 & x \end{pmatrix}$ has no multiplicative inverse , then $x =$ ....

(a) 4 (b) -4 (c) zero (d) ± 4

(6) The matrix $\begin{pmatrix} a & 12 \\ 3 & a \end{pmatrix}$ has a multiplicative inverse at ....

(a) $a = 6$ (b) $a = \pm 6$

(c) $a \in \mathbb{R} - \{6\}$ (d) $a \in \mathbb{R} - \{6 , -6\}$

(7) The matrix $\begin{pmatrix} x+3 & 0 \\ 2 & x-3 \end{pmatrix}$ has no multiplicative inverse at $x =$ ....

(a) 3 (b) ± 3 (c) 5 (d) ± 5

(8) The multiplicative inverse of the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ equals ....

(a) I (b) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$

(9) If $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, then : $A^{-1} =$ ....

(a) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 1 \\ 5 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

(10) If $X = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$, then : $X^{-1} =$ ....

(a) X (b) X^t (c) I (d) 0

(11) If $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 2 & -1 \\ x & 3 \end{pmatrix}$, then $x =$ ....

(a) 3 (b) -3 (c) 5 (d) -5

(12) If $A \times \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} = I$, then $A =$ ....

(a) $\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

(13) If the product of two matrices $A \times B = I$ and the matrix $A = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$, then the matrix $B =$ ....

(a) $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 5 \\ 3 & -8 \end{pmatrix}$

(14) If $X \times \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then : $X =$ ....

(a) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (c) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (d) $3\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

(15) If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then : $x =$ ....

(a) 1 (b) 2 (c) 3 (d) 4

(16) If $A \times \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = I$, then : $3A =$ ....

(a) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ (b) $3\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ (c) $\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

(17) If $A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix}$, then : $B =$ ....

(a) $\begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -4 \\ -4 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} -4 & 4 \\ 4 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -4 & 4 \\ -4 & 1 \end{pmatrix}$

(18) If $A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}$ and $A = A^{-1} \times B$, then : $B =$ ....

(a) $\begin{pmatrix} -7 & 4 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -7 & 8 \\ -4 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 8 \\ 4 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & -8 \\ 4 & 4 \end{pmatrix}$

(19) If $A = \begin{pmatrix} x & 2 \\ y & -2 \end{pmatrix}$ and $A \times A^{-1} = A^2$, then : $x \times y =$ ....

(a) 3 (b) 2 (c) – 2 (d) – 3

(20) If $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X = \begin{pmatrix} x & x+y \\ 0 & x \end{pmatrix}$, then : $X =$ ....

(a) $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$ (b) $\begin{pmatrix} x & x \\ 0 & y \end{pmatrix}$ (c) $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ (d) $\begin{pmatrix} x & y \\ 0 & y \end{pmatrix}$

(21) If $A = \begin{pmatrix} -1 & -2 \\ 1 & k \end{pmatrix}$ and $A^3 = -A$, then : $k =$ .... where $k \in \mathbb{Z}$

(a) 1 (b) – 1 (c) zero (d) 3

(22) If $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$, then : $a + b =$ ....

(a) 3 (b) 4 (c) 5 (d) 6

(23) When the two equations $a x + b y = 5$, $c x + d y = -1$ have been solved , it found that the multiplicative of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} 3 & 2 \\ -3 & 1 \end{pmatrix}$, then $x + y =$ ....

(a) 3 (b) zero (c) 9 (d) – 3

(24) If $A = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}$, $A^{-1} + (A^t)^t = X$, then the matrix $X =$ ....

(a) 4 A (b) 3 I (c) 6 I (d) 10 I

(25) If $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, $A^{-1} + B^t = A^t$, then the matrix B = ...

(a) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 6 \\ 6 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}$

(26) If $A^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, then $y - x$ = ...

(a) 17 (b) – 17 (c) 7 (d) – 7

(27) If $A = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$, then $\frac{1}{4}(A + A^{-1})$ = ...

(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Second Essay questions

1 **Show the matrices which have multiplicative inverse and the matrices which have not multiplicative inverse in the following , and find it if it is existed :**

(1) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ (3) $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$

(4) $\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$ (5) $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ (6) $\begin{pmatrix} x & -y \\ x & -y \end{pmatrix}$ where $xy \neq 0$

2 **What are the real values of a which make each of the following matrices have a multiplicative inverse :**

(1) $\begin{pmatrix} a & 1 \\ 6 & 3 \end{pmatrix}$ (2) $\begin{pmatrix} a & 9 \\ 4 & a \end{pmatrix}$ (3) $\begin{pmatrix} a & 4 \\ 3 & a-1 \end{pmatrix}$

(4) $\begin{pmatrix} a-1 & -2 \\ 1 & a-2 \end{pmatrix}$ (5) $\begin{pmatrix} 8 & a-3 \\ a+3 & 2 \end{pmatrix}$ (6) $\begin{pmatrix} a & i \\ i & -1 \end{pmatrix}$ where $i^2 = -1$

3 Find the real values of x which make the matrix $\begin{pmatrix} x & 27 \\ 3 & x \end{pmatrix}$ have no multiplicative inverse.

« ± 9 »

4 If $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$,

prove that : the matrix B is the multiplicative inverse of the matrix A

5 📖 If $X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **, then prove that :** $X^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

6 📖 If $B = \begin{pmatrix} x & -xy \\ 0 & y \end{pmatrix}$ **, prove that :** $B^{-1} = \begin{pmatrix} \frac{1}{x} & 1 \\ 0 & \frac{1}{y} \end{pmatrix}$ given that $x\,y \neq 0$

7 If $X = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and $Y = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a\,b \neq 0$ **, prove that each of the matrices :**

X , Y and XY has a multiplicative inverse and find it.

8 If $X = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ **, then prove that :**

(1) 📖 $(XY)^{-1} = Y^{-1} X^{-1}$ (2) $(X^{-1})^{-1} = X$ (3) $(X^{-1}Y)^{-1} = Y^{-1} X$

9 If $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ **, then prove that :** $(A^t)^{-1} = (A^{-1})^t$

10 If $A = \begin{pmatrix} 2 & x \\ y & 4 \end{pmatrix}$ and $x\,y = 7$ **, prove that :** $A + A^{-1} = 6\,I$

11 📖 If $B = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ and $A\,B = I$ **, find the matrix** A

12 **Find the matrix A in each of the following :**

(1) 📖 $A \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2) $A \begin{pmatrix} -2 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 23 \\ 8 & 13 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} 3 & 10 \\ 0 & 3 \end{pmatrix}$

13 📖 If $A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ and $AB = \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}$ **, find the matrix** B

14 If $(AB)^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ **, then find :** B^{-1}

15 **Solve each system of the following linear equations using the matrices :**

(1) 📖 $3x + 2y = 5$, $2x + y = 3$

(2) 📖 $2x - 7y = 3$, $x - 3y = 2$

(3) $3x + 2y - 1 = 0$, $x = 3y - 7$

(4) $\frac{1}{2}x - y = -1$, $x + \frac{1}{3}y = 5$

16 Use the matrices to find the two numbers whose sum equals 10 and the difference between them equals 4 « 7 , 3 »

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17 Half the difference of two numbers is 2 , the sum of the greater number and double the smaller number is 13 , use matrices to find the two numbers. « 7 , 3 »

---

18 Use the matrices to find the perimeter of the rectangle whose length increases the double of its width by 2 cm. and the double of its length increases its width by 13 cm. « 22 cm. »

---

19 If 3 and – 1 are the roots of the equation : $a x^2 + b x - 3 = 0$, then use the matrices to find the values of the constants : a and b « 1 , – 2 »

---

20 The curve whose equation is : $y = a x^2 + b x$ passes through the two points (3 , 0) and (4 , 8) , use the matrices to find the constants a and b « 2 , – 6 »

---

21 The straight line whose equation is : $y + a x = c$ passes through the two points (1 , 5) and (2 , 1) , use the matrix to find the constants a and c « 4 , 9 »

Third Problems that measure high standard levels of thinking

- **Choose the correct answer from those given :**

(1) If A is a skew symmetric matrix of order 2×2 , then A^{-1} is ....

(a) symmetric. (b) skew symmetric. (c) diagonal matrix. (d) not exist.

(2) If $A = \begin{pmatrix} x & 1 \\ y & 3 \end{pmatrix}$, $A = 3 A^{-1}$, then $x + y =$ ....

(a) – 3 (b) – 5 (c) – 7 (d) – 9

(3) If the matrix B is the multiplicative inverse of the matrix A , then ....

(a) $A + B = O$ (b) $AB = I$ (c) $AB = O$ (d) $A = \frac{1}{B}$

(4) If A is a square matrix and $A^2 = I$, then $A^{-1} =$ ....

(a) O (b) A (c) 2 A (d) A + I

(5) If $X = \begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}$, then $X^{-1} =$ ....

(a) $\cos^2\theta\, X$ (b) $\cos^2\theta\, X^t$ (c) $\sec^2\theta\, X$ (d) $\sec^2\theta\, X^t$

(6) If X is a square matrix such that $X^2 + 5X + 5I = O$, then the multiplicative inverse of the matrix $(X + 2I)$ equals ....

(a) $X - 2I$ (b) $X + 3I$ (c) $X + 2I$ (d) $X - 3I$

(7) If $A^2 - A + I = O$, then the multiplicative inverse of the matrix A is ....

(a) A (b) $A - I$ (c) $I - A$ (d) $A + I$

(8) If A and B are two matrices of order 2×2 , then which of the following always is true ?

(1) If $AB = O$, then $A = O$ or $B = O$ **(2)** If $AB = I$, then $A = B^{-1}$

(3) If $(A + B)^2 = A^2 + 2AB + B^2$

(a) 1 , 2 only. (b) 1 , 3 only. (c) 2 , 3 only. (d) 2 only.

Life applications

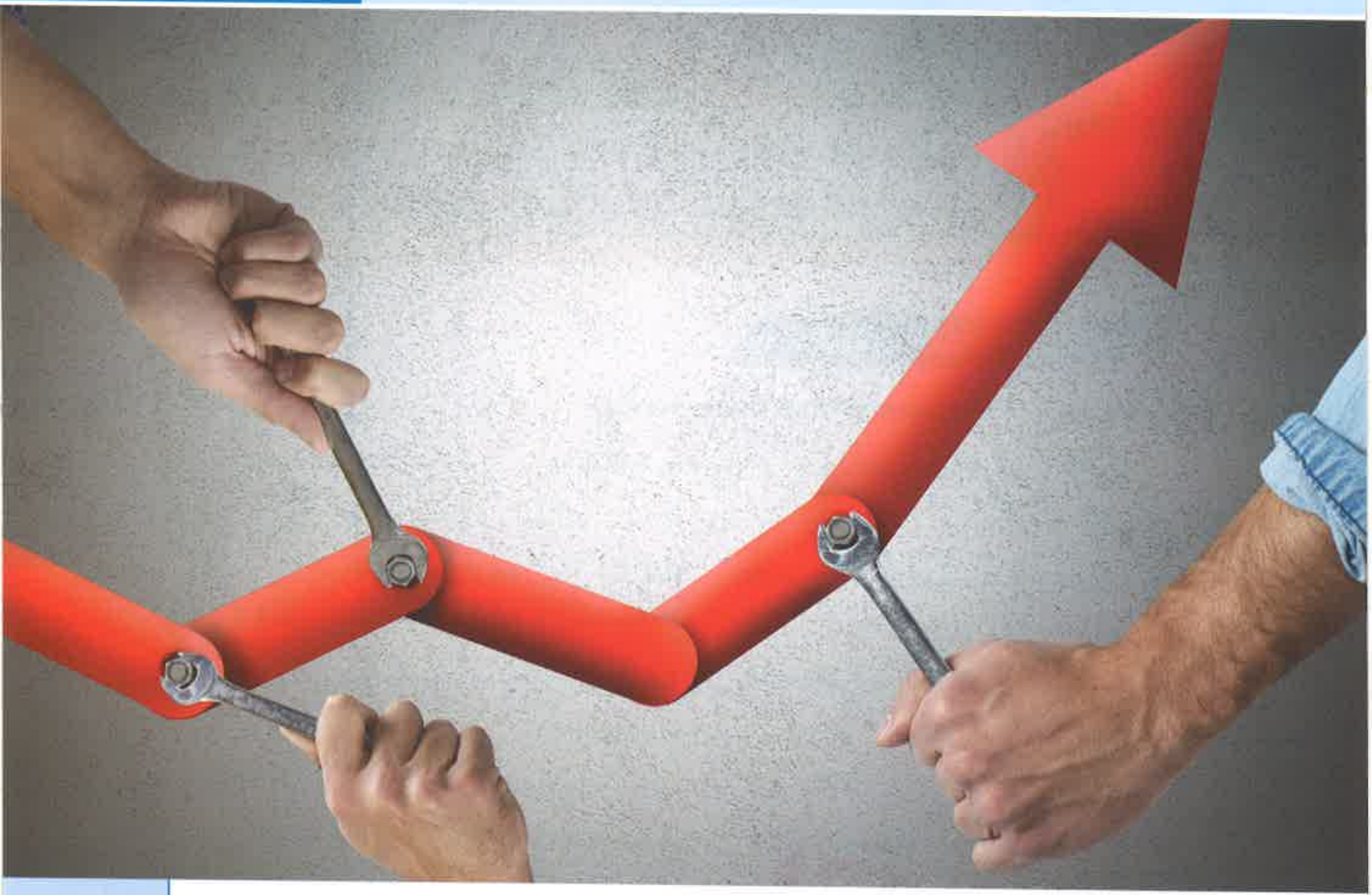
1 **Book Fair :** Hoda and Mariam went to Cairo international Book fair. Hoda bought 5 scientific books from a library and 4 historical books. She paid 120 pounds. Mariam bought from the same library 5 scientific books and 10 historical books , she paid 150 pounds. If the scientific books had the same price and also the historical books , use the matrices to find the price of each scientific book and each historical book. « 20 , 5 »

2 **Consumer :** Amal bought 8 kg of flour and 2 kg of butter for 140 pounds. Her friend Reem bought 4 kg of flour and 3 kg of butter for 170 pounds. Use the matrices to find the price of each of flour and butter. « 5 , 50 »

3 **Life :** A driver of a motorcycle buys 24 litres of gasoline and 5 litres of oil for 56 pounds to fill his motorcycle. While a driver of another motorcycle buys 18 litres of gasoline and 10 litres of oil for 67 pounds to fill his motorcycle , use matrices to find the price of each litre of gasoline and the price of each litre of oil , given that they use the same type of gasoline and oil. « $1\frac{1}{2}$, 4 »

Unit 2

Linear programming



Exercise Six : Linear inequalities - Solving systems of linear inequalities graphically.

Exercise Seven : Linear programming and optimization.

Exercise
Six

Linear inequalities-Solving systems of linear inequalities graphically

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) The S.S. of the inequality : $-1 < -x \leq 1$ in $\mathbb{R}$ is ....

(a) $]-1, 1]$ (b) $\mathbb{R} -]-1, 1]$ (c) $\{0, 1\}$ (d) $[-1, 1[$

(2) The solution set of the inequality : $1 \leq 2x - 1 < 5$ in $\mathbb{R}$ is ....

(a) $]1, 3[$ (b) $]1, 3]$ (c) $[1, 3[$ (d) $[1, 3]$

(3) The quadrant represents the solution for the system of inequalities :

$y > 0, x > 0$ in $\mathbb{R} \times \mathbb{R}$ is the .... quadrant.

(a) first (b) second (c) third (d) fourth

(4) The region representing the S.S. of the inequalities : $y > 0, x < 0$ in $\mathbb{R} \times \mathbb{R}$ is the .... quadrant.

(a) 1^{st} (b) 2^{nd} (c) 3^{rd} (d) 4^{th}

(5) The point which belongs to the solution set of the two inequalities :

$x > 0, y < 0$ is ....

(a) $(0, -3)$ (b) $(2, 0)$ (c) $(2, -3)$ (d) $(2, 3)$

(6) The point which belongs to the S.S. of the two inequalities : $x > 2, y > 1$ together is ....

(a) $(1, 2)$ (b) $(2, 1)$ (c) $(3, 1)$ (d) $(3, 2)$

(7) The point lying in the solution region of the inequality : $x + y \leq 3$ is ....

(a) $(1, 3)$ (b) $(2, -3)$ (c) $(2, 3)$ (d) $(1, 4)$

(8) The point .... does not lie in the solution region of the inequality : $2\,x + y \geq 5$

(a) $(-1 , 6)$ (b) $(5 , -1)$ (c) $(1 , 4)$ (d) $(2 , 4)$

(9) The point $(3 , 2) \in$ the S.S. of the inequality : $3\,x - y$ 1

(a) $<$ (b) $\leq$

(c) $>$ (d) (a) and (b) together

(10) If the point $(2 , 3)$ belongs to the solution set of the inequality : $x + y \leq a$, then ....

(a) $a > 5$ (b) $a \geq 5$ (c) $a < 5$ (d) $a > 0$

(11) If $(1 , y)$ belongs to the region of solution of the inequality : $x + 2\,y < 7$, then ....

(a) $y < 3$ (b) $y > 3$ (c) $y = 3$ (d) $y > 7$

(12) The two points : $(3 , 5)$, $(1 , 5)$ belong to the solution set of the inequality

$x + y$ 8

(a) $>$ (b) $\geq$ (c) $<$ (d) $\leq$

(13) Which of the following points belongs to the solution set of the system :

$x > 0$, $y \geq 0$, $2\,x + y > 6$? ....

(a) $(1 , 3)$ (b) $(0 , 0)$ (c) $(2 , 3)$ (d) $(4 , -2)$

(14) The point which does not belong to the solution set of the inequalities :

$x \geq 2$, $y \geq 0$, $x + y > 3$ is ....

(a) $(3 , 1)$ (b) $(2 , 2)$ (c) $(3 , 2)$ (d) $(2 , 1)$

(15) The point which belongs to the solution set of the inequalities :

$x > 3$, $y < 1$, $x + y \leq 5$ is ....

(a) $(6 , -2)$ (b) $(1 , -2)$ (c) $(4 , 4)$ (d) $(3 , -2)$

(16) The point which belongs to the solution set of the inequalities :

$2\,x + y < 4$, $x + 3\,y < 6$ is ....

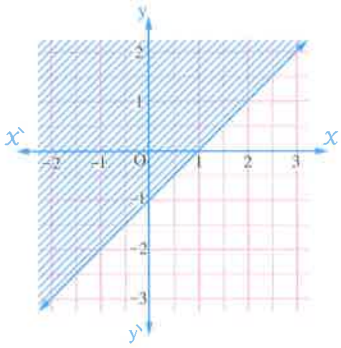
(a) $(1 , -4)$ (b) $(2 , 1)$ (c) $(1 , 2)$ (d) $(3 , -1)$

(17) The point which belongs to the S.S. of the system of the inequalities $x \geq 0$, $y \geq 0$,

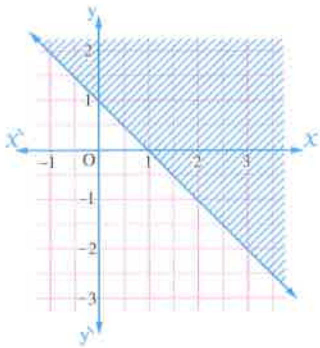
$x + 2\,y \geq 4$, $3\,x + 2\,y \geq 8$ in $\mathbb{R} \times \mathbb{R}$ is ....

(a) $(3 , 0)$ (b) $(2 , 1)$ (c) $(0 , 2)$ (d) $(0 , 3)$

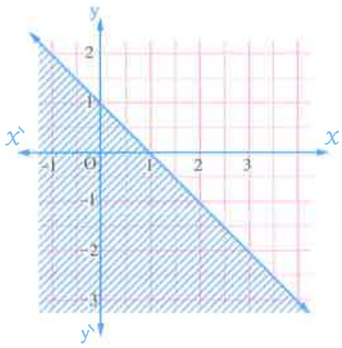
(18) Which of the following figures represents the S.S. of the inequality $x + y \geq 1$?



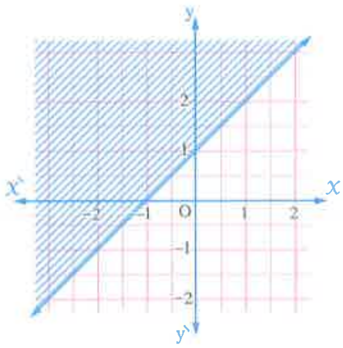
(a)



(b)



(c)

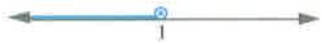


(d)

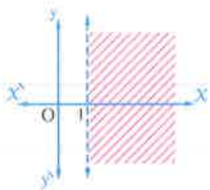
(19) Which of the following figures represents the solution set of the inequality :

$5 - 2x < 3$ in $\mathbb{R} \times \mathbb{R}$?

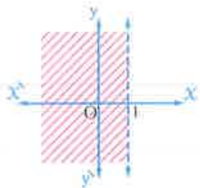
(a)



(b)



(c)



(d)

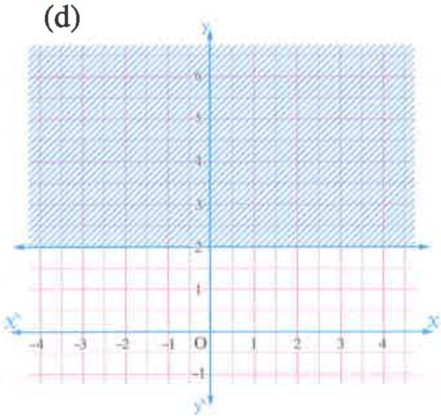
(20) The opposite figure represents the solution set of the inequality ....

(a) $y \leq 2$

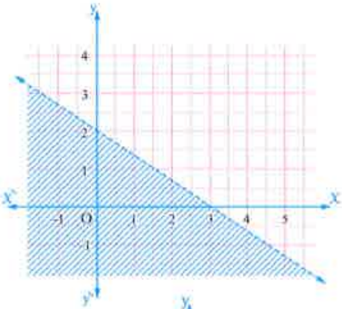
(b) $y < 2$

(c) $y \geq 2$

(d) $y > 2$



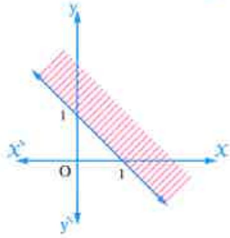
(21) The opposite figure represents the solution set of the inequality .... in $\mathbb{R} \times \mathbb{R}$



(a) $x + y < 5$ (b) $2x + 3y \le 6$

(c) $3x + 2y < 6$ (d) $2x + 3y < 6$

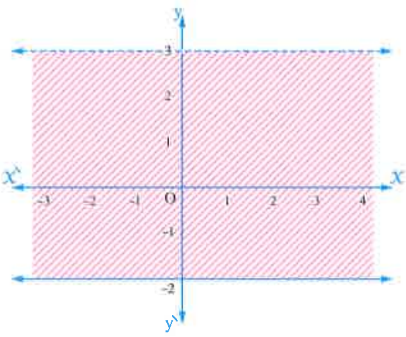
(22) In the opposite figure the shaded region represents the solution for the inequality ....



(a) $x + y \ge 1$ (b) $x + y \le 1$

(c) $x + y > 1$ (d) $x - y < 1$

(23) The opposite figure represents the S.S. of the inequality .... in $\mathbb{R} \times \mathbb{R}$

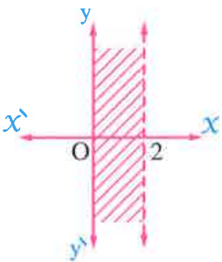


(a) $-2 \le x < 3$ (b) $-2 < x \le 3$

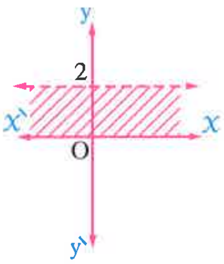
(c) $-2 \le y < 3$ (d) $-2 < y \le 3$

(24) Which of the following figures represents the solution set of the inequality :

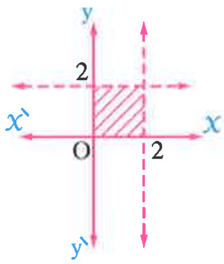
$0 \le x < 2$ in $\mathbb{R} \times \mathbb{R}$?



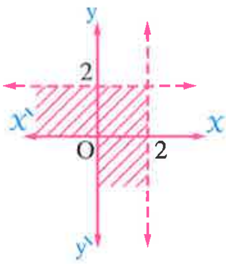
(a)



(b)



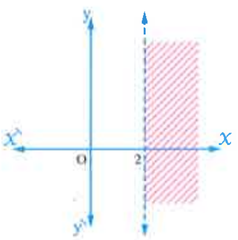
(c)



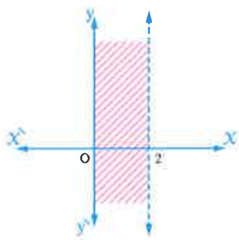
(d)

(25) Which of the following figures represents the solution set of the inequality :

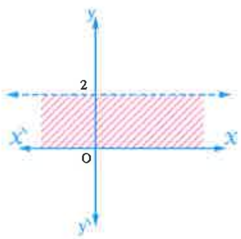
$0 \le y < 2$ in $\mathbb{R} \times \mathbb{R}$?



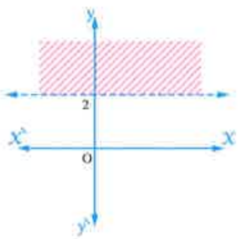
(a)



(b)



(c)



(d)

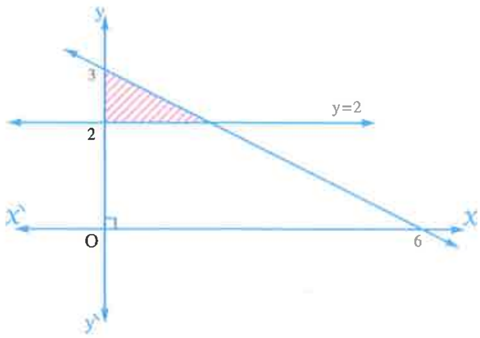
(26) In the opposite figure : the shaded region represents the solution set of the system of the inequalities $y \geq 2$, $x \geq 0$, ...

(a) $x + 2y - 6 \leq 0$

(b) $x + 2y + 6 \leq 0$

(c) $x + 2y - 6 \geq 0$

(d) $x + 2y + 6 \geq 0$



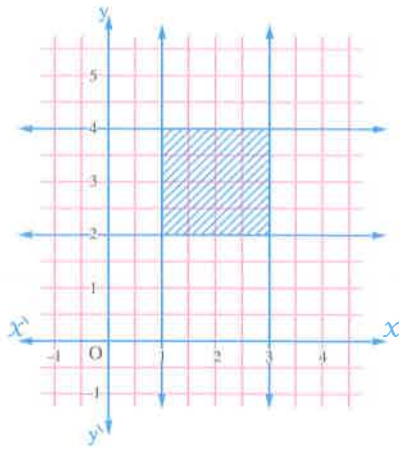
(27) The shaded region in the opposite figure represents the solution set of the inequalities ....

(a) $x > 1$, $y > 2$

(b) $1 < x < 3$, $2 < y < 4$

(c) $1 \leq x \leq 3$, $2 \leq y \leq 4$

(d) $x + y \geq 3$, $x - y \leq 7$



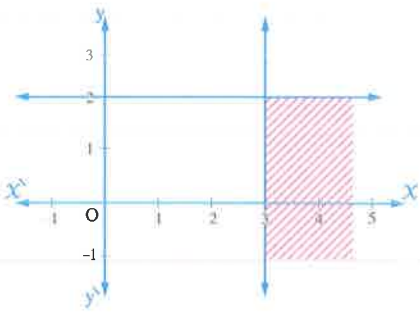
(28) The shaded region in the opposite figure represents the solution set of the inequalities ....

(a) $x > 3$, $y < 2$

(b) $x \geq 3$, $y \geq 2$

(c) $x + 1 < 4$, $y + 1 < 3$

(d) $x + 1 \geq 4$, $2y \leq 4$



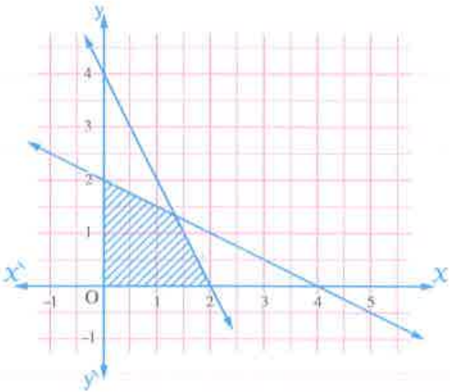
(29) The shaded region in the opposite figure represents the solution set of the inequalities ....

(a) $x \geq 0$, $y \geq 0$, $x + 2y \leq 4$, $2x + y \leq 4$

(b) $x + 2y \leq 4$, $2x + y \leq 4$

(c) $x > 0$, $y > 0$, $x + 2y < 4$, $2x + y < 4$

(d) $4x + 2y \leq 0$, $2x + 4y \leq 0$



Second Essay questions

1 **Find the S.S. in $\mathbb{R}$ for each of the following inequalities, representing it on the number line :**

(1) $2x - 3 \leq 5$
(2) $4 - 2x \leq 6$
(3) $3x - 9 > 6x$
(4) $6 + x < 3x + 2 \leq 14 + x$
(5) $2x - 1 < x + 3 < 3x + 7$

2 **Find graphically the S.S. in $\mathbb{R} \times \mathbb{R}$ for each of the following inequalities :**

(1) $2x - y \geq 6$
(2) $x + y < 3$
(3) $x \geq -2$
(4) $y \leq 5$
(5) $y > 2x - 3$
(6) $x < 2y - 4$
(7) $y \leq 2x$
(8) $-2 < x \leq 4$
(9) $-1 \leq y \leq 2$

3 **Solve graphically each system of the following linear inequalities in $\mathbb{R} \times \mathbb{R}$:**

(1) $x \geq 1$, $y < 3$
(2) $x - 2 < 0$, $y > 1$
(3) $x \geq 0$, $x + 2y > 4$
(4) $-1 \leq x < 2$, $0 \leq y < 3$
(5) $y \geq 2x + 6$, $y + 3x < -1$
(6) $y > x$, $x - y > 1$
(7) $2x - y \geq 5$, $2y \geq 20 + 4x$
(8) $x + y \leq 3$, $x - y > 1$
(9) $x < 1$, $x + y \leq -1$
(10) $y < x + 1$, $y > x - 1$

4 **Solve graphically each system of the following linear inequalities in $\mathbb{R} \times \mathbb{R}$:**

(1) $x \geq 0$, $y \geq 0$, $x + y \leq 5$
(2) $x \leq 4$, $y < x + 2$, $x + 2y \geq -2$
(3) $x \geq 0$, $y \geq 0$, $y \geq 7 - 2x$, $x + 2y \geq 8$
(4) $x \geq 0$, $y \geq 0$, $2x + y \leq 6$, $x + y \leq 4$
(5) $y - x > 0$, $2x + 2y \leq 12$, $y < 6 + 2x$
(6) $x + 4y > 4$, $4x + y \geq 2$, $x - y < 1$
(7) $0 \leq x \leq 5$, $0 \leq y \leq 3$, $x \geq y - 1$
(8) $x \leq 4$, $y \leq 6$, $2y - x \geq 2$, $y + 2x \geq 6$
(9) $x + 4y < 8$, $x - 2y < 6$, $0 \leq x < 4$

Unit 2

Third Problems that measure high standard levels of thinking

Choose the correct answer from those given :

(1) If the point (a , b) does not belong to the solution set of the inequality $2x + y > 3$, then ....

(a) $2a + b > 3$ (b) $2a + b < 3$ (c) $2a + b \leq 3$ (d) $2a - b > 3$

(2) Which of the following inequalities has a solution set does not lie in the second or third quadrant , then ....

(a) $x > 0$ (b) $x < 0$ (c) $y > 0$ (d) $y < 0$

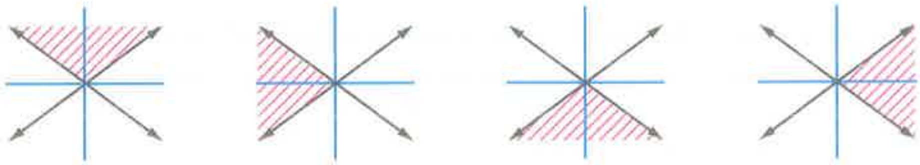
(3) If the solution set of the inequality $ax + y > 2$ does not lie in the third or fourth quadrant , then ....

(a) $a > 0$ (b) $a < 0$ (c) $a = 0$ (d) $a > 2$

(4) The solution set of the inequalities $x + y > 4$, $x - y < 4$ does not lie in the .... quadrant(s).

(a) first (b) first or second (c) second or third (d) third or fourth

(5) The solution set of the inequality $-x \leq y \leq x$ is ....



(a) (b) (c) (d)

(6) If x , y are integers , then the solution set of the system of inequalities :

$x > 0$, $y > 0$, $x + y < 3$ is ....

(a) $\{(0,0), (0,1), (0,2), (0,3), (1,0), (2,0), (3,0), (1,1)\}$

(b) $\{(1,1), (1,2), (2,1)\}$

(c) $\{(1,1)\}$

(d) $\varnothing$

(7) If x , y are two integers , then the number of solutions of the system of inequalities :

$x > 0$, $y > 0$, $x + 2y \leq 6$, $2x + y \leq 6$ equals ....

(a) 3 (b) 4

(c) 6 (d) infinite number of solutions.

(8) If the two points (1 , 4) , (4 , 1) belong to the solution set of the inequality $a\,x + b\,y \le c$, which of the following points must belong to the solution set also ?

(a) (0 , 5) (b) (2 , 4) (c) (2 , 3) (d) (4 , 2)

(9) If the point (4 , k) lies on the axis of symmetry of the region represents the solution set of the inequalities $x + y > a$, $x - y > a$, then k = ...

(a) 4 (b) – 4 (c) a (d) zero

(10) If $x + y \ge a$, $x + y \le b$ and the solution set of the system equals $\varnothing$, then ....

(a) $a > b$ (b) $a < b$ (c) $a = b$ (d) $a \le b$

Life applications

1 **Life :** A shepherd wants to make a rectangular sheep barn. The length of the barn must not be less than 80 metres and its perimeter must not increase than 310 metres. What are the possible dimensions of the barn ? (Write four possible dimensions)

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2 **Professions :** A carpenter wants to buy two types of nails. He does not want to pay more that 48 pounds for the purchase. If the carpenter needs at least 3 kilograms from the first type, and at least 1 kilogram from the second type, how much will the carpenter pay as a price for each type given that the price of one kilogram of the first type is 6 pounds and the price of one kilogram of the second type is 8 pounds ?

(1) Write a system of linear inequalities describing this situation.

(2) Represent graphically this system to show the possible solutions.

(3) Determine a point to be a solution of this system.

(4) Determine a point not to be a solution of this system.

---

3 Mr. Karim gave 60 minutes to his students to solve a test in mathematics. The students have to solve at least 4 questions from section "A" and at least 3 questions from section "B" such that the number of the answered questions is at least 10 questions from both sections. If Hanaa answered her questions in 4 minutes to every question in section "A" , 5 minutes to every questions in section "B"

How many questions in each section did Hanaa try to solve ?

Exercise Seven

Linear programming and optimization

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) The point at which the function : $P = 40\,x + 20\,y$ has a maximum value is ....

(a) $(0 , 0)$ (b) $(0 , -4)$ (c) $(15 , 10)$ (d) $(25 , 0)$

(2) The point at which the function : $P = 35\,x + 10\,y$ has a minimum value is ....

(a) $(0 , 0)$ (b) $(0 , 10)$ (c) $(0 , 40)$ (d) $(20 , 10)$

(3) If two times the number x is not less than three times the number y then ....

(a) $2\,x < 3\,y$ (b) $2\,x \leq 3\,y$ (c) $2\,x > 3\,y$ (d) $2\,x \geq 3\,y$

(4) Which of the following sentences represents the inequality : $x + y \leq 15$?

(a) Two numbers , sum of them is less than 15

(b) Two numbers , sum of them is not less than 15

(c) Two numbers , sum of them is more than 15

(d) Two numbers , sum of them is not more than 15

(5) Which of the following expressions represents the following sentence "two numbers , the sum of one of them and twice of the other does not exceeds 20" ?

(a) $x + 2\,y > 20$ (b) $x + 2\,y \geq 20$ (c) $x + 2\,y < 20$ (d) $x + 2\,y \leq 20$

(6) Which of the following sentences represents the inequality $y \geq 2\,x$?

(a) Two numbers , one of them more than twice of the other.

(b) Two numbers , one of them not more than twice of the other.

(c) Two numbers , one of them less than twice of the other.

(d) Two numbers , one of them is not less than twice of the other.

(7) The point which belongs to the region of the solution set of the inequalities :
$x + y \geq 5$, $x \geq 1$, $y \geq 2$ and makes the objective function $P = 2x + y$ has minimum value is ....

(a) (0 , 0) (b) (4 , 3) (c) (3 , 2) (d) (1 , 4)

(8) The maximum value of the function :
$P = 5x + 2y$ under conditions : $x \geq 0$, $y \geq 0$, $x + y \leq 7$, $x + 2y \leq 10$ is ....

(a) 10 (b) 26 (c) 35 (d) 70

(9) The point which belongs to the feasible region of the inequalities : $0 \leq x \leq 5$, $0 \leq y \leq 2$ and makes the objective function $P = 2x + 3y$ has maximum value is ....

(a) (4 , 5) (b) (6 , 1) (c) (0 , 0) (d) (5 , 2)

(10) The smallest value of the expression $3x - 2y$ under the conditions :
$-3 \leq x \leq 7$, $-6 \leq y \leq 5$ equals ....

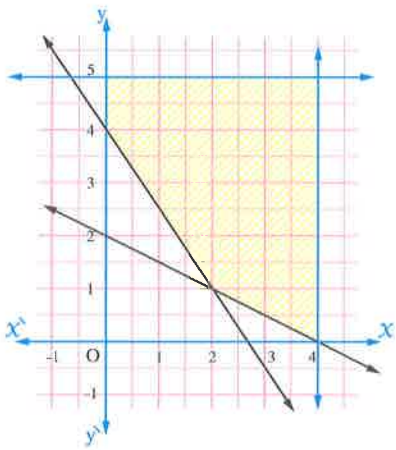
(a) 3 (b) – 19 (c) – 28 (d) 11

(11) If (a , b) belongs to the solution set of the inequality $x + 2y \geq 5$ where a , b are integers , then the least value of $2a + 4b =$ ....

(a) 5 (b) – 5 (c) 10 (d) 6

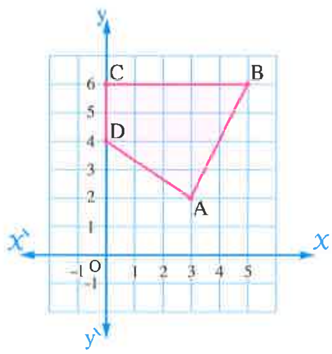
(12) The opposite figure represents the solution set of a system of inequalities , then the objective function $P = x + y$ is as small as possible at the point ....

(a) (0 , 0) (b) (1 , 2)
(c) (2 , 1) (d) (4 , 5)



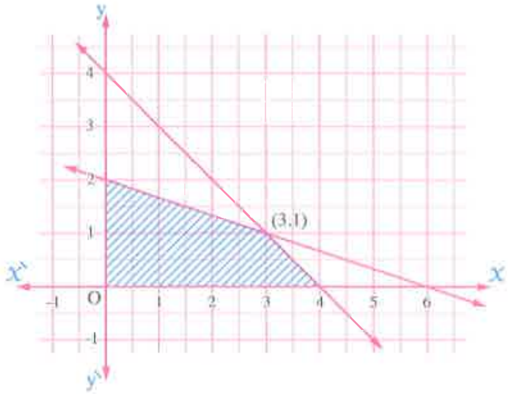
(13) The opposite figure :
Represents the solution set of a system of inequalities , then the smallest value of the objective function $P = 3x + 2y$ is ....

(a) 6 (b) 8
(c) 12 (d) 13



(14) **In the opposite figure :**

The shaded region represents the solution set of the system of the inequalities $x \geq 0$, $y \geq 0$, $x + 3y \leq 6$, $x + y \leq 4$, then the maximum value of the objective function $P = 2x + y$ equals ....



(a) 7 (b) 8

(c) 3 (d) 4

(15) A small factory produces metal furniture 20 cupboard weekly at most of two different kinds and the factory sells from the first kind at least 3 times what it sells from the second kind. Let the number of the first kind of cupboards = x and the number of the second kind of cupboards = y. Which of the following systems of inequalities modelling the previous data and rules ?

(a) $x \geq 0$, $y \geq 0$, $x + y \geq 20$, $x \leq 3y$

(b) $x \geq 0$, $y \geq 0$, $x + y \geq 20$, $x \geq 3y$

(c) $x \geq 0$, $y \geq 0$, $x + y \leq 20$, $3x \leq y$

(d) $x \geq 0$, $y \geq 0$, $x + y \leq 20$, $x \geq 3y$

(16) A factory produces 120 units at most from two different kinds of goods x and y respectively. If the number of units sold from the second kind is not less than half what is sold from the first kind. Which of the following inequalities represents the data and the previous constrains ?

(a) $x \geq 0$, $y \geq 0$, $x + y \leq 120$, $2y \leq x$

(b) $x \geq 0$, $y \geq 0$, $x + y \geq 120$, $y \leq 2x$

(c) $x \geq 0$, $y \geq 0$, $x + y \leq 120$, $2y \geq x$

(d) $x \geq 0$, $y \geq 0$, $x + y \leq 120$, $y \geq 2x$

(17) A bakehouse produces two kinds of cake , first kind needs 200 gm. of flour and 25 gm. of butter and the second kind needs 100 gm. of flour and 50 gm. of butter and if the available quantity of flour is only 4 kg. and available quantity of butter is only $1\frac{1}{4}$kg. Let the number of the first kind of cake = x and the number of the second kind of cake = y , then which of the following systems is modelling the given rules ?

(a) $x \geq 0$, $y \geq 0$, $2x + y \geq 40$, $x + 2y \geq 50$

(b) $x \geq 0$, $y \geq 0$, $2x + y \leq 40$, $x + 2y \leq 50$

(c) $x \geq 0$, $y \geq 0$, $x + 3y \geq 40$, $3x + y \geq 50$

(d) $x \geq 0$, $y \geq 0$, $x + 2y \geq 30$, $2x + y \leq 70$

Second Essay questions

1 **Represent each of the following systems graphically , then find the point that satisfies the objective function in each case :**

(1) $x + y \le 5$, $y \ge 1$, $x \ge 2$, the objective function $P = 2x + 3y$ is as small as possible. « (2 , 1) »

(2) $x \ge 0$, $y \ge 0$, $y + 2x \le 10$, $x + 4y \le 12$, the objective function $P = 2y + 5x$ is as great as possible. « (5 , 0) »

(3) $x \ge 0$, $y \ge 0$, $3y + x \ge 15$, $4x + 3y \ge 24$, the objective function $P = 3y + 2x$ is as small as possible. « (3 , 4) »

(4) $x - y \le 3$, $3x + 2y \ge -6$, $x \ge -2$, $y \le 5$, the objective function $P = 2x - 3y$ is as great as possible. « (0 , – 3) »

---

2 Find the maximum and minimum values of the objective function : $P = x + 3y - 5$ where (x , y) belongs to the feasible region of the inequalities system : $-3 \le x \le 3$, $-4 \le y \le 4$, $4x + 3y \le 12$, $4x + 3y \ge -12$ « 7 , – 17 »

---

3 Youssef knows that if he wants to keep fit, he should burn the extra calories by walking and running. He found that walking for one minute burns 6 calories, running for one minute burns 15 calories, if Youssef walks 10 – 20 minutes daily and runs 30 – 45 minutes daily and the available time for walking and running daily does not increase one hour, then how many minutes does Youssef need to walk and run to burn the maximum amount of calories ? « 15 , 45 »

---

4 A small factory produces metal furniture 20 cupboards weekly at most of two different kinds A and B. If the profit from kind "A" is 80 pounds, and the profit from kind B is 100 pounds. The factory sells from kind A at least 3 times what it sells from kind B. Find the number of cupboards , from each kind to satisfy the greatest possible profit to the factory. « 15 , 5 »

---

5 A farmer wants to breed chickens and ducks. The poultry place that he has can accommodate only 300 of them; while he wants the number of chickens not to be less than twice the number of ducks. If he earns one pound from each chicken and two pounds from each duck , then find how many chickens and ducks he can breed to get the maximum profit. « 200 , 100 »

---

6 One of the seafood shops sells two types of cooked fish A and B , and the requests from the shop owner are not less than 50 fish , as he does not consume more than 30 fish from the type (A) , and no more than 35 fish from the type (B). If the price of a fish from type A is 4 pounds and 3 pounds from type B. How much fish from each of the two types A and B must be used to achieve the lowest cost possible to buy ? « 15 , 35 »

7 One of the factories of musical instruments produces two types of blowing instruments , the first type needs 25 units of copper , 4 units of nickel and the second type needs 15 units of copper , 8 units of nickel. If the available quantities in the factory on a day were 95 units of copper , 32 units of nickel and the profit of the factory from the first type was 60 pounds and 48 pounds from the second type. Find the number of instruments which the factory should produce from each type to get the maximum profit. « 2 , 3 »

---

8 A farmer found that he can improve the quality of planting , if he used at least 16 units of Nitrates , 9 units of Phosphates in the process of fertilization for one kirate, there are two types of fertilizer A , B , its contents and cost of each shown in the following table :

The fertilizer	Number of units for each kilogram		Cost for each kilogram
	Nitrates	Phosphates	
A	4	1	170 Pt
B	2	3	150 Pt

Find the least cost of a mixture of the two fertilizers A and B , such that the farmer can provide a sufficient number of units of Nitrates and Phosphates to improve the quality of his plants. « 3 , 2 »

---

9 Suppose you manufacture and sell skin moisturizer, if manufacturing a unit of the normal moisturizer requires 2 cm^3 of oil, 1 cm^3 of cocoa butter, and manufacturing a unit of the excellent moisturizer requires 1 cm^3 of oil, 2 cm^3 of cocoa butter. You will gain 10 pounds for every unit of the normal kind, 8 pounds for every unit of the excellent kind. If you had 24 cm^3 of oil, 18 cm^3 of cocoa butter. What is the number of units you can manufacture from each kind to get the maximum possible profit and what is this profit ?

« 10 , 4 »

---

10 There are two packages of food substances. The first has 5 units of vitamins and gives 3 calories , while the second has 2 units of vitamins and gives 6 calories. Given that we need at least 25 units of vitamins and 39 calories, and the price of the unit of the first article is P.T 6 and of the second is P.T 8, then find the number of each article that should be bought to obtain what we need at the least cost. « 3 , 5 »

---

11 A factory produces two types of iron sheet offices, one of the workers collects each type and then another worker paints it. The first worker takes 2 hours to collect a unit of the first type, and 3 hours to collect a unit of the second type. While the second worker takes $1\frac{1}{2}$ hours to paint a unit of the first type , and 2 hours to paint a unit of the second type , if the first worker works at least 6 hours daily , while the second worker works at most 6 hours daily , the profit of the factory is 50 pounds to each unit of the two types. What is the number of units the factory should produce daily from each of the two types to achieve the maximum possible profit ? « 4 from the first type »

12 A factory produces two kinds of soap A and B. The production whose value is L.E. 100 of the kind A needs 30 kg. of raw material and 18 hours of working on machines and the production of whose value is L.E. 100 of the second kind B needs 20 kg. of the same raw material and 24 hours of working on machines.

Find the greatest value of the production that is produced of 75 kg. of raw material and the available time is 72 hours of working on machines. « L.E. 325 »

---

13 Two tailors produce two styles of blouses A and B

The first tailor designs clothes , while the second tailor sews them.

The first tailor takes one hour to design the style A and two hours to design the style B

The second tailor takes 3 hours to sew the style A and one hour to sew the style B

The first tailor works for 8 hours / day at most , while the second works for 9 hours / day at most. The profit of selling the blouse of style A is L.E. 10 and their profit of selling the blouse of style B is L.E. 15

Find the number of blouses of each style that they should produce daily to gain the maximum profit. « 2 , 3 »

Third Problems that measure high standard levels of thinking

1 A workshop for making furniture in which 72 workers at most can work. Some of them are trained and the others are under training. If it must be for each two trained workers , one worker under training works with them at least. If the quantity of production by the trained worker equals two and half time as the production of untrained worker ,

find the number of each of trained and untrained workers such that the workshop realizes the maximum quantity of production. « 48 , 24 »

---

2 Yousef and Samy work on one of machines to produce a certain production.

If Yousef produces the unit of production in an hour , while Samy produces two units of this production in an hour , but he can work for 2 hours at most more than the number of hours worked by Yousef.

If we know that the machine should be operated at least 6 hours daily to cover its expenses and 8 units should be produced of the production at least daily , find the least daily wages paid to Yousef and Samy if Yousef takes L.E. 5 per hour and Samy takes L.E. 8 per hour.

« 20 , 16 »

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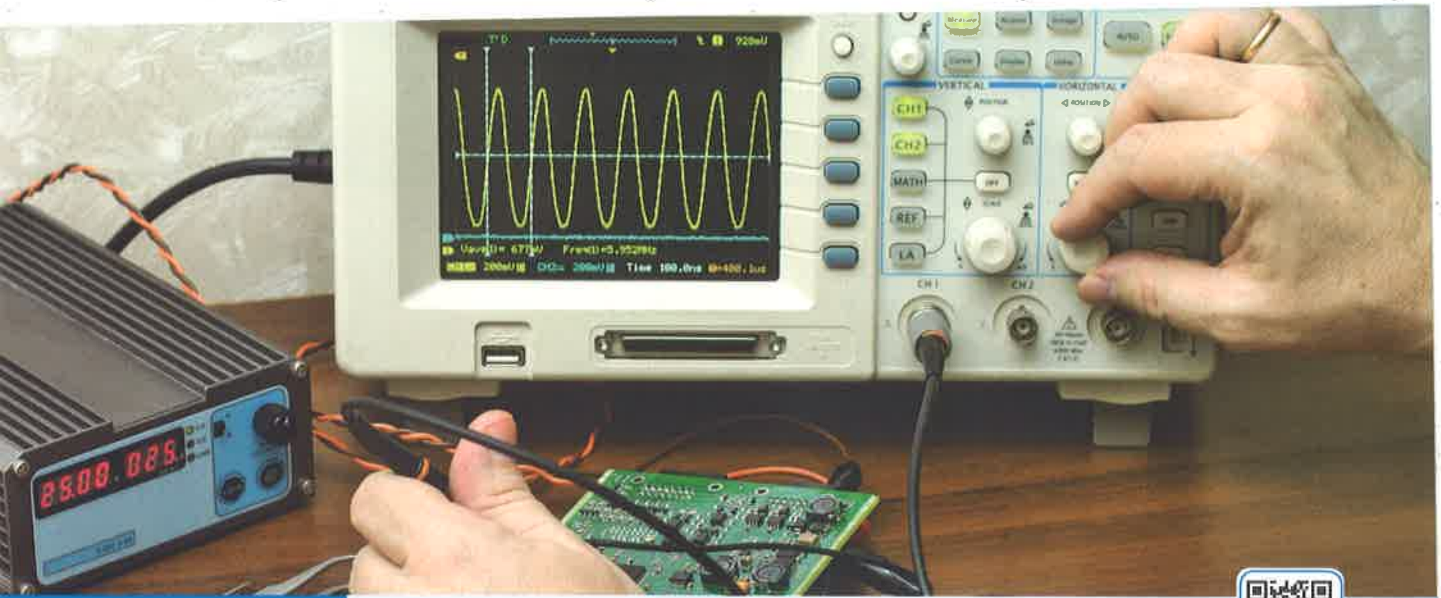
3 It is wanted to put two kinds of books A and B on a library shelf with length 96 cm. and can carry 20 kg.

If the weight of the book of any kind is 1 kg. and the thickness of the book of the kind A is 6 cm. and of kind B is 4 cm , find the number of books of each kind that should be put on the shelf such that its number is maximum.

(Give reasons for having more than one solution).

Unit 3

TRIGONOMETRY



Exercise Eight

Trigonometric identities

Test yourself

📖 From the school book

First Multiple choice questions

● **Choose the correct answer from the given ones :**

(1) Which of the following represents an identity ?

(a) $\cos \theta = \frac{\sqrt{3}}{2}$ (b) $\sin \theta = \cos \theta$

(c) $\sin (\pi - \theta) = \sin \theta$ (d) $\sin (2\pi - \theta) = -\frac{1}{2}$

(2) Which of the following relations represents an equation ?

(a) $\tan \left(\frac{3\pi}{2} + \theta\right) = -\cot \theta$ (b) $\cos \left(\frac{3\pi}{2} - \theta\right) = -\sin \theta$

(c) $\cos (-\theta) = \cos \theta$ (d) $\cot \theta = -\tan 30°$

(3) $\tan \theta \csc \theta =$ ....

(a) 1 (b) $\cos \theta$ (c) $\sec \theta$ (d) $\csc \theta$

(4) 📖 $\frac{\tan \theta \cot \theta}{\sec \theta}$ in the simplest form equals ....

(a) $\sin \theta$ (b) $\cos \theta$ (c) $\sec \theta$ (d) $\csc \theta$

(5) $1 - \frac{1}{\sec^2 37°} =$ ....

(a) $\tan^2 37°$ (b) $\cot^2 37°$ (c) $\cos^2 37°$ (d) $\sin^2 37°$

(6) $5 \cos^2 30° + 5 \sin^2 30° =$ ....

(a) 5 (b) 1 (c) 25 (d) 10

(7) $\frac{3}{\sin \theta} \times \frac{-2}{\csc \theta} =$ ....

(a) 6 (b) -6 (c) $\frac{1}{6}$ (d) $\frac{-1}{6}$

(8) $\sin^2 (180° - \theta) + \sin^2 (270° - \theta) =$ ....

(a) $2 \sin^2 \theta$ (b) $\cos^2 \theta$ (c) 1 (d) 10

(9) $(1 + \cot \theta)^2 - 2 \cot \theta =$ ....

(a) $\sec^2 \theta$ (b) $\cos^2 \theta$ (c) $\csc^2 \theta$ (d) $-\csc^2 \theta$

(10) $\sin \theta \cos \theta \tan \theta$ in the simplest form equals ....

(a) $\sin^2 \theta$ (b) $\cos^2 \theta$ (c) $\tan^2 \theta$ (d) $1 - \sin^2 \theta$

(11) $\frac{1 - \sin^2 \theta}{1 - \cos^2 \theta}$ in the simplest form equals ....

(a) -1 (b) 1 (c) $\tan^2 \theta$ (d) $\cot^2 \theta$

(12) $\frac{\sec^2 \theta - \tan^2 \theta}{\sin^2 \theta + \cos^2 \theta}$ in the simplest form equals ....

(a) 1 (b) $\tan^2 \theta$ (c) -1 (d) $\sec^2 \theta$

(13) $\cos (90° - \theta) \sec (\theta - 90°)$ in the simplest form equals ....

(a) 1 (b) -1 (c) $\sin^2 \theta$ (d) $\cot^2 \theta$

(14) The expression : $\frac{1 - \cos^2 \beta}{\sin^2 \beta - 1}$ in the simplest form equals ....

(a) $-\tan^2 \beta$ (b) $-\cos^2 \beta$ (c) $\tan^2 \beta$ (d) $\cot^2 \beta$

(15) $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$ in the simplest form equals ....

(a) $\tan^2 \theta$ (b) $\cot^2 \theta$ (c) 1 (d) $\cos^2 \theta$

(16) $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta =$ ....

(a) 1 (b) $\cot^2 \theta$ (c) $\csc^2 \theta$ (d) $\sec^2 \theta$

(17) $(\tan^2 \theta - \sec^2 \theta)^5 =$ ....

(a) 1 (b) -1 (c) 5 (d) -5

(18) $2 \sin^2 \theta + \cos^2 \theta + \frac{1}{\sec^2 \theta} =$ ....

(a) 2 (b) 1 (c) $\tan^2 \theta$ (d) $\sec^2 \theta$

(19) $\sin \theta \csc \theta + 2 \cos \theta \sec \theta + 3 \tan \theta \cot \theta =$ ....

(a) 1 (b) 3 (c) 5 (d) 6

(20) $(1 - \cos \theta)(1 + \cos \theta)(1 + \cot^2 \theta) =$ ....

(a) -1 (b) 1 (c) $\sin \theta$ (d) $\tan \theta$

(21) If $\csc^2 \theta = \frac{25}{9}$, then $\cot^2 \theta =$ ....

(a) $\frac{4}{3}$ (b) $\frac{5}{3}$ (c) $\frac{16}{9}$ (d) $\frac{9}{16}$

(22) If $\tan^2 \theta = 15$, then $\sec^2 \theta =$ ....

(a) 225 (b) 226 (c) 15 (d) 16

(23) If $\cot \theta = \frac{1}{3}$, then $\csc^2 \theta =$ ....

(a) $\frac{1}{9}$ (b) $\frac{4}{3}$ (c) $\frac{10}{9}$ (d) $\frac{3}{4}$

(24) If $\sin \theta + \csc \theta = 5$, then $\sin^2 \theta + \csc^2 \theta =$ ....

(a) 1 (b) 5 (c) 23 (d) 25

(25) If $\sin \theta - \cos \theta = \frac{4}{5}$ where $\theta \in]0 , \frac{\pi}{2}[$, then $\sin \theta \cos \theta =$ ....

(a) $\frac{1}{5}$ (b) $\frac{9}{25}$ (c) $\frac{41}{50}$ (d) $\frac{9}{50}$

(26) If $\csc \theta - \cot \theta = \frac{1}{3}$, then $\csc \theta + \cot \theta =$ ....

(a) 3 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

(27) If $\tan \theta + \cot \theta = 3$, then $\tan^2 \theta + \cot^2 \theta =$ ....

(a) 9 (b) 8 (c) 7 (d) 1

(28) $\frac{\sin^4 \theta - \cos^4 \theta}{\sin^2 \theta - \cos^2 \theta} =$ ....

(a) $\tan \theta \sec \theta$ (b) $\sin^2 \theta - \cos^2 \theta$ (c) $\csc^2 \theta - \cot^2 \theta$ (d) $\tan^2 \theta - \sec^2 \theta$

(29) If $\sin^n \theta \times \csc^6 \theta = \sin \theta$, then $n =$ ....

(a) 4 (b) 5 (c) 6 (d) 7

(30) $\frac{\sin^3 \theta + \cos^3 \theta}{1 - \sin \theta \cos \theta} =$ ....

(a) $\cos \theta - \sin \theta$ (b) $\cos \theta + \sin \theta$ (c) $2 \csc^2 \theta$ (d) $2 \sin^2 \theta$

(31) $\frac{1 - \cos X}{\sin X} + \frac{\sin X}{1 - \cos X} =$ ....

(a) $2 \sec X$ (b) $2 \csc X$ (c) $2 \cot X$ (d) $2 \sin X$

(32) If $A = 2 \sin X + \cos X$, $B = 2 \cos X - \sin X$, then $A^2 + B^2 =$ ....

(a) 4 (b) 5

(c) $8 \sin X \cos X$ (d) $4 \sin^2 X + 5 \cos^2 X$

(33) If θ , $\propto$ are two acute angles and $\theta + \propto = 90°$, then $\sin^2 \theta + \sin^2 \propto =$ ....

(a) zero (b) 1 (c) $2 \sin^2 \theta$ (d) $2 \cos^2 \theta$

(34) In Δ ABC , if $\sin^2 A + \cos^2 B = 1$, then Δ ABC is ....

(a) equilateral. (b) isosceles. (c) scalene. (d) right-angled.

(35) If $\tan \theta = 4$ then $\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} =$ ....

(a) $\frac{17}{15}$ (b) 1 (c) $\frac{-7}{15}$ (d) -1

(36) $\frac{\cos^2 \theta - \sin \theta - 1}{1 + \sin \theta} =$ ....

(a) -1 (b) $\cos \theta$ (c) $-\sin \theta$ (d) $\sin \theta$

Second Essay questions

1 **Write in the simplest form each of the following expressions "where θ is the measure of an angle in which all trigonometric functions and their reciprocals are defined at it" :**

(1) $\frac{1}{\cos^2\theta} - \frac{1}{\cot^2\theta}$

(2) $\cos\left(\frac{\pi}{2}-\theta\right)\sec\left(\frac{\pi}{2}-\theta\right)$

(3) $\sin(\pi-\theta)\csc(\pi-\theta)$

(4) $\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos(2\pi-\theta)}$

(5) $(\sin\theta+\cos\theta)^2 - 2\sin\theta\cos\theta$

(6) $\sin\left(\frac{\pi}{2}+\theta\right)\sec(-\theta)$

(7) $\cos^2\theta\sec\theta\csc\theta$

(8) $\sin\theta\csc\theta - \cos^2\theta$

(9) $\sin\theta\sin\left(\frac{\pi}{2}-\theta\right)\tan\theta$

(10) $\sin\left(\frac{\pi}{2}-\theta\right)\cos\theta - \cos\left(\frac{\pi}{2}+\theta\right)\sin(\pi-\theta)$

(11) $\sin\left(\frac{\pi}{2}-\theta\right)\cot\left(\frac{\pi}{2}-\theta\right)\csc(\pi-\theta)$

(12) $\sin\theta\cos\theta(\tan\theta+\cot\theta)$

(13) $\frac{1+\cot^2\left(\frac{3\pi}{2}-\theta\right)}{1+\tan^2\left(\frac{\pi}{2}-\theta\right)}$

(14) $\frac{\sin\theta\cos\theta}{\tan\theta} + \frac{\tan\theta}{\sec\theta\csc\theta}$

2 **Prove the validity of each of the following identities :**

(1) $\tan\theta+\cot\theta = \sec\theta\csc\theta$

(2) $\csc\theta-\sin\theta = \cos\theta\cot\theta$

(3) $2\sin^2\theta+\cos^2\theta = 1+\sin^2\theta$

(4) $\sin^2\propto + \tan^2\propto\sin^2\propto = \tan^2\propto$

(5) $\sin(90^\circ-\mu)\cos\mu = 1-\sin^2\mu$

(6) $\cot^2\mu - \cos^2\mu = \cot^2\mu\cos\mu^2$

(7) $\sec^2\beta+\csc^2\beta = \sec^2\beta\csc^2\beta$

(8) $\sec\theta - \sin\theta\tan\theta = \cos\theta$

(9) $\tan^2\theta+\cot^2\theta-(\sec^2\theta+\csc^2\theta) = -2$

(10) $\tan^2\theta\cos^2\theta+\cot^2\theta\sin^2\theta = 1$

(11) $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2 = 2$

(12) $\sin^3\theta\csc\theta+\cos^3\theta\sec\theta = 1$

(13) $(1-\sin\theta)(\sec\theta+\tan\theta) = \cos\theta$

(14) $\sin\theta\sin(90^\circ-\theta)\tan\theta = 1-\cos^2\theta$

(15) $\cos^2\theta\tan^2\theta+\cos^2\theta+\cot^2\theta = \csc^2\theta$

(16) $\sin^4\theta-\cos^4\theta+\cos^2\theta-\sin^2 0$

(17) $\sin^2\propto\sin^2\beta+\sin^2\propto\cos^2\beta+\cos^2\propto = 1$

3 **Prove the validity of each of the following identities :**

(1) $\frac{\cos\theta\times\tan\theta}{\csc\theta} = 1-\cos^2\theta$

(2) $\frac{\cot\theta}{1+\cot^2\theta} = \sin\theta\times\cos\theta$

(3) $\frac{\cos\theta}{1+\sin\theta}+\tan\theta = \sec\theta$

(4) $\frac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

(5) $\frac{1-\sin^2\theta}{1-\cos^2\theta} = \csc^2\theta - 1$

(6) $\frac{1-\tan^2\theta}{1+\tan^2\theta} = 2\cos^2\theta - 1$

(7) 📖 $\frac{\csc\theta}{\cos\theta}(1-\sin^2\theta) = \cot\theta$

(8) 📖 $\frac{1}{\sin^2(90^\circ-\theta)} - \tan^2\theta = 1$

(9) 📖 $\frac{1+\tan^2\theta}{\sec^4\theta} = 1-\sin^2\theta$

(10) 📖 $\frac{1}{1+\tan^2\alpha} - \frac{1}{1+\tan^2\beta} = \cos^2\alpha - \cos^2\beta$

(11) $\frac{\cos^4\theta-\cos^2\theta}{\sin^4\theta-\sin^2\theta} = \sec^2\theta - \tan^2\theta$

(12) 📖 $(\sec\theta-\tan\theta)^2 = \frac{1-\sin\theta}{1+\sin\theta}$

(13) 📖 $\frac{1}{1+\cot\theta} = \frac{\tan\theta}{1+\tan\theta}$

(14) $\frac{\sin^3\alpha+\cos^3\alpha}{\sin\alpha+\cos\alpha} + \frac{\sin^3\alpha-\cos^3\alpha}{\sin\alpha-\cos\alpha} = 2$

(15) 📖 $\frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos^2\theta+\cos\theta\sin^2\theta} = \csc\theta - \sec\theta$

(16) $\frac{\sin\theta\sin(90^\circ-\theta)}{\tan\theta} + \frac{\tan(180^\circ+\theta)}{\sec\theta\csc\theta} = 1$

(17) $(\sin\theta+\csc\theta)^2 + (\cos\theta+\sec\theta)^2 = \tan^2\theta + \cot^2\theta + 7$

(18) $\frac{1-(\sin\theta+\cos\theta)^2}{\sin\theta\cos\theta-\cot\theta} = 2\tan^2\theta$

---

4 If $\frac{3\cos\theta-2\sin\theta}{3\cos\theta+2\sin\theta} = \frac{3}{2}$ **, then find the value of :** $\tan\theta$ « $-\frac{3}{10}$ »

---

5 If $\sin\theta+\cos\theta = \frac{3}{2}$ **, then find the value of :** $\sin\theta\cos\theta$, where $\theta \in \left]0 , \frac{\pi}{2}\right[$ « $\frac{5}{8}$ »

---

6 If $\sec\theta - \tan\theta = \frac{1}{4}$ **, then find the value of each of :** $\sec\theta$ and $\tan\theta$ « $\frac{17}{8}$, $\frac{15}{8}$ »

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7 If $\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} = \frac{5}{4}$ **, then prove that :** $\sin\theta\cos\theta = \frac{9}{32}$

---

8 If $\tan\theta+\cot\theta = 5$ **, then find the numerical value of each of the following :**

(1) $\tan^2\theta+\cot^2\theta$ « 23 »

(2) $\tan^3\theta+\cot^3\theta$ « 110 »

(3) $\tan\theta-\cot\theta$ « $\pm\sqrt{21}$ »

(4) $\tan^2\theta-\cot^2\theta$ « $\pm 5\sqrt{21}$ »

Unit 3

Third Problems that measure high standard levels of thinking

1 Choose the correct answer from those given :

(1) If $x + y = 30°$, then : $\tan(x + 2y) \tan(2x + y) =$ ....

(a) 1 (b) zero (c) -1 (d) $2\sqrt{3}$

(2) If $\sin\theta = \frac{a}{b}$, $\theta \in [0 , \frac{\pi}{2}[$, then $\sqrt{1 + \tan^2\theta} =$ ....

(a) $\frac{a}{\sqrt{a^2 - b^2}}$ (b) $\frac{a}{\sqrt{a - b}}$ (c) $\frac{a}{\sqrt{1 + a^2}}$ (d) $\frac{b}{\sqrt{b^2 - a^2}}$

(3) If $\frac{\pi}{2} < \theta < \pi$, then $\sqrt{4 + 4\tan^2\theta} =$ ....

(a) $2\sec\theta$ (b) $2\cos\theta$ (c) $-2\sec\theta$ (d) $-2\cos\theta$

(4) If $\sin\theta$, $\cos\theta$ are the two roots of the equation $2x^2 + bx - 1 = 0$, then $b =$ ....

(a) zero (b) 2 (c) 3 (d) -4

(5) If $3\sin\theta + 4\cos\theta = 5$, then $3\cos\theta - 4\sin\theta =$ ....

(a) 5 (b) 4 (c) 2 (d) zero

(6) If $\theta \in]0 , \frac{\pi}{2}[$ and $\tan\theta + \cot\theta = 8$, then $\sin\theta + \cos\theta =$ ....

(a) $\frac{\sqrt{5}}{4}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{5}{2}$ (d) $\frac{5}{4}$

(7) If $\theta \in]0 , \frac{\pi}{2}[$, then $\sqrt{\sec^2\theta + \csc^2\theta} =$ ....

(a) $\frac{\sec\theta}{\csc\theta}$ (b) $\tan\theta + \cot\theta$ (c) $\sec\theta - \tan\theta$ (d) $\sec\theta + \tan\theta$

(8) If $\sin A + \sin^2 A = 1$, then $\cos^2 A + \cos^4 A =$ ....

(a) -1 (b) 1 (c) $\sin^2 A$ (d) 2

(9) $\sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 90°$ equals ....

(a) $44\frac{1}{2}$ (b) 45 (c) $45\frac{1}{2}$ (d) 46

(10) $\frac{\cos^2 1° + \cos^2 2° + \cos^2 3° + \cdots + \cos^2 180°}{\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 180°} =$ ....

(a) -90 (b) -88 (c) 88 (d) 90

---

2 Prove that : $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

---

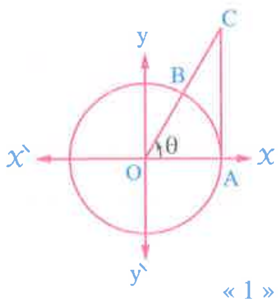
3 In the opposite figure :

A unit circle of centre O ,

$BC = \sin\theta$, $\overline{AC}$ is a tangent to

the circle at A

Find the value of : $\cos^3\theta + \sin\theta$



« 1 »

Exercise Nine
Solving trigonometric equations

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) If $0° \le \theta < 360°$ and $\sin \theta + 1 = 0$, then $\theta =$ ....

(a) 0° (b) 90° (c) 180° (d) 270°

(2) If $0° \le \theta < 360°$ and $\cos \theta + 1 = 0$, then $\theta =$ ....

(a) 90° (b) 180° (c) 270° (d) 360°

(3) If $0° \le \theta < 360°$ and $\csc \theta - 1 = 0$, then $\theta =$ ....

(a) 0° (b) 90° (c) 180° (d) 270°

(4) If $2 \sin \theta - \sqrt{3} = 0$, where $\theta \in]0 , 2\pi[$, then $\theta =$ ....

(a) 30° or 150° (b) 60° or 120° (c) 150° or 210° (d) 120° or 240°

(5) If $2 \cos \theta + 1 = 0$, where θ is the measure of the greatest positive angle , $\theta \in [0 , 2\pi[$, then $\theta =$ ....

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

(6) If $\sec(-\theta) = 2$ where $\theta \in [0 , \pi[$, then $\theta =$ ....

(a) 60° (b) 30° (c) 120° (d) 150°

(7) The solution set of the equation : $\sin \theta - \sqrt{3} \cos \theta = 0$, where $\theta \in]\pi , \frac{3}{2}\pi[$ is ....

(a) $\{\frac{4}{3}\pi\}$ (b) $\{\frac{7}{6}\pi\}$ (c) $\{\frac{5}{4}\pi\}$ (d) $\{\frac{1}{3}\pi\}$

(8) The solution set of the equation : $\sqrt{3} \tan \theta = 1$, where $90° < \theta < 270°$ is ....

(a) {30°} (b) {150°} (c) {210°} (d) {240°}

(9) The solution set of the equation : $\sin \theta + \cos \theta = 0$, where $180° < \theta < 360°$ is ....

(a) $\{210°\}$ (b) $\{225°\}$ (c) $\{240°\}$ (d) $\{315°\}$

(10) If $0° \le \theta < 180°$, $\cot \theta = 1$, then $\theta =$ ....

(a) 30° (b) 45° (c) 60° (d) 135°

(11) If $\theta \in [0 , \frac{\pi}{2}[$, $\sin \theta \cot \theta = \frac{1}{2}$, then the solution set is ....

(a) $\varnothing$ (b) $\{\frac{\pi}{3}\}$ (c) $\{\frac{4\pi}{3}\}$ (d) $\{\frac{5\pi}{3}\}$

(12) The solution set of the equation : $\sin^2 \theta + 1 = 0$, $\theta \in [0 , \pi[$ is ....

(a) $\{\frac{\pi}{2}\}$ (b) $\{\frac{\pi}{4}\}$ (c) $\{\pi\}$ (d) $\varnothing$

(13) If $180° \le \theta < 360°$ and $2 \cos \theta + 1 = 0$, then $\theta =$ ....

(a) 210° (b) 240° (c) 300° (d) 330°

(14) The general solution of the equation : $\sqrt{3} \tan \theta - 1 = 0$ is .... (where $n \in \mathbb{Z}$)

(a) $\frac{\pi}{6} + n\pi$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $\frac{\pi}{3} + n\pi$ (d) $2n\pi \pm \frac{\pi}{3}$

(15) The general solution of the equation : $\cos \theta = \frac{1}{2}$ is .... (where $n \in \mathbb{Z}$)

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $\frac{\pi}{6} + n\pi$ (d) $\frac{\pi}{3} + n\pi$

(16) The general solution of the equation :

$\cot\left(\frac{\pi}{2} - \theta\right) = \sqrt{3}$ is .... (where $n \in \mathbb{Z}$)

(a) $\frac{\pi}{3} + n\pi$ (b) $\frac{\pi}{3} + 2n\pi$

(c) $\frac{4\pi}{3} + 2n\pi$ (d) $\frac{\pi}{2} + 2n\pi$ or $\frac{4\pi}{3} + 2n\pi$

(17) If $5 \sin x = 12 \cos x$ where $x \in [0 , \pi]$, then x equals approximately ....

(a) 157° 22` 48`` (b) 112° 37` 12`` (c) 22° 37` 12`` (d) 67° 22` 48``

(18) If $3^{\sin \theta} = 1$ where $\theta \in]0 , 2\pi[$, then : $\theta =$ ....°

(a) 45 (b) 90 (c) 180 (d) 270

(19) If $\theta \in [0 , 2\pi[$, then the solution set of the equation : $7^{\sin \theta} = \frac{1}{49}$ is ....

(a) $\{30°\}$ (b) $\{30° , 150°\}$ (c) $\{210° , 330°\}$ (d) $\varnothing$

(20) If $\sin x = \frac{-1}{2}$, $\cos x = \frac{\sqrt{3}}{2}$ where $x \in [0 , 2\pi[$, then $x =$ ....

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

(21) If $X \in [0, 2\pi[$, then the solution set of the equation : $\cos X = \frac{1}{2}$ is the same solution set of the equation ....

(a) $\tan X = 2 \sin X$ (b) $2 \cos^2 X = \cos X$

(c) $2 \cos^2 X + 3 \cos X = 2$ (d) $\cos\left(X - \frac{1}{2}\right) = 0$

(22) If $\theta \in [0, 2\pi[$, then the number of solutions of the equation $2 \sin \theta = 3$ is ....

(a) zero (b) 1 (c) 2 (d) 3

(23) If $\theta \in [0, 2\pi[$, then the solution set of the equation : $\tan(\theta - 90°) = \sqrt{3}$ is ....

(a) $\{30°, 210°\}$ (b) $\{150°, 210°\}$ (c) $\{150°, 330°\}$ (d) $\{210°, 330°\}$

(24) If $0 \le X < 2\pi$, then the solution set of the equation $\cos\left(X - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ is ....

(a) $\left\{\frac{\pi}{4}, \frac{-\pi}{4}\right\}$ (b) $\left\{\frac{\pi}{12}, \frac{\pi}{4}\right\}$ (c) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}\right\}$ (d) $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$

(25) If $\theta \in [0, 2\pi[$, then the solution set of the equation : $2 \cos^2 \theta - 1 = 0$ is ....

(a) $\{45°, 135°, 225°, 315°\}$ (b) $\{45°, 315°\}$

(c) $\{135°, 225°\}$ (d) $\{45°, 225°\}$

(26) If $\theta \in [0, 2\pi[$, then the solution set of the equation : $\cos^2 \theta - \cos \theta = 0$ is ....

(a) $\{0°, 90°\}$ (b) $\{0, 90°, 180°\}$ (c) $\{0°, 90°, 270°\}$ (d) $\{90°, 270°\}$

(27) If $\theta \in [0, 2\pi[$, then the solution set of the equation : $2 \sin \theta \cos \theta + 3 \cos \theta = 0$ is ....

(a) $\{0°, 90°, 270°\}$ (b) $\{90°, 270°\}$

(c) $\{0°, 90°\}$ (d) $\{0°, 180°\}$

(28) If $0 \le X < 2\pi$, then the solution set of the equation $\frac{\cos^2 X - \sin^2 X}{1 - \sin^2 X} + 2 = 0$ is ....

(a) $\left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$ (b) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

(c) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ (d) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{3}\right\}$

(29) If $0° < \theta \le 360°$, then the solution set of the equation $\sin(2\theta + 30°) = \sqrt{3} \cos(2\theta + 30°)$ is ....

(a) $\{30°, 120°\}$ (b) $\{60°, 240°\}$

(c) $\{15°, 60°, 120°, 240°\}$ (d) $\{15°, 105°, 195°, 285°\}$

(30) If $\theta \in [0, 2\pi[$, then the solution set of the equation $\sin \theta + \csc \theta = 2$ equals ....

(a) $\left\{\frac{\pi}{2}\right\}$ (b) $\left\{\frac{\pi}{3}\right\}$ (c) $\{\pi\}$ (d) $\left\{\frac{3\pi}{2}\right\}$

(31) The number of solutions of the equation : $\cos^2\theta - 4\cos\theta + 4 = 0$ equals ....

(a) zero (b) 1 (c) 2 (d) 3

(32) If $\theta \in [0 , 2\pi[$, then the solution set of the equation : $2\cos^2\theta - 5\cos\theta - 3 = 0$ is ....

(a) $\{30° , 150°\}$ (b) $\{60° , 120°\}$ (c) $\{60° , 240°\}$ (d) $\{120° , 240°\}$

(33) If $\theta \in [0 , \frac{3}{2}\pi[$, then the solution set of the equation : $2\sin^2\theta + 3\sin\theta + 1 = 0$ is ....

(a) $\{240°\}$ (b) $\{210°\}$ (c) $\{225°\}$ (d) $\{330°\}$

(34) If $\theta \in [0 , \pi[$, then the solution set of the equation : $3\sec^2\theta - 7\sec\theta + 2 = 0$ is ....

(a) $\{30°\}$ (b) $\{60°\}$ (c) $\{150°\}$ (d) $\{120°\}$

Second Essay questions

1 Find the general solution of each of the following equations :

(1) $\sin\theta = \frac{1}{2}$

(2) $\cos\theta = \frac{\sqrt{2}}{2}$

(3) $\tan\theta = \sqrt{3}$

(4) $\sin\theta = \frac{-\sqrt{3}}{2}$

(5) $\cos\theta = \frac{-1}{2}$

(6) $\tan\theta = -1$

(7) $\csc\theta = -2$

(8) $\sec\theta = \sqrt{2}$

(9) $\cot\theta = -\sqrt{3}$

(10) $2\cos\theta - \sqrt{3} = 0$

(11) $2\sin\theta + \sqrt{2} = 0$

(12) $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$

(13) $2\sin\theta\cos\theta + 3\cos\theta = 0$

(14) $\sin\theta\cos\theta = \frac{\sqrt{3}}{2}\sin\theta$

(15) $2\sin^2\theta - \sqrt{2}\sin\theta = 0$

(16) $\cos^2\theta - \cos\theta = 0$

(17) $2\sin^2\theta = \sin\theta$

(18) $\sqrt{2}\sin\theta\cos\theta - \sin\theta = 0$

(19) $2\cos^2\theta + \cos\theta = 0$

(20) $2\cos^2\theta + 3\cos\theta + 1 = 0$

2 If $\theta \in [0 , 2\pi[$, find the solution set of each of the following equations :

(1) $2\cos\theta - 1 = 0$

(2) $2\sin\theta + 1 = 0$

(3) $2\cos\theta + \sqrt{3} = 0$

(4) $\tan\theta - 1 = 0$

(5) $\sqrt{2}\cos\theta + 2 = 0$

(6) $3\csc\theta + 2 = 0$

(7) $4\sin\theta + 3 = 0$

(8) $\sec\theta + \sqrt{2} = 0$

(9) $4 \tan \theta - 5 = 0$

(10) $\sin \theta - 2 \cos \theta = 0$

(11) $\cos \theta + 1 = 0$

(12) $\sin \theta \cos \theta = 0$

(13) $\frac{\sin 25^\circ}{\cos 65^\circ} + \tan (90^\circ - \theta) = 0$

(14) $\cos (\theta - 50^\circ) = \frac{1}{2}$

(15) $4 \sin^2 \theta = 3$

(16) $2 \sin^2 \theta = \sin \theta$

(17) $3 \sec^2 \theta = 4$

(18) $\sin^2 \theta - \cos^2 \theta = 0$

(19) $3 \cos^2 \theta + 2 \sin \theta \cos \theta = 0$

(20) $4 \sin^2 \theta - 3 \sin \theta \cos \theta = 0$

(21) $4 \tan^2 \theta + 3 \tan \theta = 0$

(22) $2 \sin^2 \theta + \sin \theta - 1 = 0$

(23) $\tan^2 \theta - \frac{1}{\cot \theta} = 0$

3 **Solve each of the following equations in the interval $\left[0 , \frac{3\pi}{2}\right[$:**

(1) $\tan^2 \theta - \tan \theta = 0$

(2) $2 \sin \theta \cos \theta - \cos \theta = 0$

4 **Solve the equation :** $\sin \theta \cos \theta - \frac{1}{2} \cos \theta = 0$, where $0^\circ < \theta < 180^\circ$

5 **Find the general solution of each of the following equations :**

(1) $\cos \theta = \sin 2\theta$

(2) $\cos 2\theta = \sin \theta$

(3) $\cos 5\theta = \sin 4\theta$

(4) $\sec 4\theta = \csc 2\theta$

6 **Find the solution set of each of the following equations in the interval $[0 , \pi[$:**

(1) $2 \sin^2 \theta - 5 \sin \theta + 2 = 0$

(2) $6 \cos^2 \theta - 5 \cos \theta + 1 = 0$

(3) $4 \sin^2 \theta + 8 \sin \theta + 3 = 0$

(4) $2 \tan^2 \theta - \tan \theta - 1 = 0$

(5) $3 \sec^2 \theta - 7 \sec \theta + 2 = 0$

7 **Find the measure of the smallest positive angle which satisfies the two equations :**

$2 \cos \theta + 1 = 0$, $\tan \theta - \sqrt{3} = 0$ « 240° »

8 **Find the solution set of the equation :**

$\sin \left(\frac{9\theta}{4}\right) = \frac{1}{\sqrt{2}}$, where $\theta \in \left]0 , \frac{\pi}{2}\right[$ « {20° , 60°} »

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) The number of solutions of the equation : $\sin x = 0$ where $x \in [0 , 6\pi[$ is ....

(a) 2 (b) 4 (c) 6 (d) 8

(2) If $\sin A + \sin B = 2$, then ....

(a) $\cos A + \cos B =$ zero (b) $\cos A - \cos B = 1$

(c) $\sin A - \sin B = 1$ (d) $\sin (A + B) = -1$

(3) The solution set of the equation : $\cos X + \sin X = 2$ where $X \in [0 , 2\pi[$ is ....

(a) $\left\{\frac{\pi}{2}\right\}$ (b) {zero} (c) $\left\{0 , \frac{\pi}{2}\right\}$ (d) $\varnothing$

(4) If $0 \leq X \leq 360°$, then the number of solutions of the equatain $3 \sin X = \tan X$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(5) If $\tan \theta + \cot \theta = 2$, then $\tan^{2019} \theta + \cot^{2019} \theta =$ ....

(a) zero (b) 1 (c) 2 (d) 3

(6) The values of θ which makes the roots of the quadratic equation :

$X^2 + 2X + 2 \cos \theta = 0$ are equal where $\theta \in [0 , 2\pi[$ are ....

(a) $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ or $\frac{5\pi}{3}$ (d) $\frac{\pi}{6}$ or $\frac{5\pi}{6}$

(7) The sum of the solutions of the equation :

$\sin^2 X - \sin X \cos X = \cos X - \sin X$, where $X \in]0 , 2\pi[$ is ....

(a) $\frac{3}{2}\pi$ (b) 2π (c) 3π (d) 4π

(8) The sum of the solutions of the equation :

$\cos 2X = -\sin 2X$ where $X \in [0 , 2\pi[$ is ....

(a) $\frac{3\pi}{2}$ (b) $\frac{3\pi}{8}$ (c) $\frac{9\pi}{2}$ (d) $\frac{11\pi}{2}$

(9) If $\theta \in \left[0 , \frac{\pi}{2}\right]$ and the equation $X^2 - \left(\tan \theta + \frac{1}{\tan \theta}\right) X + 1 = 0$

has positive unique root , then $\sin \theta \cos \theta =$ ....

(a) 4 (b) $\frac{1}{4}$ (c) 2 (d) $\frac{1}{2}$

2 If $\theta \in [0 , 2\pi[$, then find the solution set of each of the following equations :

(1) $2 \cos^2 \theta + 11 \sin \left(\frac{\pi}{2} - \theta\right) - 6 = 0$

(2) $4 \sin \theta + \csc \theta - 4 = 0$

(3) $\cos \theta + \sec \theta \cos^2 \theta = \sqrt{3}$

(4) $\tan^4 \theta - 3 \tan^2 \theta + 2 = 0$

(5) $2 \sin \theta \cos \theta + 2 \sin \theta - \cos \theta - 1 = 0$

(6) $\sqrt{3} \tan^2 \theta - \left(1 + \sqrt{3}\right) \tan \theta + 1 = 0$

(7) $\tan^2 \theta + \sec^2 \theta = 7$

(8) $6 \tan^2 \theta + 5 \sec \theta + 5 = 0$

Exercise Ten

Solving the right-angled triangle

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) The right angled triangle can be solved in each of the following cases except if the given is ....

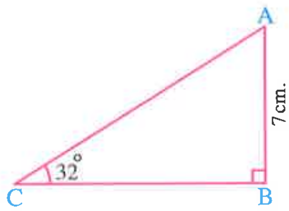
(a) the length of two sides in the triangle.

(b) the length of two sides and the measure of their included angle.

(c) measure of two angles.

(d) the length of one side of the right angle and the length of the hypotenuse.

(2) **In the opposite figure :**

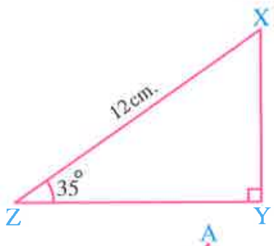


AC $\simeq$ .... cm.

(a) 13.2 (b) 8.3

(c) 3.7 (d) 5.9

(3) **In the opposite figure :**

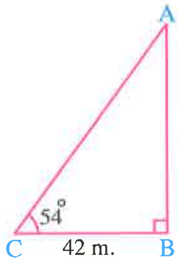


XY = ... cm.

(a) 9.8 (b) 6.9

(c) 8.4 (d) 14.6

(4) **In the opposite figure :**



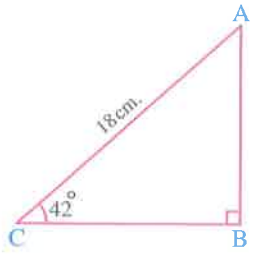
The length of $\overline{AB} \simeq$ .... to the nearest metre.

(a) 56 (b) 57

(c) 58 (d) 59

Unit 3

(5) In the opposite figure :

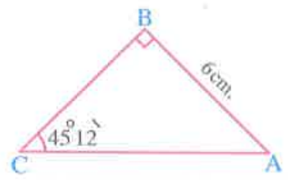


The length of $\overline{BC} \simeq$ .... cm.

(a) 12 (b) 13

(c) 16 (d) 24

(6) In the opposite figure :

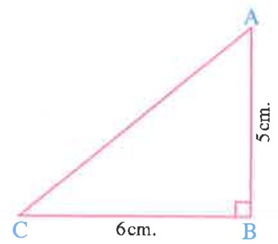


The length of $\overline{BC} \simeq$ .... cm.

(a) 6 (b) 4

(c) 9 (d) 5

(7) In the opposite figure :



m (∠ C) = ... °

(a) $56° 27'$ (b) $39° 48'$

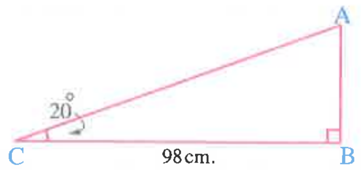
(c) $33° 33'$ (d) $50° 12'$

(8) Triangle ABC is a right angled triangle at B , AB = 5 cm. , BC = $5\sqrt{3}$ cm.

, then m (∠ C) = ...

(a) 60° (b) 30° (c) 45° (d) 56°

(9) In the opposite figure :



AB = ... cm.

(a) 98 cot 20° (b) 98 sin 20°

(c) 98 csc 20° (d) 98 tan 20°

(10) If Δ ABC is right angled triangle at B , m (∠ A) = 0.925^{rad} , BC = 8 cm.

, then AC ≃ .... cm.

(a) 10 (b) 13 (c) 6 (d) 11

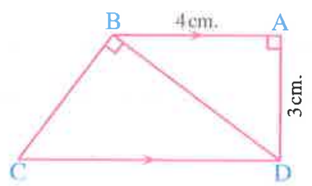
(11) If Δ ABC is right angled triangle at B , m (∠ C) = $54° 13'$, BC = 20 cm.

, then the length of $\overline{AB} \simeq$ .... cm.

(to the nearest 1 decimal place).

(a) 16.2 (b) 11.7 (c) 14.4 (d) 27.7

(12) In the opposite figure :

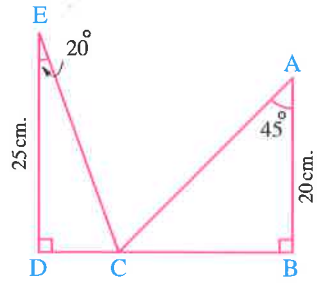


Length of $\overline{BC}$ = ...

(a) 5 cm. (b) $6\frac{2}{3}$ cm.

(c) $3\frac{3}{4}$ cm. (d) 3 cm.

(13) In the opposite figure :



Length of $\overline{BD} \simeq$ ....

(a) 9 cm.

(b) 29 cm.

(c) 23 cm.

(d) 28.5 cm.

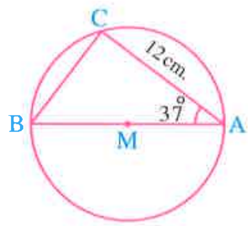
(14) ABC is an isosceles triangle , AB = AC = 14.8 cm. , m ($\angle$ A) = $64^\circ 32'$

, then the length of $\overline{BC} \simeq$ .... cm.

(to the nearest 1 decimal place).

(a) 25.2 (b) 15.8 (c) 18.7 (d) 25.8

(15) 📖 The opposite figure shows the circle of centre M , $\overline{AB}$ is a diameter in it , if AC = 12 cm. , m ($\angle$ A) = 37°

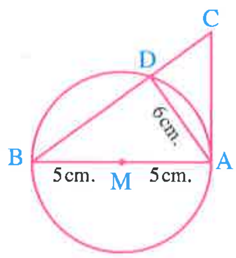


, then the length of the radius of the circle to the nearest two decimal places $\simeq$ .... cm.

(a) 7.51 (b) 9.97

(c) 7.96 (d) 4.79

(16) In the opposite figure :



M is a circle with radius length is 5 cm.

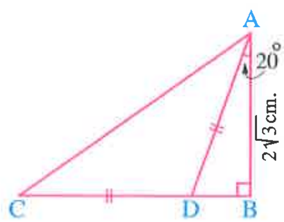
, $\overleftrightarrow{AC}$ is a tangent at A

, AD = 6 cm. , then m ($\angle$ CAD) $\simeq$ ....

(a) 53° (b) 31°

(c) 39° (d) 37°

(17) In the opposite figure :



The length of $\overline{AC} \simeq$ .... cm.

(a) 6 (b) 10

(c) 4 (d) 5

(18) 📖 ABC is a triangle , draw $\overrightarrow{AD} \perp \overline{BC}$, if AD = 6 cm.

, m ($\angle$ B) = 52°

, m ($\angle$ C) = 28° , then the length of $\overline{BC} \simeq$ .... cm. (to the nearest cm.)

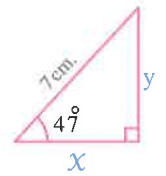
(a) 20 (b) 16

(c) 17 (d) 18

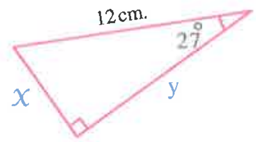
Second Essay questions

1 Find the value of each of x and y in each of the following figures :

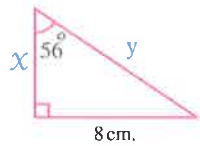
(1)



(2)

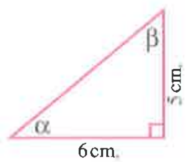


(3)

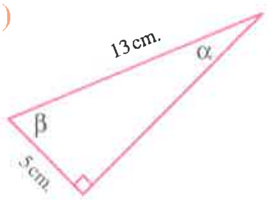


2 Find the value of each of the angles α and β in degree measure in each of the following figures :

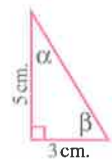
(1)



(2)



(3)



3 ABC is a right-angled triangle at B. Find AB to one decimal , if :

(1) m (∠ C) = $32^\circ\ 18'$ and AC = 25 cm. « 13.4 cm. »

(2) m (∠ A) = $62^\circ\ 44'$ and BC = 16 cm. « 8.2 cm. »

(3) m (∠ A) = $42^\circ\ 8'$ and AC = 24 cm. « 17.8 cm. »

4 ABC is a right-angled triangle at B. Find m (∠ C) to the nearest minute , if :

(1) AB = 12.6 cm. and AC = 18.6 cm. « $42^\circ\ 39'$ »

(2) BC = 54 cm. and AC = 88 cm. « $52^\circ\ 9'$ »

(3) AB = 27.2 cm. and BC = 20.4 cm. « $53^\circ\ 8'$ »

5 Solve the triangle ABC which is right-angled at B approximating the measures of angles to the nearest degree and the lengths of sides to the nearest cm. where :

(1) AB = 4 cm. , BC = 6 cm.

(2) AB = 12.5 cm. , BC = 17.6 cm.

(3) AB = 5.3 cm. , AC = 12.2 cm.

(4) BC = 31 cm. , AC = 42 cm.

6 Solve the right-angled triangle ABC at B in which :

(1) AC = 24.6 cm. , AB = 16.2 cm.

(2) AB = 39 cm. , BC = 62 cm.

(3) m (∠ C) = 62° , AC = 76 cm.

(4) AB = 12 cm. , m (∠ A) = $42^\circ\ 24'$

7 **Solve the triangle ABC which is right-angled at B approximating the measures of the angles to the nearest thousandth in radian measure , and the lengths of the sides to the nearest thousandth cm. where :**

(1) m (∠ A) = 1.169^{rad} , AB = 18 cm. | (2) m (∠ C) = 0.646^{rad} , AC = 15.7 cm.

(3) m (∠ C) = 1.082^{rad} , AC = 35.8 cm.

---

8 An isosceles triangle , the length of each of its legs is 7 cm. and its base length is 10 cm. Calculate the measures of its angles. « 44° 24′ 55″ , 44° 24′ 55″ , 91° 10′ 10″ »

---

9 ABC is a triangle in which AB = AC , BC = 20 cm. and m (∠ B) = 48° 54′
Find the length of $\overline{AB}$ to the nearest cm. « 15 cm. »

---

10 XYZ is a triangle in which XY = 11.5 cm. , YZ = 27.6 cm. , XZ = 29.9 cm.
Prove that the triangle is right-angled at Y , then find the measure of angle X « 67° 23′ »

---

11 A circle of radius length 8 cm. , $\overline{AC}$ is a diameter drawn in it , then the chord $\overline{AB}$ was drawn of length 10 cm.
Find the measures of the angles of the triangle ABC « 51° 19′ 4″ , 90° , 38° 40′ 56″ »

---

12 A circle of radius length 7 cm. , a chord was drawn in it opposite to a central angle of measure 110°. Calculate the length of this chord to the nearest thousandth.
« 11.468 cm. »

---

13 ABCD is a rhombus , the lengths of its diagonals $\overline{AC}$ and $\overline{BD}$ are 18.8 cm. and 24.6 cm.
Find : m (∠ ADC) to the nearest minute. « 74° 47′ »

---

14 A piece of land is in the shape of a rhombus ABCD. Its side length is 10 m. and m (∠ ABC) = 104° 16′ . Find the length of each of its two diagonals.
« 15.79 m. approximately , 12.28 m. approximately »

---

15 ABCD is a rectangle in which the length of its diagonal $\overline{AC}$ = 24.8 cm. and m (∠ ACB) = 23° 36′ **Find the length of each of :** $\overline{AB}$ and $\overline{BC}$
« 9.9 cm. approximately , 22.7 cm. approximately »

---

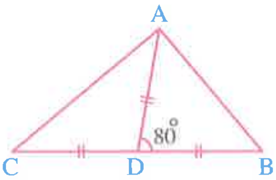
16 ABCD is an isosceles trapezium in which $\overline{AD}$ // $\overline{BC}$, AB = CD = 5 cm. , AD = 4 cm. , BC = 10 cm. Find the measure of each of its four angles.
« 53° 8′ , 126° 52′ , 53° 8′ , 126° 52′ »

Unit 3

Third Problems that measure high standard levels of thinking

- **Choose the correct answer from those given :**

(1) **In the opposite figure :**



If $D \in \overline{BC}$ such that DA = DB = DC = 5 cm.

, m ($\angle$ ADB) = 80° , then AC = ... cm.

(a) 10 sin 40° (b) 10 sin 50° (c) 5 sin 80° (d) 5 sin 40°

(2) If the side lengths of right-angled triangle ABC are a , a + 1 , a – 1 where a > 1 , then the measure of its greatest acute angle approximately is ....

(a) $36° 52'$ (b) $48° 18'$ (c) $53° 8'$ (d) $62° 42'$

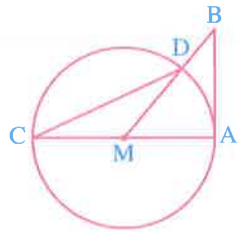
(3) If ABC is a right-angled triangle at B , AB = 6 cm. and the perimeter of Δ ABC = 24 cm. , then m ($\angle$ C) $\simeq$ ....

(a) 14° (b) 18° (c) 37° (d) 53°

(4) If ABC is a right-angled triangle at B and AB > BC and the area of Δ ABC = 30 cm^2 , AB + BC = 20 cm. , then m ($\angle$ A) = ...

(a) $77° 19'$ (b) $54° 37'$ (c) $26° 18'$ (d) $12° 41'$

(5) **In the opposite figure :**



If $\overline{AC}$ is a diameter in circle M

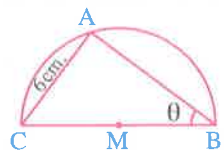
, $\overline{AB}$ is a tangent , AB = 6 cm , r = 5 cm.

, then m ($\angle$ DCM) = ...

(a) $50° 12'$ (b) $25° 6'$

(c) $18° 31'$ (d) $37° 39'$

(6) **In the opposite figure :**



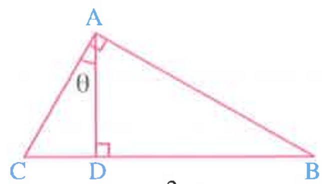
$\overline{BC}$ is a diameter in circle M

, AC = 6 cm. , m ($\angle$ ABC) = θ

, then the area of Δ ABC = ... cm^2

(a) 6 sin θ (b) 6 tan θ (c) 18 tan θ (d) 18 cot θ

(7) **In the opposite figure :**



If ABC is a right-angled triangle

at $\angle$ A , $\overline{AD} \perp \overline{BC}$, AD = $\sin^2 \theta$, then BC = ...

(a) sin θ (b) tan θ (c) cos θ (d) $\tan^2 \theta$

Exercise Eleven

Angles of elevation and angles of depression

From the school book

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) From a point on the ground surface 40 m. away from a tower base , the measure of elevation angle of the top of the tower is 72° , then the height of the tower to the nearest metre is .... m.

(a) 120 (b) 121 (c) 122 (d) 123

(2) A plane 1000 metres high was observed by a person at an angle of elevation of measure 40° , then the distance between the plane and the observer to the nearst metre ....

(a) 643 (b) 1192 (c) 1305 (d) 1556

(3) From the top of a tower 80 m high. , the measure of depression angle of a body lies on the horizontal plane that passing through the tower base equals $24^\circ 12'$, then the distance between the body from the base of the tower approximately equal ....

(a) 195 m. (b) 178 m. (c) 88 m. (d) 36 m.

(4) From the top of a light house 80 metres high , the measure of the angle of depression of a fixed target on the see equals 80° , then the distance between the fixed target and the top of the light house equals .... to nearest metre.

(a) 78 (b) 79 (c) 80 (d) 81

(5) A light pole of height 8 metres gives a shade on the ground of length 5 metres , then the measure of the elevation angle of the sun at that moment to the nearest degree equals ....

(a) 32° (b) 51° (c) 39° (d) 58°

(6) From the top of a rock 100 metres high , the depression angle of the boat which is 200 m. away from the base of the rock equals (in radian) ≈ .... rad.

(a) 0.08 (b) 0.46 (c) 0.25 (d) 0.24

(7) If a person walks 1 km. on a road inclined to the horizontal by an angle of measure $22^\circ\ 15'$, then his height above the horizontal equals approximately .... m.

(a) 925.5 (b) 409.1 (c) 378.6 (d) 376.8

(8) The length of the thread of a kite is 42 metres. If the measure of the angle which the thread makes with the horizontal ground equals 63° , then the height of the kite from the surface of the ground ≃ .... m.

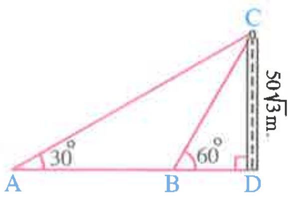
(a) 37 (b) 19 (c) 82 (d) 80

(9) A man of height 160 cm. was standing on the ground at a point which is 20 m. from a tree. He found that the measure of the angle of elevation of its top is $31^\circ\ 48'$, then the height of the tree ≃ .... m.

(a) 13 (b) 14 (c) 12 (d) 11

(10) **In the opposite figure :**

The angle of elevation of the top of a tower of length $50\sqrt{3}$ m. is measured from two points A and B on the same horizontal line as the tower base , their measures are 30° , 60° respectively , then the distance between the two points equals .... m.



(a) $100\sqrt{3}$ (b) $50\sqrt{3}$ (c) 100 (d) 50

(11) From the roof of a house 8 metres high , a person found that the elevation angle of the top of an opposite building was of measure 63° , and observed the depression angle of its base , it was of measure 28° , then the height of the building to the nearest metre equals .... m.

(a) 30 (b) 38 (c) 29 (d) 31

(12) From the top of a rock 40 metres high , two ships were observed in one ray on the sea with the rock base and their depression angles were measured to be $35^\circ\ 12'$ and $53^\circ\ 6'$, then the distance between the two ships ≈ .... m.

(a) 19.4 (b) 17.7 (c) 26.7 (d) 86.7

Second Essay questions

1 From a point 8 metres apart from the base of a tree , it was found that the measure of the elevation angle of the top of the tree is 22°
Find the height of the tree to the nearest hundredth. « 3.23 m. »

2 A man found that the measure of the angle of elevation of the top of a tower , at a distance of 50 m. from its base , is $39^\circ\ 21'$ Find the height of the tower. « 41 m. »

3 From a point on the ground at 20 metres from the base of a house , it was found that the measure of the angle of elevation of the top of the house is 27° 43̀
Find the height of the house to the nearest metre. « 11 m. »

---

4 An airplane 1000 metres high was observed by a person at an angle of elevation of measure 25° 17̀ Find the distance between the plane and the observer. « 2341.4 m. »

---

5 From the top of a rock 180 metres high from the sea level , the depression angle of a boat 300 metres apart from the base of the rock was measured. What is the radian measure of the depression angle ? « 0.54^{rad} »

---

6 A person observed , from the top of a hill 2.56 km. high , a point on the ground. He found its depression angle measure was 63°. Find the distance between the point and the observer to the nearest metre. « 2873 m. »

---

7 From the top of a lighthouse 200 metres high , it was found that the measure of the depression angle of a boat is 31° 14̀ Find the distance between the boat and the base of the lighthouse , knowing that the boat and the base of the lighthouse are collinear. « 329.8 m. »

---

8 From the top of a tower 60 metres high , the measure of the angle of depression of a body located in a horizontal level which passes through the base of the tower equals 28° 36̀. Find how far was the body from the base of the tower to the nearest metre. « 110 m. »

---

9 A light pole of height 7.2 metres gives a shade on the ground of length 4.8 metres. Find in radian the measure of the elevation angle of the sun at that moment. « 0.983^{rad} »

---

10 Find the measure of the angle of elevation of the sun when the shadow of a flagpole of height 3.5 m. is 2 m. « 60° 15̀ »

---

11 From the top of a tower , a person found that the measure of the angle of depression of a point in the same horizontal plane passing through its base is 35° If the height of the tower is 160 m. , then find the distance between the point and each of the base and the top of the tower. « 229 m. , 279 m. »

---

12 The upper end of a ladder rests on a vertical wall , it is 3.8 m. far from the surface of the ground , the lower end rests on a horizontal ground. If the measure of the angle of inclination of the ladder to the ground is 64°

, find to the nearest hundredth each of :

(1) The distance between the lower end and the wall.

(2) The length of the ladder. « 1.85 m. , 4.23 m. »

Unit 3

13 If the measure of the elevation angle of the top of a minaret from a point 140 metres distance from its base was $26° 46'$,
what is the height of the minaret to the nearest metre ?
If it is measured from 110 metres distance from its base,
find to the nearest minute, the measure of its elevation angle at that distance.

« 71 m., $32° 50'$ »

---

14 An observer measured the angle of elevation of a fixed balloon to be $\frac{\pi}{6}$,
when he walked in a horizontal plane towards the balloon a distance 800 metres,
he measured its angle of elevation to be $\frac{\pi}{4}$
Find the height of the balloon to the nearest metre. « 1093 m. »

---

15 Two persons stand on opposite sides of a flagpole fixed vertically on the ground such that the two persons and the base of the flagpole are collinear. They find that the angles of elevation of the top of the flagpole are of measure $54° 16'$ and $47° 12'$ If the length of the flagpole is 12 m., then find the distance between the two persons. (Let the heights of the persons be neglected) « 19.7 m. »

---

16 $\overline{AB}$ is a tower of 50 m. high where A is its top and B is its base. Two persons stand, one of them at C and the other at D where B, C and D are collinear and $C \in \overline{BD}$ If the measures of elevation angles of A from C and D are $52° 13'$ and $45° 36'$ respectively, then find the length of $\overline{CD}$ (Let the heights of the persons be neglected) « 10.2 m. »

---

17 From the top of a tower of 60 m. high, a person found two boats on one ray with the base of the tower. He found that the measures of the depression angles of the two boats are $47°$ and $41° 35'$ respectively.
Find the distance between the two boats to the nearest metre. « 12 m. »

---

18 A man was standing on the ground at a point which is 85 m. from the base of a tower. On the top of the tower there is a flagpole. He noticed that the measures of the angles of elevation of the top of the flagpole and its base are $56°$ and $54°$ respectively
Find the length of the flagpole to the nearest metre. (Let the height of the man be neglected). « 9 m. »

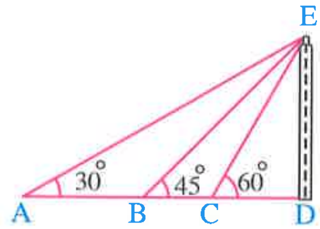
---

19 A ship approaches a lighthouse 50 metres high. At a moment, it was found that the measure of the elevation angle of the top of the lighthouse is 0.11^{rad} After 15 minutes, it was found again that the measure of its elevation angle is 0.22^{rad}.
Calculate the uniform velocity of the ship. « 15.3 m./min. »

Third Problems that measure high standard levels of thinking

● **Choose the correct answer from those given :**

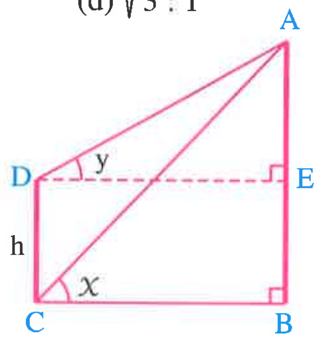
(1) **In the opposite figure :**



If the elevations angles of the top of a tower from three points on a line leads to the bottom of the tower are 30° , 45° , 60° respectively , then AB : BC = ...

(a) $1 : \sqrt{3}$ (b) $2 : 3$ (c) $\sqrt{3} : \sqrt{2}$ (d) $\sqrt{3} : 1$

(2) **In the opposite figure :**



The elevation angle of a top of a mountain from the base and the top of a house $\overline{CD}$ whose height is h , their measures was x , y respectively , then AB = ...

(a) $\frac{h \tan x}{\tan y}$ (b) $\frac{h \tan x}{\tan x - \tan y}$ (c) $h (\tan x - \tan y)$ (d) $h \tan x \tan y$

Exercise Twelve

Circular sector

From the school book

Test yourself

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) The perimeter of the circular sector in which the length of its arc is 4 cm. and the length of the diameter of its circle is 10 cm. equals ....

(a) 14 cm. (b) 20 cm. (c) 30 cm. (d) 40 cm.

(2) The area of the circular sector which the radius length of its circle is 4 cm. and the length of its arc is 6 cm. equals .... cm^2.

(a) 24 (b) 12 (c) 10 (d) 8

(3) The area of a sector whose arc is of length 10 cm. and the length of the diameter of its circle = 10 cm. equals ....

(a) 50 cm^2. (b) 25 cm^2. (c) 12.5 cm^2. (d) 100 cm^2.

(4) The area of the circular sector in which the measure of its angle is 1.2^{rad} and the length of the radius of its circle is 4 cm. equals ....

(a) 4.8 cm^2. (b) 9.6 cm^2. (c) 12.8 cm^2. (d) 19.6 cm^2.

(5) The area of the circular sector in which the measure of its angle is 120° , the length of the radius of its circle is 3 cm. equals ....

(a) 3π cm^2. (b) 6π cm^2. (c) 9π cm^2. (d) 12π cm^2.

(6) If the perimeter of a sector is 8 cm. and its arc is of length 2 cm. , then its circle is of radius length .... cm.

(a) 6 (b) 2 (c) 3 (d) 4

(7) The perimeter of a sector is 44 cm. Its circle is of radius length 14 cm. ,
then the length of the arc of the sector = ·············· cm.

(a) 16 (b) 8 (c) 32 (d) 4

(8) 📖 The area of the circular sector in which , its perimeter is 12 cm. , length of its arc is 6 cm. equals ·············· cm^2.

(a) 6 (b) 9 (c) 12 (d) 18

(9) 📖 The area of the circular sector in which r = 4 cm. and its perimeter 20 cm. equals ·············· cm^2.

(a) 40 (b) 32 (c) 24 (d) 48

(10) The arc of a sector is of length 3 cm. and the area of this sector is 15 cm^2. , then its circle radius is of length ·············· cm.

(a) 5 (b) 10 (c) 2.5 (d) 15

(11) The area of a sector is 400 cm^2. If its radius length is 20 cm. , then its arc length equals ·············· cm.

(a) 10 (b) 5 (c) 20 (d) 40

(12) 📖 If the area of the circular sector equals 110 cm^2. , the measure of its angle equals 2.2^{rad} , then the length of the radius of its circle equals ··············

(a) 2 cm. (b) 5 cm. (c) 10 cm. (d) 20 cm.

(13) 📖 The perimeter of the circular sector whose area is 24 cm^2. , length of its arc is 8 cm. equals ··············

(a) 20 (b) 14 (c) 32 (d) 24

(14) The area of a circular sector is 45 cm^2. and the length of the diameter of its circle is 20 cm. , then the length of its perimeter equals ·············· cm.

(a) 29 (b) 19 (c) 39 (d) 49

(15) The area of a circular sector is 27 cm^2. and the length of its radius is 6 cm. , then the measure of its central angle = $(\cdots\cdots)^{rad.}$

(a) 1.5 (b) 2 (c) 3 (d) 4.5

(16) The perimeter of a circular sector = 4 r cm. where r is radius length of its circle , then the radian measure of its central angle equals $(\cdots\cdots)^{rad.}$

(a) $\frac{1}{2}$ (b) 8 (c) 2 (d) $\frac{1}{3}$

(17) The length of an arc of a circular sector (ℓ) and the measure of its angle $(1.2)^{rad.}$ which drawn inside a circle of radius length (r) , then its perimeter = ·············· cm.

(a) 1.2 r (b) 3.2 r (c) 1.2 r^2 (d) 3.2 r^2

(18) The area of a sector is $\frac{\pi r^2}{6}$ cm^2. , where r is the length of the radius of its circle , then the measure of its angle equals ··············

(a) 30° (b) 60° (c) 90° (d) 45°

(19) The perimeter of a circular sector equals 24 cm. and the length of its arc equals 10 cm. , then the area of its surface = ... cm^2.

(a) 7π (b) 14π (c) 49π (d) 154π

(20) If the area of a circle is 53.6 cm^2 , then the area of the circular sector of this circle such that the measure of its central angle equals $67° 30'$ $\simeq$ .... cm^2.

(a) 10 (b) 11 (c) 12 (d) 13

(21) The area of a circle is $490\frac{5}{8}$ cm^2 , then the area of a circular sector from this circle whose arc is of length 32 cm. $\simeq$ .... cm^2.

(a) 100 (b) 200 (c) 400 (d) 300

(22) The arc of a sector is of length 4ℓ cm. and the length of the radius of its circle equals r cm. , then its perimeter = ... cm.

(a) $\ell + 2r$ (b) $r + 2\ell$ (c) $2(r + 2\ell)$ (d) $2(\ell + 2r)$

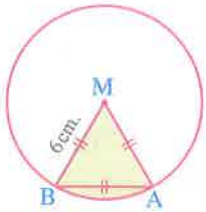
(23) The length of an arc in a circular sector is (ℓ) , the measure of its angle (θ^{rad}) and the length of the radius of its circle is (r) , then the perimeter of this sector ....

(a) $r + \ell$ (b) $r + 2\ell$ (c) $r(2 + \theta^{rad})$ (d) $2r(1 + \theta^{rad})$

(24) The perimeter of a circular sector is 35 cm. and its area is 75 $cm.^2$, then the measure of its central angle in radians = ... rad.

(a) $\frac{3}{2}$ or $\frac{7}{3}$ (b) $\frac{4}{3}$ or $\frac{7}{3}$ (c) $\frac{3}{2}$ or $\frac{8}{3}$ (d) 1 or $\frac{8}{3}$

(25) **In the opposite figure :**

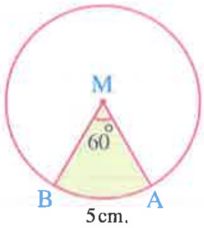


The area of the shaded sector = ... cm^2.

(a) 18 (b) $18\sqrt{3}$

(c) $9\sqrt{3}\pi$ (d) 6π

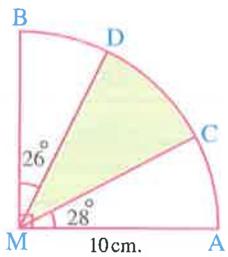
(26) **In the opposite figure :**



The area of the shaded sector = ... cm^2.

(a) 30π (b) $\frac{225}{\pi}$ (c) $\frac{75}{2\pi}$ (d) 50π

(27) **In the opposite figure :**



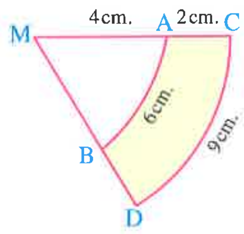
Quarter of circle whose centre M

, then area of the shaded part = ... cm^2.

(a) 10π (b) 20π

(c) 30π (d) 40π

(28) In the opposite figure :



Two concentric circles with centre (M)

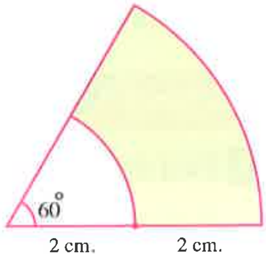
Their radii 4 cm. , 6 cm. and the

length of $\overset{\frown}{CD}$ = 9 cm. , the length of $\overset{\frown}{AB}$ = 6 cm.

, then the area of the shaded part = ... cm^2.

(a) 10 (b) 9π (c) 12π (d) 15

(29) In the opposite figure :

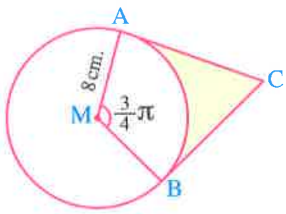


The area of the shaded part = ... cm^2.

(a) π (b) 2π

(c) $\frac{\pi}{3}$ (d) $\frac{2}{3}\pi$

(30) In the opposite figure :



$\overrightarrow{CA}$, $\overrightarrow{CB}$ are two tangent segments of the circle M

, the radius length of the circle M = 8 cm.

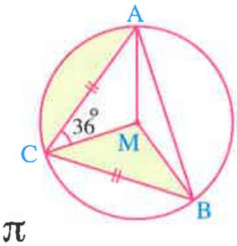
if m ($\angle$ AMB) = $\frac{3}{4}\pi$

, then the area of shaded region = ... cm^2.

(a) 79.1 (b) 97.1

(c) 7.91 (d) 9.71

(31) In the opposite figure :



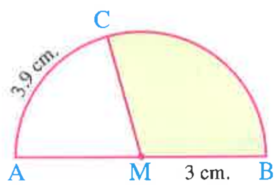
The radius of circle M is 10 cm.

, BC = AC , m ($\angle$ ACM) = 36°

, then the area of the shaded region = ... cm^2.

(a) 20π (b) 30π (c) 40π (d) 50π

(32) In the opposite figure :



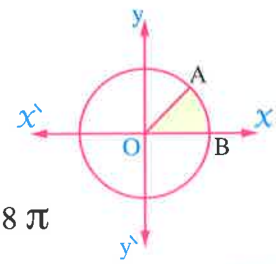
A semicircle whose centre M

, then area of the shaded part ≈ .... cm^2.

(a) 8.29 (b) 16.6

(c) 5.52 (d) 11.04

(33) In the opposite figure :



If A $\left(4\sqrt{2}\ , 4\sqrt{2}\right)$, then :

the area of the shaded part equals .... cm^2.

(a) 64π (b) 16π (c) 4π (d) 8π

(34) **In the opposite figure :**

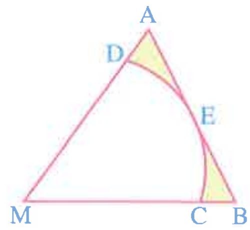
$\overline{AB}$ is a tangent to the circle M

which passes through C , D , E

If AB = 8 cm. , MD = 11 cm.

the length of $\overset{\frown}{CED}$ = 6 cm.

, then the area of the shaded part = ... cm^2.



(a) 22 (b) 18 (c) 12 (d) 11

Second Essay questions

1 Find the area of a sector where the length of its arc is 12 cm. and the length of its radius is 8 cm. « 48 cm^2. »

2 Find the area of the circular sector in which the length of its arc is 16 cm. , and the length of the radius of its circle is 9 cm. « 72 cm^2. »

3 Find to the nearest cm^2. the area of a circular sector , where the measure of its central angle is 30° and the radius of its circle is of length 3.5 cm. « 3 cm^2. approximately »

4 Find the area of the circular sector in which the length of the diameter of its circle is 20 cm. and the measure of its angle is 120°. « 104.7 cm^2. approximately»

5 Find the area of a sector whose central angle is of measure 40° and the the radius of its circle is of length 6 cm. « 13 cm^2. »

6 The central angle of a circular sector is 48° and the length of the radius of its circle is 6 cm. Find the area of the sector to the nearest cm^2. « 15 cm^2. »

7 Find the area of the circular sector in which the length of the radius of its circle is 10 cm. and the measure of its angle is 1.2^{rad} « 60 cm^2. »

8 Find the area of the circular sector in which the length of its arc is 7 cm. and its perimeter equals 25 cm. « 31.5 cm^2. »

9 The perimeter of a circular sector is of length 28 cm. If the length of its radius is 7 cm. , then find its area and the measure of its central angle in degrees and in radians.

« 49 cm^2. , 114° 35̀ , 2^{rad} »

10 A circular sector of area 270 cm.^{2} , the length of the radius of its circle equals 15 cm. , find the length of the arc of the sector and measure of its central angle in radian meausre.

« 36 cm. , $2.4^{rad.}$ »

---

11 The area of a sector is 40 cm^2. Find its perimeter if the length of its arc is 8 cm. « 28 cm. »

---

12 The area of a circular sector is 25 cm^2. If its central angle is of measure 0.5^{rad} , then find the length of the radius of its circle and the length of its arc. « 10 cm. , 5 cm. »

---

13 The area of a circular sector equals $\frac{2}{5}$ the area of its circle.

Find the measure of the central angle of this sector in radians and in degrees and if the length of the radius of the circle is 10 cm. , then find the perimeter of the sector to the nearest cm. « 2.51^{rad} , 144° , 45 cm. »

---

14 **Find in terms of π the area of the shaded part in each of the following figures :**

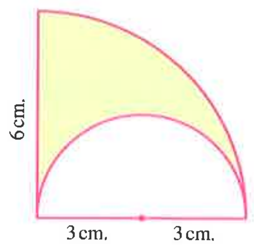


Figure (1)

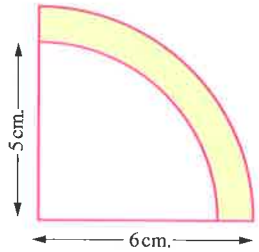


Figure (2)

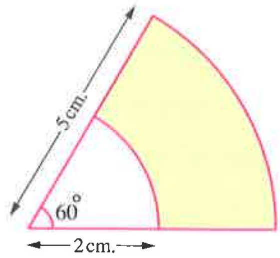


Figure (3)

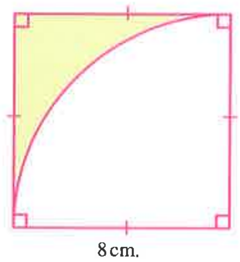


Figure (4)

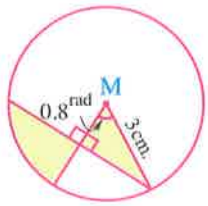


Figure (5)

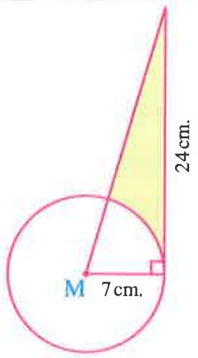


Figure (6)

---

15 A circle M of radius length 7.5 cm. , $\overline{MA}$, $\overline{MB}$ are radii where A B = 12 cm.

Find the area of the minor circular sector M A B to the nearest square centimetre.

« 52 cm^2. approximately »

---

16 Three congruent circles are drawn to touch each other. If the length of each of their radii is 5 cm. , then find the surface area of the included part between these circles.

« 4 cm^2. approximately »

17 $\overline{AB}$ and $\overline{AC}$ are two tangent segments from A to the circle M to touch it at B and C , so that MA = 12 cm. Find to the nearest cm^2 the area of the part between the two tangents and the smaller arc $\overset{\frown}{BC}$, knowing that the radius of the circle is of length 6 cm.

« 25 cm^2 approximately »

---

18 ABC is an equilateral triangle with side of length $8\sqrt{3}$ cm. A circle of centre A is drawn to touch $\overline{BC}$ at D and cuts $\overline{AB}$ and $\overline{AC}$ at X and Y respectively. Find to the nearest tenth of cm^2

the area of the part between $\overline{BC}$ and $\overset{\frown}{XY}$ « 7.7 cm^2 approximately »

---

19 $\overline{AB}$ and $\overline{AC}$ are two chords in a circle M such that AB = AC = 8 cm. and m ($\angle$ A) = 60°

Find to the nearest cm^2 the area of the minor sector MBC « 22 cm^2 »

Third Problems that measure high standard levels of thinking

1 Choose the correct answer from those given :

(1) If the roots of the equation $3x^2 - 19x + 13 = 0$ equals the diameter length and arc length of a sector in a circle , then the perimeter of this sector = ... cm.

(a) 19 (b) 13 (c) $\frac{19}{3}$ (d) $\frac{13}{3}$

(2) If the roots of the equation $x^2 - 13x + 19 = 0$ equals the diameter length and arc length of a sector in a circle , then the area of this sector = ... cm^2.

(a) 19 (b) $\frac{19}{2}$ (c) $\frac{13}{4}$ (d) $\frac{19}{4}$

(3) **In the opposite figure :**

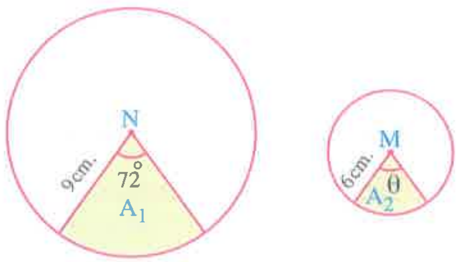
Two disjoint circles M and N

, A_1 , A_2 are the areas of the two sectors

If $\frac{A_1}{A_2} = \frac{9}{5}$, then θ = ...

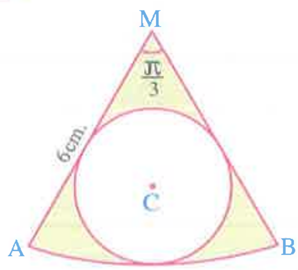
(a) 72° (b) 80°

(c) 90° (d) 100°



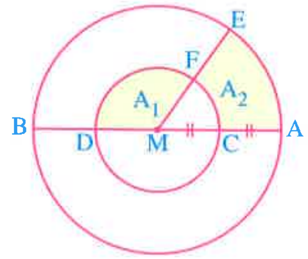
(4) **In the opposite figure :**

MAB is a circular sector from a circle whose centre (M) , the length of its radius is 6 cm. , m ($\angle$ AMB) = $\frac{\pi}{3}$, A circle C is drawn inside the sector such that it touches $\overline{MA}$, $\overline{MB}$, $\overset{\frown}{AB}$, then the area of the shaded region = ... cm^2



(a) π (b) 2 π (c) 4 π (d) 6 π

(5) In the opposite figure :



Two concentric circles at centre (M)

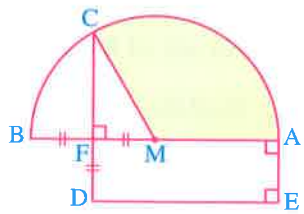
MC = CA , A_1 , A_2 are the areas of the two shaded parts

, if $A_1 = A_2$, then m (∠ AME) = ...

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{12}$

(6) In the opposite figure :



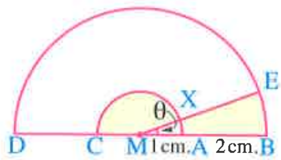
If the area of the rectangle AEDF = 27 cm^2.

, then the area of the shaded part = ... cm^2.

(a) 9π (b) 12π

(c) 15π (d) 18π

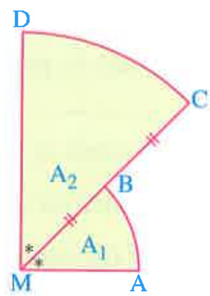
(7) In the opposite figure :



Two semicircle sharing the same centre (M) , if the area of the two shaded regions are equal , MA = 1 cm. AB = 2 cm. , then θ = ...°

(a) 15 (b) 20 (c) 30 (d) 45

(8) In the opposite figure :



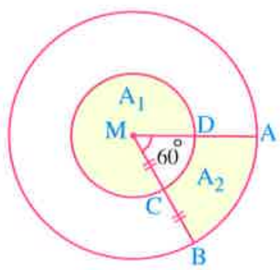
$\overset{\frown}{AB}$ and $\overset{\frown}{DC}$ are two arcs in two concentric circles with centre M

MB = BC , m (∠ AMB) = m (∠ CMD)

, A_1 , A_2 are the areas of the two sectors , then $\frac{A_1}{A_2}$ = ...

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

(9) In the opposite figure :



Two concentric circles with centre (M) , if m (∠ AMB) = 60°

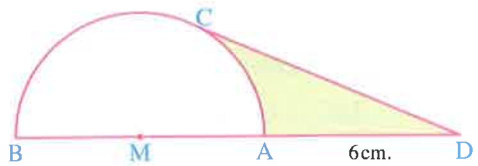
MC = CB , A_1 , A_2 are the areas of the two shaded parts , then $\frac{A_1}{A_2}$ = ...

(a) 2 (b) $\frac{5}{3}$

(c) $\frac{4}{3}$ (d) 1

(10) **In the opposite figure :**

Semicircle (M) if $DB = \sqrt{3}\, DC$,
$AD = 6$ cm. , then the area of the
shaded area = ... cm^2.



(a) $2\sqrt{3}$ (b) $12\sqrt{3} - 3\pi$ (c) $18\sqrt{3} - 6\pi$ (d) $18\sqrt{3} - 8\pi$

---

2 ABC is a right-angled triangle at B in which AB = 4 cm. and BC = 6 cm.
An arc of the circle of centre A touches $\overline{BC}$ at B and cuts $\overline{AC}$ at D
Find the area of the zone bounded by $\overline{BC}$, $\overline{CD}$ and $\overset{\frown}{BD}$ « 4.1 cm^2. »

---

3 M and N are two circles touching each other externally at A Let $\overleftrightarrow{BC}$ be a common tangent touching them at B and C Let the radii of these two circles be 5 cm. and 15 cm. respectively , so find the area of the part bounded by the two circles and the common tangent $\overleftrightarrow{BC}$ $\left(\sqrt{3} = 1.732\right)$ « 29 cm^2. approximately »

Life applications

1 **Agriculture :** A flower bed is in the shape of a circular sector , its area equals 48 m^2. , length of its arc equals 6 m.
Find its perimeter and the length of the radius of its circle. « 38 m. , 16 m. »

---

2 A piece of paper is in the shape of a square. If we cut from it a quarter of a circle of centre on the vertex of the square and the length of its radius is equal to the side length of the square and if the area of the remained part from the square equals 48.285 cm^2. , then find the side length of the square. « 15 cm. »

Exercise Thirteen

Circular segment

From the school book

Test yourself

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) The area of a circular segment , the radius of its circle is 8 cm. and the measure of its central angle is 120° approximately equals .... cm^2.

(a) 95 (b) 51 (c) 83 (d) 39

(2) The area of the circular segment whose length of the diameter of its circle is 8 cm. and the measure of its central angle is 1.2^{rad} equals approximately .... cm^2.

(a) 8.57 (b) 2.14 (c) 4.28 (d) 1.07

(3) The area of the circular segment whose radius length is 10 cm. and its arc is of length 5 cm. is approximately .... cm^2.

(a) 1.03 (b) 2.06 (c) 0.01 (d) 0.05

(4) The area of the circular segment whose central angle measure is 30° and the radius length of its circle is $2\sqrt{3}$ cm. equals .... cm^2.

(a) $\frac{\pi}{3} + 2$ (b) $\pi - 3$ (c) $\pi + 3$ (d) $\frac{\pi}{3} - 2$

(5) The area of the circular segment inscribed in a circle , its radius length is 10 cm. and subtend an inscribed angle of measure 60° equals approximately .... cm^2.

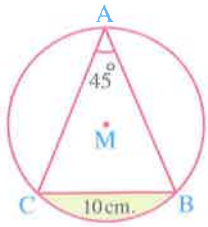
(a) 18 (b) 55 (c) 61 (d) 27

(6) The area of the circular segment whose chord length is 18 cm. , and the radius length of its circle 18 cm. approximately equals .... cm^2.

(a) 29 (b) 28 (c) 30 (d) 60

(7) **In the opposite figure :**

The area of the shaded part approximately equals .... cm^2.



(a) 7.1 (b) 28.5
(c) 14.3 (d) 2.02

(8) The area of the major circular segment in which the length of its chord equals the length of the radius of its circle equals 12 cm. equals $\simeq$ .... cm^2.

(a) 439 (b) 315 (c) 137 (d) 13

(9) ABC is an equilateral triangle inscribed in a circle where the length of its radius is 7.5 cm. Find the area of the minor segment whose chord is $\overline{BC} \simeq$ .... cm^2.

(a) 35 (b) 72 (c) 45 (d) 5

(10) If the measure of the central angle of circular segment is 90° and its area equals 56 cm^2, then the radius length of its circle equals approximately .... cm.

(a) 9.9 (b) 19.8 (c) 7 (d) 14

(11) The area of a circle is 706.5 cm^2. Find the area of a segment of this circle where the measure of its angle is 135° ($\pi = 3.14$) $\simeq$ .... cm^2.

(a) 264.9 (b) 185.5 (c) 12.4 (d) 344.6

(12) If the height of a circular segment is 5 cm. and the radius length of its circle is 10 cm. , then the area of the segment approximately equals .... cm^2.

(a) 9.1 (b) 122.8 (c) 12.3 (d) 61.4

(13) The area of a circular segment whose chord is of length 8 cm. and the length of the perpendicular from the centre of the circle to this chord is 5 cm. equal approximately .... cm^2.

(a) 48 (b) 121 (c) 7 (d) 8

(14) The area of circular segment whose chord 16 cm. and height 4 cm. $\simeq$ .... cm^2.

(a) 141 (b) 45 (c) 79 (d) 107

(15) The area of the circular segment equals the area of the circular sector subtended by the same arc if its central angle is of measure ....

(a) 90° (b) 180° (c) 270° (d) 45°

(16) ABC is a triangle in which : AB = 5 cm. , BC = 8 cm. , m ($\angle$ B) = 60° , then the area of Δ ABC = ... cm^2.

(a) 10 (b) 20 (c) $10\sqrt{3}$ (d) $20\sqrt{3}$

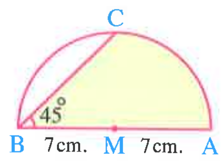
(17) In the opposite figure :

$m(\angle ABC) = 45°$

and $\overline{AB}$ is a diameter in

the circle whose length is 14 cm.

, then the area of the shaded part = ... cm^2 where $\left(\pi = \frac{22}{7}\right)$



(a) 77 (b) 63

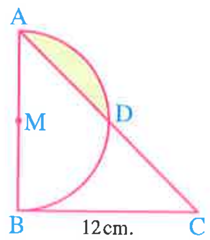
(c) 14 (d) 91

(18) In the opposite figure :

A semicircle M , $\overleftrightarrow{BC}$ is a tangent to the circle at B

, AB = BC = 12 cm.

, then the area of the shaded part ≈ .... cm^2



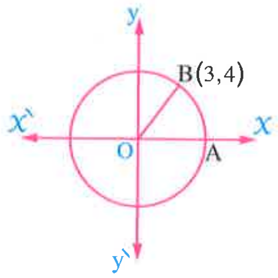
(a) 20.55 (b) 3.42

(c) 10.27 (d) 1.4

(19) In the opposite figure :

The area of the minor circular segment

whose chord $\overline{AB} \simeq$ .... square unit.



(a) 0.3 (b) 0.6

(c) 1.3 (d) 1.6

Second Essay questions

1 **Find the area of the circular segment in which :**

(1) The length of the radius of its circle is 12 cm. , and the measure of its angle equals 1.4^{rad}. « 30 cm^2 approximately »

(2) The length of the radius of its circle equals 8 cm. , and the measure of its angle equals 135° « 53 cm^2 approximately »

---

2 Find the area of the circular segment if the measure of its central angle is 115° 24` and the length of its radius is 20 cm. « 222 cm^2 approximately »

---

3 Find the area of the circular segment in which the length of the radius of its circle equals 14 cm. and the length of its arc equals 22 cm. « 56 cm^2 approximately »

---

4 The area of a circle is $490\frac{7}{8}$ cm^2. Find to the nearest cm^2 , the area of a segment of this circle if the length of its arc is 26.18 cm. « 96 cm^2 approximately »

---

5 $\overline{AB}$ is a chord in a circle in which its length is 10 cm. and opposite to a central angle of measure 60°

Find the area of the major circular segment whose chord is $\overline{AB}$ « 305 cm^2 approximately »

Unit 3

6 **Find the area of the circular segment in which :**

(1) The length of its chord equals 6 cm. , and the length of the radius of its circle equals 5 cm. « 4 cm^2 approximately »

(2) Its height equals 5 cm. , and the length of the radius of its circle equals 10 cm. « 61 cm^2 approximately »

---

7 A chord of length 6 cm. is drawn in a circle of radius length 6 cm. Find the area of the minor segment. « 3.26 cm^2 approximately »

---

8 Find the area of the major segment drawn in a circle of radius length 10.5 cm. and its chord is of length 14 cm. « 321 cm^2 approximately »

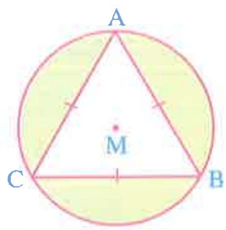
---

9 A chord of length 8 cm. in a circle is at a distance 3 cm. from its centre. Find the area of the minor circular segment resulting from the intersection of this chord with the surface of the circle. « 11 cm^2 approximately »

---

10 **In the figure drawn :**

ABC is an equilateral triangle drawn in the circle M in which , the length of its radius equals 8 cm.
Find the area of each shaded circular segment.



« 39 cm^2 approximately »

---

11 ABC is an equilateral triangle of side length 24 cm. A circle is drawn passing through its vertices. Find the length of the radius of the circle, then find the area of the minor circular segment whose chord is $\overline{BC}$ « 118 cm^2 approximately »

---

12 ABC is a triangle inscribed in a circle. If AB = AC = 15 cm. and BC = 18 cm. , then find the areas of the minor segments whose chords are $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$

« 39.3 , 89.5 , 39.3 cm^2 approximately »

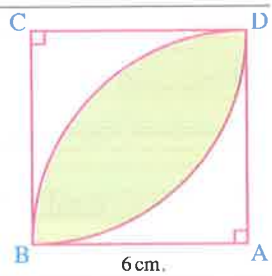
---

13 $\overline{AB}$ and $\overline{AC}$ are two equal chords in length in the circle M in which the length of each one is $6\sqrt{3}$ cm. and m ($\angle$ BAC) = 60° Find the area of the included part of the circle between the two chords and the minor arc $\overset{\frown}{BC}$ « 69 cm^2 approximately »

---

14 **In the opposite figure :**

ABCD is a square of side length 6 cm. ,
two arcs whose centres are A and C are drawn and the radius length of each of them = 6 cm.
Find the area of the shaded part.



« 21 cm^2 approximately »

15 Two congruent circles in which the radius length of each one is 12 cm. and one of them passes through the centre of the other. Find the area of the included part between them. « 177 cm^2 approximately »

---

16 ABC is a right-angled triangle at B in which AB = 6 cm. and BC = 8 cm. , drawn inside a circle. Find to the nearest cm^2 the area of each of the minor segments whose chords are the sides of the triangle. « 4 cm^2 , 11 cm^2 , 39 cm^2 approximately »

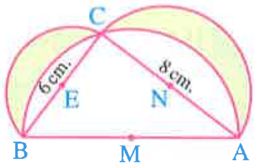
---

17 **In the opposite figure :**

M , N and E are centres of semicircles ,

AC = 8 cm. , CB = 6 cm.

Find the area of the shaded part.



« 24 cm^2 »

---

18 A is an external point of the circle M such that AM = 10 cm. Two tangents $\overline{AB}$ and $\overline{AC}$ are drawn to the circle from A to touch it at B and C Find the area of the minor segment whose arc is $\overset{\frown}{BC}$ if the radius of the circle is 5 cm. « 15.355 cm^2 approximately »

---

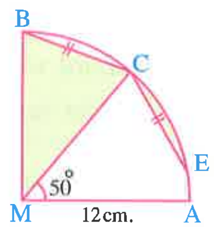
19 The lengths of radii of two circles are 6 cm. and 8 cm. If the distance between the two centres is 10 cm. , then find the area of the common zone between the two circles. « 26.57 cm^2 approximately »

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) **In the opposite figure :**

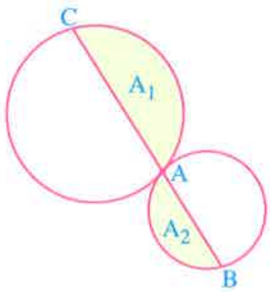
A quarter of a circle whose center M , m (∠ AMC) = 50° , CE = CB , then the area of the shaded parts = ... cm^2



(a) $3\pi + 18$ (b) 16π (c) $9 + 8\pi$ (d) 12π

(2) **In the opposite figure :**

Two circles touching externally at point A if AB = 4 cm. , AC = 6 cm. and A_1 , A_2 are the areas of the two shaded parts , then $\frac{A_2}{A_1}$ = ...

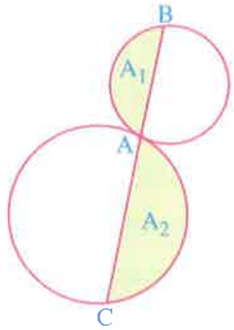


(a) $\frac{2}{3}$ (b) $\frac{2}{5}$

(c) $\frac{4}{9}$ (d) $\frac{4}{25}$

(3) **In the opposite figure :**

Two circles touching externally at point A and A_1 , A_2 are the areas of the two shaded parts if $4 A_2 = 9 A_1$, $BC = 20$ cm. , then the length of $\overline{AB}$ = ... cm.



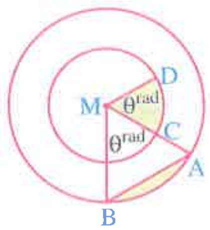
(a) 4 (b) 6

(c) 8 (d) 12

2 If the chord of the intersection of two circles is a diameter of one of the two circles and its length is equal to 10 cm. and equals the length of the radius of the other , then find the area of the common zone between the two circles. « 48.33 cm^2 approximately »

3 **In the opposite figure :**

Two concentric circles at M , if r is the radius length of the smaller circle and MD = r , MA = 2 r where A lies on the greater circle , m (∠ AMB) = m (∠ CMD) = θ^{rad}



Find the ratio between θ^{rad} and sin θ given that the areas of the two shaded parts are equal. « 4 : 3 »

Life applications

1 **Decoration :** A flower bed is in the shape of a circle whose radius equals 8 metres. A chord was drawn in the circle of length 8 metres. Calculate the area of the minor circular segment to the nearest tenth. « 5.8 m^2 approximately »

2 **Agriculture :** A piece of agricultural land is in the shape of a circle whose radius equals 4 metres is divided into four parts by an equilateral triangle whose vertices lie on the circle.
Calculate the area of each minor circular segment to the nearest hundredth.

« 9.83 m^2 approximately »

Exercise
Fourteen

Areas

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from the given ones :**

(1) The area of the triangle ABC in which AB = 7 cm. , BC = 8 cm. and m ($\angle$ B) = 50° equals .... cm^2

(a) 21.4 (b) 42.9 (c) 18 (d) 33.4

(2) The area of the isosceles triangle in which the length of one of its legs is 10 cm. and the measure of its vertex angle is 60° equals .... cm^2

(a) 25 (b) $50\sqrt{3}$ (c) $25\sqrt{3}$ (d) 50

(3) The area of the isosceles triangle in which the length of its base is 6 cm. and the length of one of its legs is 5 cm. equals .... cm^2

(a) 15 (b) 12 (c) 10 (d) 20

(4) The area of the equilateral triangle whose side length is 6 cm. equals .... cm^2

(a) 18 (b) $18\sqrt{3}$ (c) 9 (d) $9\sqrt{3}$

(5) The area of the quadrilateral in which the lengths of its diagonals are 6 cm. and 8 cm. , and the measure of the included angle between them is 30° equals .... cm^2

(a) 12 (b) 24 (c) $12\sqrt{3}$ (d) $24\sqrt{3}$

(6) The area of a quadrilateral is 30 cm^2 and the lengths of its diagonals are 10 cm. and 12 cm. , then the measure of the acute angle between its diagonals equals ....

(a) 30° (b) 60° (c) 150° (d) 45°

(7) If X is the length of the side of an equilateral triangle whose area is $9\sqrt{3}$ cm^2 , then X = ... cm.

(a) 36 (b) 6 (c) $6\sqrt{3}$ (d) $3\sqrt{2}$

(8) The area of the regular hexagon whose side length is 4 cm. equals .... cm^2

(a) $12\sqrt{3}$ (b) 12 (c) $24\sqrt{3}$ (d) 24

(9) The area of the regular pentagon whose side length is 10 cm. to the nearest hundredth ≈ .... cm^2

(a) 172.05 (b) 90.82 (c) 688.19 (d) 137.64

(10) The area of the rhombus in which the measure of one of its interior angles is 50° and its side length is 6 cm. ≈ .... cm^2

(a) 13.79 (b) 110.31 (c) 27.6 (d) 11.57

(11) The area of the equilateral triangle whose side length is x cm. equals .... cm^2

(a) x^2 (b) $\frac{\sqrt{3}}{2}x^2$ (c) $\frac{\sqrt{3}}{4}x^2$ (d) $\frac{1}{2}x^2$

(12) The area of the square whose diagonal length is x cm. equals .... cm^2

(a) x^2 (b) $\frac{1}{2}x^2$ (c) $\sqrt{2}x^2$ (d) $\frac{\sqrt{2}}{2}x^2$

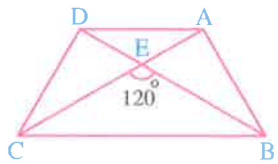
(13) The area of the regular hexagon whose side length is x cm. equals .... cm^2

(a) $\frac{3\sqrt{3}}{2}x^2$ (b) $\frac{3\sqrt{3}}{4}x^2$ (c) $\frac{\sqrt{3}}{2}x^2$ (d) $\frac{3}{2}x^2$

(14) The area of the regular octagon whose side length is x cm. equals .... cm^2

(a) $2x^2 \cot 45°$ (b) $2x^2 \tan 45°$ (c) $8x^2 \cot 22.5°$ (d) $2x^2 \cot 22.5°$

(15) **In the opposite figure :**



ABCD is a quadrilateral in which BD = 6 cm.

, the area of the figure ABCD = $24\sqrt{3}$ cm^2

, then AC = ... cm.

(a) 12 (b) 14 (c) 15 (d) 16

(16) The area of the triangle whose side lengths are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cm. equals .... cm^2

(a) $\sqrt{6}$ (b) $\frac{1}{2}\sqrt{6}$ (c) $\sqrt{30}$ (d) $\frac{1}{2}\sqrt{30}$

(17) An acute-angled triangle whose area is 14.4 cm^2, the lengths of two sides are 6 cm. , 8 cm. , then the cosine of the angle between these two sides = ...

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

(18) The area of the triangle whose side lengths 4 cm. , 6 cm. , 8 cm. ≃ .... cm^2

(a) 173.9 (b) 11.6 (c) 13.9 (d) 41.6

(19) The area of the acute angled Δ ABC is 40.13 cm^2, if AB = 9 cm. , BC = 12 cm.
, then m ($\angle$ B) $\simeq$ ....° (to the nearest degree).

(a) 32 (b) 42 (c) 48 (d) 88

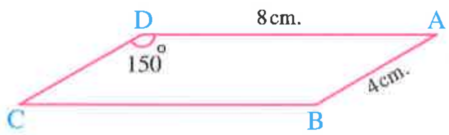
(20) The length of a side in an equilateral triangle that has area of $36\sqrt{3}$ cm^2 = ... cm.

(a) $6\sqrt{3}$ (b) 24 (c) 6 (d) 12

(21) **In the opposite figure :**

ABCD is a parallelogram

Its area = ... cm^2



(a) 16 (b) 20 (c) 24 (d) 36

(22) The area of quadrilateral in which the length of its diagonal 12 cm. and 13 cm. and cosine the angle between them is $\frac{5}{13}$ equals .... cm^2

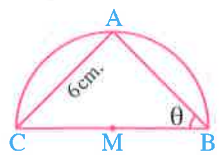
(a) 30 (b) 72 (c) 60 (d) 144

(23) If the area of a regular hexagon is $54\sqrt{3}$ cm^2, then its side length equals .... cm.

(a) 6 (b) 12 (c) $6\sqrt{3}$ (d) $12\sqrt{3}$

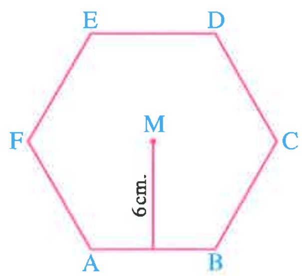
(24) **In the opposite figure :**

$\overline{BC}$ is a diameter of the circle M , AC = 6 cm.
, m ($\angle$ ABC) = θ , then the area of Δ ABC = ... cm^2



(a) 6 sin θ (b) 6 tan θ (c) 18 tan θ (d) 18 cot θ

(25) If the length of the perpendicular from the centre of the regular hexagon to one of its edges equals 6 cm.
, then the area of the hexagon = ... cm^2

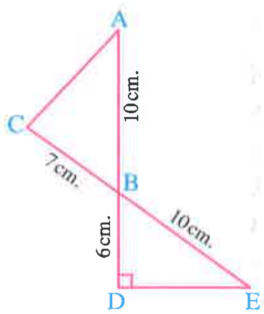


(a) $27\sqrt{3}$ (b) $36\sqrt{3}$

(c) $54\sqrt{3}$ (d) $72\sqrt{3}$

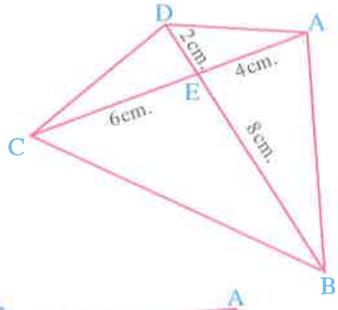
(26) **In the opposite figure :**

The area of Δ ABC equals .... cm^2



(a) 24
(b) 28
(c) 32
(d) 35

(27) **In the opposite figure :**



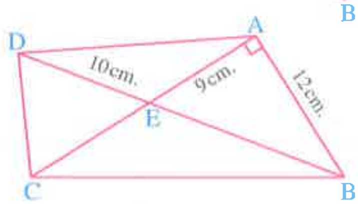
If the area of the quadrilateral

ABCD = 50 cm^2, then m (∠ AEB) = ...°

(a) 30 (b) 60

(c) 75 (d) 90

(28) **In the opposite figure :**



If the area of the quadrilateral

ABCD = 190 cm^2

, then the length of $\overline{EC}$ = ... cm.

(a) 9 (b) 10 (c) 11 (d) 12

(29) In Δ ABC :

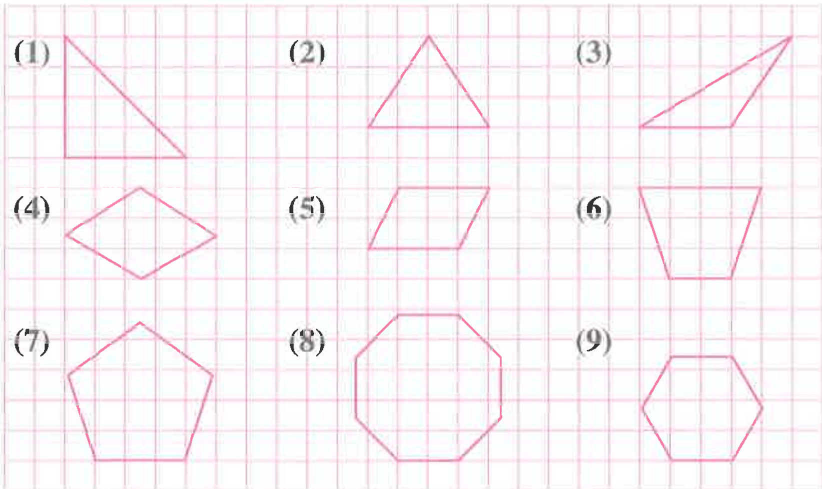
If half of its perimeter is denoted by S and S – AB = 6 cm.

, S – BC = 8 cm. , S – AC = 10 cm. , then area of (Δ ABC) = ... cm^2

(a) $8\sqrt{30}$ (b) $24\sqrt{5}$ (c) $4\sqrt{30}$ (d) $48\sqrt{5}$

Second Essay questions

1 **Find the area of each of the following figures , given that □ expresses a unit of the area :**



2 **Find the area of the triangle ABC in each of the following cases :**

(1) AB = 6 cm. , BC = 8 cm. , m (∠ B) = 90° « 24 cm^2 »

(2) AC = 12 cm. , length of perpendicular drawn from B to $\overline{AC}$ equals 7 cm. « 42 cm^2 »

(3) AB = 16 cm. , BC = 20 cm. , m (∠ B) = 46° « 115 cm^2 approximately »

3 Find the area of the triangle ABC in which : BC = 16 cm. , BA = 22 cm. and m (∠ B) = 63° approximating the result to the nearest thousandth. « 156.817 cm^2 »

4 Find to the nearest tenth the area of an isosceles triangle , the length of one of its legs equals 12 cm. and the measure of the included angle between them is 64° « 64.7 cm^2 »

5 Find the area of the quadrilateral in which the lengths of its diagonals are 12 cm. , 16 cm. and the measure of the included angle between them is 68° approximating the result to the nearest square centimetre. « 89 cm^2 »

6 **Find the area of the figure ABCD in each of the following cases :**

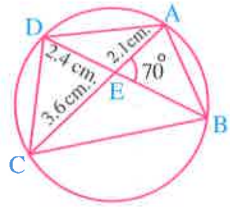
(1) A parallelogram in which AB = 8 cm. , BC = 11 cm. , m (∠ B) = 60° « $44\sqrt{3}$ cm^2 »

(2) A trapezium in which , the lengths of its parallel bases $\overline{AD}$ and $\overline{BC}$ are 7 cm. , 11 cm. respectively , the length of the perpendicular drawn from D to $\overline{BC}$ equals 6 cm. « 54 cm^2 »

(3) A rhombus in which AB = 8 cm. , and the measure of the included angle between two adjacent sides in it equals 58° « 54 cm^2 »

7 The area of the parallelogram ABCD is 192 cm^2 and the lengths of its diagonals $\overline{AC}$ and $\overline{BD}$ are 16 cm. and 24 cm. respectively , **then find :** m (∠ AMD) « 90° »

8 **In the opposite figure :**



ABCD is an inscribed quadrilateral in a circle , $\overline{AC} \cap \overline{BD} = \{E\}$, if AE = 2.1 cm. , EC = 3.6 cm. , ED = 2.4 cm. and m (∠ AEB) = 70° , then calculate the area of the quadrilateral ABCD « 15 cm^2 approximately »

9 **Find the area of each of the following regular polygons approximating the result to the nearest tenth :**

(1) A regular pentagon of side length 16 cm. « 440.4 cm^2 »

(2) A regular hexagon of side length 12 cm. « 374.1 cm^2 »

(3) A regular octagon of side length 8 cm. « 309 cm^2 »

(4) A regular heptagon of side length 10 cm. « 363.4 cm^2 »

10 Find the area of a regular polygon of 12-sides of side length 10 cm. approximating the result to the nearest tenth. « 1119.6 cm^2 »

11 Find the area of the triangle ABC in which : AB = 8 cm. , BC = 7 cm. and AC = 11 cm. « 28 cm^2 approximately »

Unit 3

Third Problems that measure high standard levels of thinking

1 Choose the correct answer from those given :

(1) The perimeter of Δ ABC = 14 cm. and its area = $2\sqrt{14}$ cm^2. and the length of one of its sides 3 cm. , then the difference between the length of the other two sides = ... cm.

(a) 1 (b) $2\frac{1}{4}$ (c) 7 (d) 11

(2) A regular hexagon whose area (A_1) is drawn inside a circle whose area (A_2) , then $A_2 : A_1$ = ...

(a) $\pi : \sqrt{3}$ (b) $2\pi : \sqrt{3}$ (c) $\pi : 2\sqrt{3}$ (d) $2\pi : 3\sqrt{3}$

(3) Two regular polygons are drawn inside the same circle one of them formed from 6 sides and of area (A_1) and the other formed from 12 sides and of area (A_2) , then $A_1 : A_2$ = ...

(a) $1 : \sqrt{2}$ (b) 1 : 2 (c) $\sqrt{2} : 3$ (d) $\sqrt{3} : 2$

(4) The area of a regular polygon formed from 400 sides and its side length $\sqrt{\tan\left(\frac{9}{20}\right)^\circ}$ length unit equals .... square unit.

(a) 50 (b) 100 (c) $100 \cot\left(\frac{9}{20}\right)^\circ$ (d) $50 \tan\left(\frac{9}{20}\right)^\circ$

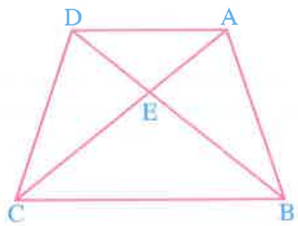
(5) **In the opposite figure :**

ABCD is a quadrilateral in which

$\overline{AC} \cap \overline{BD} = \{E\}$

, the area of (Δ AED) = 9 cm^2.

, the area of (Δ AEB) = 18 cm^2. , the area of (Δ CEB) = 16 cm^2. , then the area of Δ DEC = ... cm^2.



(a) 6 (b) 8 (c) 10 (d) 12

(6) If ABCDE is a regular pentagon of side length = ℓ cm. and the length of $\overline{AC}$ = m cm. , then $\frac{\text{the area of } (\Delta \text{ ABC})}{\text{the area of } (\Delta \text{ ACD})}$ = ...

(a) $\frac{\ell}{m}$ (b) $\frac{m}{\ell}$ (c) $\frac{2\ell}{3m}$ (d) $\frac{3\ell}{2m}$

(7) **In the opposite figure :**

ABCD is a quadrilateral in which

$\overline{AC} \cap \overline{BD} = \{E\}$

If the area of (Δ DEC) = A cm^2

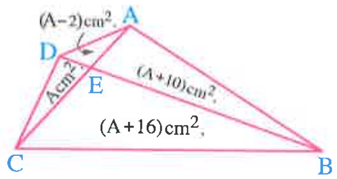
, the area of (Δ AED) = (A – 2) cm^2

, the area of (Δ AEB) = (A + 10) cm^2

, the area of (Δ BEC) = (A + 16) cm^2

, then the area of the figure ABCD

= .... cm^2



(a) 8 (b) 32 (c) 56 (d) 88

---

2 **In the opposite figure :**

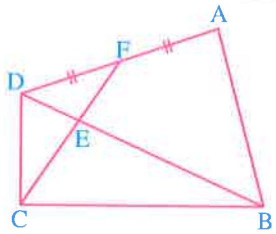
If the area of (Δ FED) = 3 cm^2

, the area of (Δ EDC) = 8 cm^2

, the area of (Δ EBC) = 24 cm^2

and F is the midpoint of $\overline{AD}$

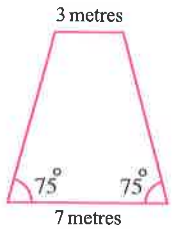
Find the area of the figure ABCD



« 56 cm^2 »

Life applications

1 **Constructions :** The opposite figure represents a set of stairs lead to the entrance of the residential compound is in the shape of an isosceles trapezium , its larger base is down and its length equals 7 metres , its smaller base is up and its length equals 3 metres , each leg inclines by an angle of measure 75° to the larger base.



Find :

(1) The length of its base at the midpoints of the two legs (middle base).

(2) The length of each of its legs to the nearest tenth.

(3) The area of the trapezium to the nearest square metre. « 5 m. , 7.7 m. , 38 m^2 »

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2 **Basins decorations :** Basin is designed to fish decoration , its base is in the shape of a regular pentagon , the length of its diagonal equals 72 cm.

Find to the nearest square centimetre the area of its base. « 3407 cm^2 »

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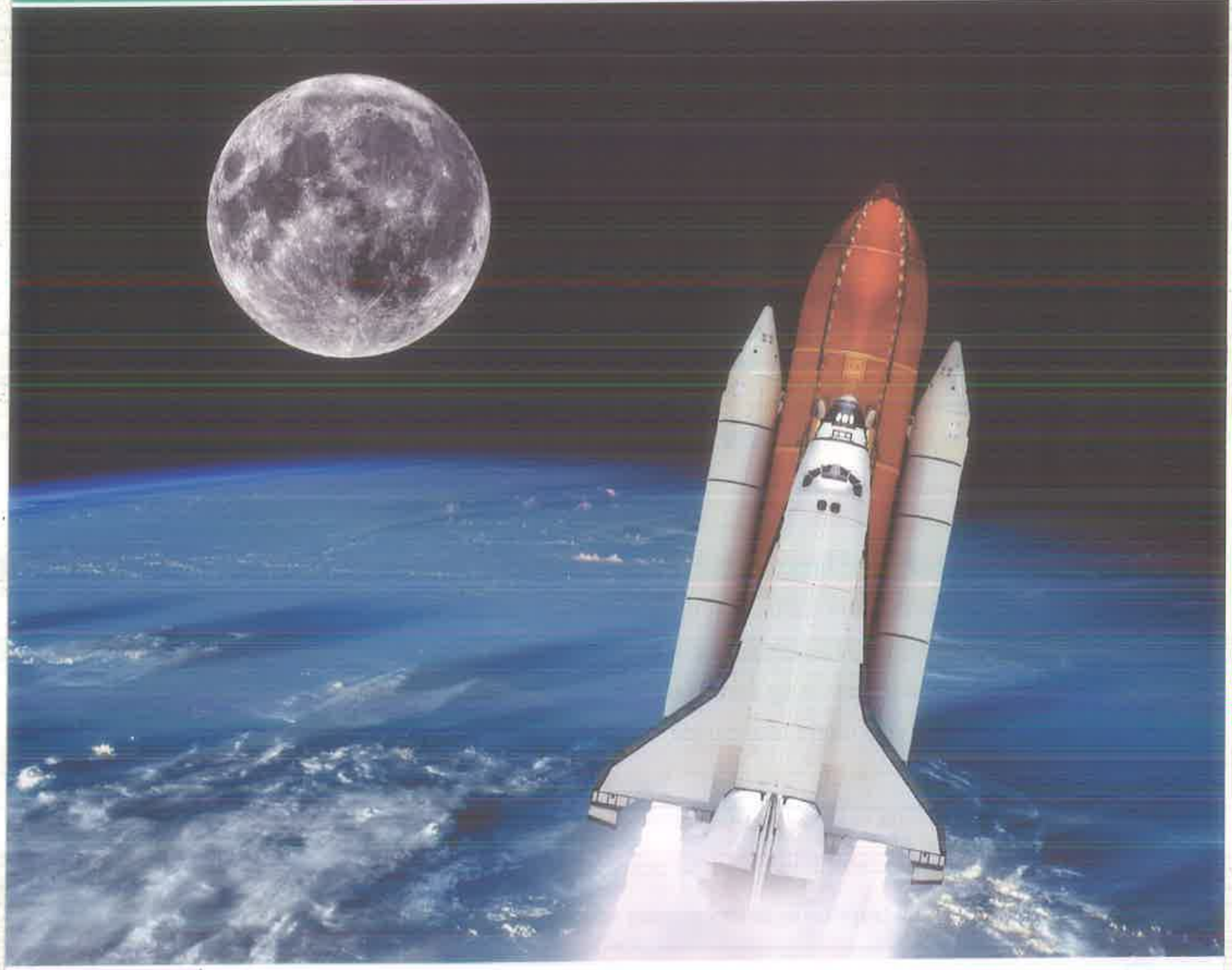
3 **Flowers :** Karim designs a garden to his house , and hopes to determine a special part for flowers , is in the form of a regular hexagon of area $54\sqrt{3}$ m^2

Find the length of its side. « 6 m. »

Second

Analytic Geometry

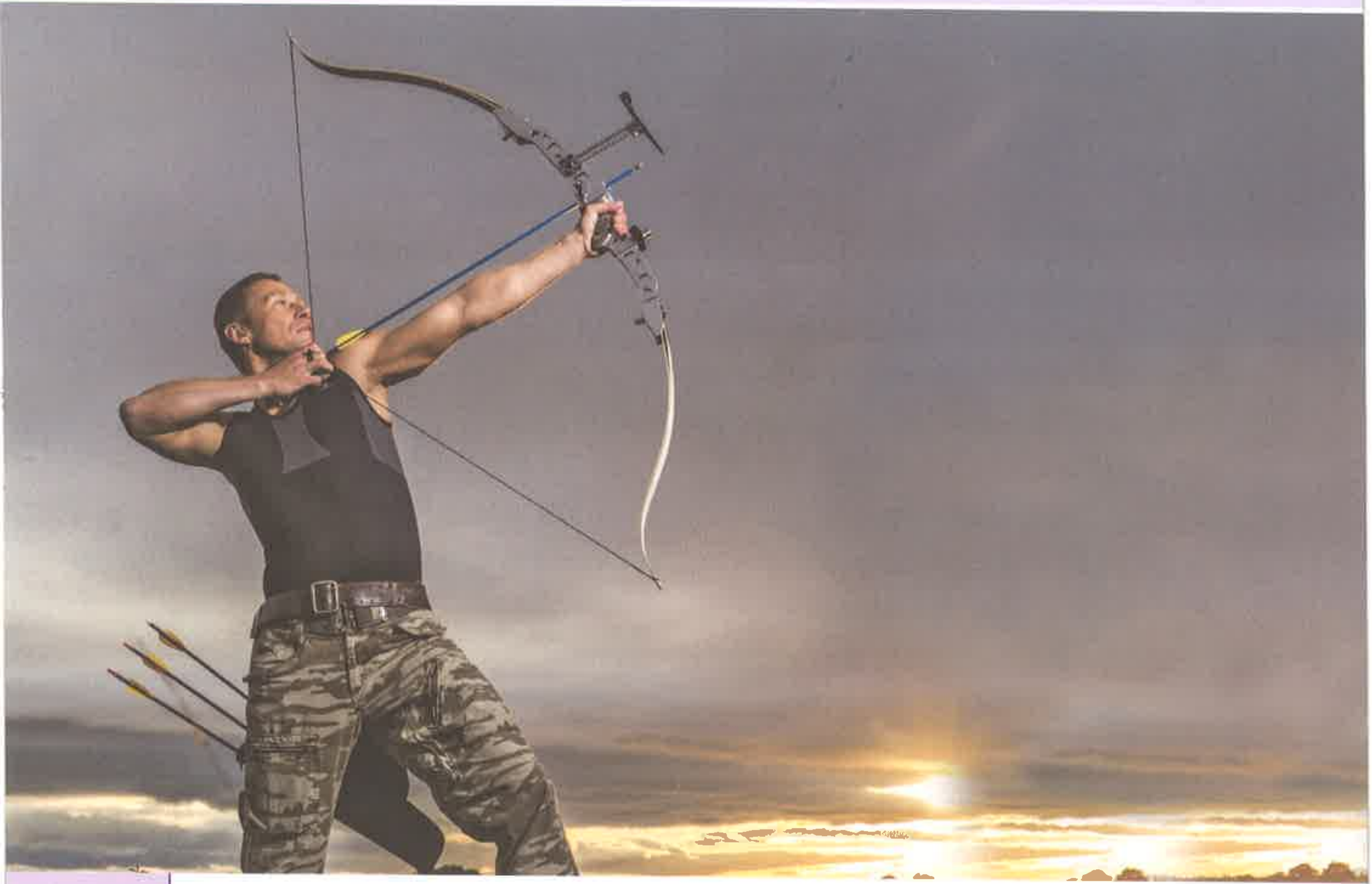


Unit Four Vectors

Unit Five Straight line

Unit 4

VECTORS



Exercise One	:	Scalars , vectors and directed line segment.
Exercise Two	:	Vectors.
Exercise Three	:	Operations on vectors.
Exercise Four	:	Applications on vectors.

Exercise **One**

Scalars , vectors and directed line segment

📖 From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) 📖 Which of the following represents a vector ?

(a) Time. (b) Temperature degree.

(c) The displacement. (d) The mass.

(2) If ABCD is a parallelogram , its diagonals intersecting at M , then

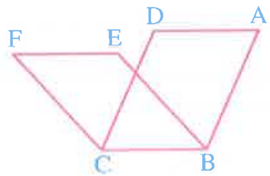
First : $\overrightarrow{CD}$ is equivalent to ....

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BA}$ (c) $\overrightarrow{CA}$ (d) $\overrightarrow{AD}$

Second : $\overrightarrow{MD}$ is equivalent to ....

(a) $\overrightarrow{DM}$ (b) $\overrightarrow{MB}$ (c) $\overrightarrow{BM}$ (d) $\overrightarrow{MA}$

(3) **In the opposite figure :**



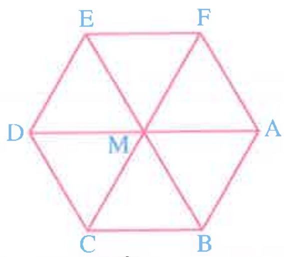
If ABCD and EBCF are two parallelograms

, then $\overrightarrow{AD}$ is equivalent to each .... and ....

(a) $\overrightarrow{BA}$, $\overrightarrow{CD}$ (b) $\overrightarrow{BC}$, $\overrightarrow{CF}$

(c) $\overrightarrow{BC}$, $\overrightarrow{EF}$ (d) $\overrightarrow{BC}$, $\overrightarrow{BA}$

(4) 📖 **In the opposite figure :**



ABCDEF is a regular hexagon , its centre is M

Then :

First : $\overrightarrow{AB}$ is equivalent to each of the following directed segments except ....

(a) $\overrightarrow{ED}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{MC}$ (d) $\overrightarrow{FM}$

Second : $\overrightarrow{MD}$ is equivalent to ....

(a) $\overrightarrow{MA}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{CF}$ (d) $\overrightarrow{CB}$

(5) M is the point of intersection of the diagonals of square ABCD , then each two directed line segments of the following are equivalent except ....

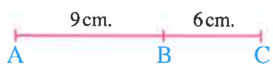
(a) $\overrightarrow{AB}$, $\overrightarrow{DC}$ (b) $\overrightarrow{AM}$, $\overrightarrow{MC}$ (c) $\overrightarrow{BC}$, $\overrightarrow{AD}$ (d) $\overrightarrow{AM}$, $\overrightarrow{MD}$

(6) If ABCDEF is a regular hexagon whose centre (M) which of the following directed line segments are not equivalent ?

(a) $\overrightarrow{AB}$, $\overrightarrow{FM}$ (b) $\overrightarrow{AB}$, $\overrightarrow{ED}$ (c) $\overrightarrow{AB}$, $\overrightarrow{MC}$ (d) $\overrightarrow{AB}$, $\overrightarrow{MD}$

(7) **In the opposite figure :**

If a body moved from the point A in direction of east to the point C , then return to the point B in direction of west , then



First : The distance covered by the body = ... cm.

(a) 6 (b) 9 (c) 15 (d) 21

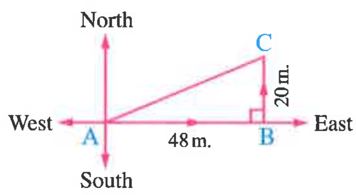
Second : The displacement = ... cm.

(a) 9 cm. in $\overrightarrow{AB}$ direction (b) 6 cm. in $\overrightarrow{CB}$ direction

(c) 9 cm. in $\overrightarrow{BA}$ direction (d) 21 cm. in $\overrightarrow{BA}$ direction

(8) **In the opposite figure :**

If a body moved from the point A a distance 48 m. east , then changed its direction and moved a distance 20 m. north and stopped at the point C , then



First : The distance covered by the body = ... m.

(a) 52 (b) 68 (c) 48 (d) 28

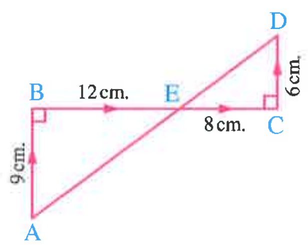
Second : The displacement = ...

(a) 68 m. in $\overrightarrow{AC}$ direction. (b) 68 m. in $\overrightarrow{CA}$ direction.

(c) 52 m. in $\overrightarrow{AC}$ direction. (d) 52 m. in $\overrightarrow{CA}$ direction.

(9) **In the opposite figure :**

If each of $\overline{DC}$ and $\overline{AB}$ is perpendicular to $\overline{BC}$ and a body moved from the point A to the point B , then the point C and stopped at the point D , then



First : The distance covered by the body = ... cm.

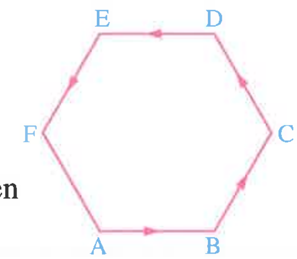
(a) 25 (b) 35 (c) 29 (d) 20

Second : The displacement = ··············· cm.

(a) 35 cm. in $\overrightarrow{AD}$ direction (b) 35 cm. in $\overrightarrow{DA}$ direction

(c) 25 cm. in $\overrightarrow{AD}$ direction (d) 25 cm. in $\overrightarrow{DA}$ direction

(10) **In the opposite figure :**



ABCDEF is a regular hexagon of side length 8 m.

, if a body moved from the point A to the point B ,

then to C , then to D , then to E and stopped at the point F , then

First : The distance covered by the body = ··············· m.

(a) 8 (b) 48 (c) 32 (d) 40

Second : The displacement = ···············

(a) 8 m. in $\overrightarrow{AF}$ direction. (b) 40 m. in $\overrightarrow{FA}$ direction.

(c) 8 m. in $\overrightarrow{FA}$ direction. (d) 40 m. in $\overrightarrow{AF}$ direction.

(11) A car covered 20 meters due north then it covered the same distance due west , then the displacement of the car is ··········

(a) 40 meters due west. (b) 40 meters due western north.

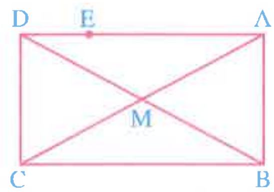
(c) $20\sqrt{2}$ meters due western north. (d) $20\sqrt{2}$ meters due western south.

(12) In a coordinate orthogonal plane , if A (1 , 3) , B (– 3 , 1) , C (0 , 4) and $\overrightarrow{AB}$ is equivalent to $\overline{CD}$, then the coordinate of D is ··········

(a) (– 4 , 2) (b) (4 , – 2) (c) (– 3 , 1) (d) (– 4 , – 2)

Second Essay questions

1 **In the opposite figure :**



ABCD is a rectangle , its diagonals are intersecting at M , E $\in \overline{AD}$

Show whether the two rays have the same , opposite or different directions in each of the following :

(1) $\overrightarrow{AB}$ and $\overrightarrow{CD}$

(2) $\overrightarrow{AE}$ and $\overrightarrow{DE}$

(3) $\overrightarrow{AM}$ and $\overrightarrow{AC}$

(4) $\overrightarrow{BM}$ and $\overrightarrow{MC}$

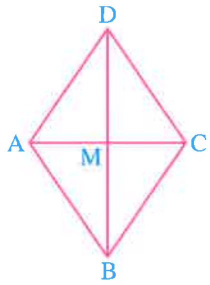
(5) $\overrightarrow{DE}$ and $\overrightarrow{CB}$

(6) $\overrightarrow{BM}$ and $\overrightarrow{MD}$

2 **In the opposite figure :**

ABCD is a rhombus in which $\overline{AC} \cap \overline{BD} = \{M\}$

Write the directed line segments equivalent to each of the following :



(1) $\overrightarrow{MA}$ | (2) $\overrightarrow{MD}$

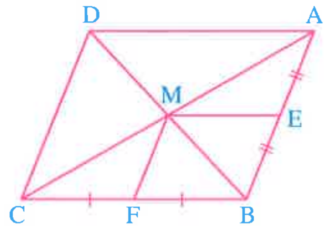
(3) $\overrightarrow{AD}$ | (4) $\overrightarrow{BA}$

---

3 **In the opposite figure :**

ABCD is a parallelogram in which $\overline{AC} \cap \overline{BD} = \{M\}$

, E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{BC}$



First : Determine the directed line segments (if existed) which are equivalent to :

(1) $\overrightarrow{BE}$ | (2) $\overrightarrow{MB}$

(3) $\overrightarrow{ME}$ | (4) $\overrightarrow{AC}$

(5) $\overrightarrow{AB}$ | (6) $\overrightarrow{BC}$

Second : Show why the following directed line segments are not equivalent :

(1) $\overrightarrow{ME}$ and $\overrightarrow{BF}$ | (2) $\overrightarrow{AC}$ and $\overrightarrow{MC}$

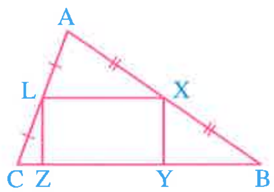
(3) $\overrightarrow{FB}$ and $\overrightarrow{FC}$ | (4) $\overrightarrow{BA}$ and $\overrightarrow{DC}$

---

4 **In the opposite figure :**

ABC is a triangle in which X is the midpoint of $\overline{AB}$

, L is the midpoint of $\overline{AC}$ and XYZL is a rectangle.



Write the directed line segments (if existed) which are equivalent to each of the following :

(1) $\overrightarrow{XY}$ | (2) $\overrightarrow{XL}$ | (3) $\overrightarrow{AL}$

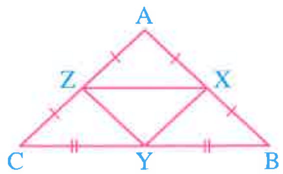
(4) $\overrightarrow{BX}$ | (5) $\overrightarrow{BY}$

---

5 **In the opposite figure :**

ABC is a triangle in which AB = AC

, X , Y and Z are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively



First : Which of the following statements is true ?

(1) $\| \overrightarrow{XY} \| = \| \overrightarrow{ZY} \|$ | (2) $\overrightarrow{XY}$ is equivalent to $\overrightarrow{ZY}$

(3) $\overrightarrow{BY}$ is equivalent to $\overrightarrow{ZX}$

Second : Write all directed line segments (if found) which are equivalent to :

(1) $\overrightarrow{BX}$ | (2) $\overrightarrow{AZ}$ | (3) $\overrightarrow{XZ}$

(4) $\overrightarrow{CY}$ | (5) $\overrightarrow{XY}$ | (6) $\overrightarrow{ZY}$

6 **On the lattice , if A (2 , 3) , B (– 3 , 1) and C (5 , – 1)**

(1) Draw $\overrightarrow{CD}$ such that it is equivalent to $\overrightarrow{AB}$ and determine the coordinates of D

(2) Determine the coordinates of the point M which is the midpoint of $\overrightarrow{BC}$, then determine the directed line segments which are equivalent to each of :

[a] $\overrightarrow{BM}$ **[b]** $\overrightarrow{AM}$ **[c]** $\overrightarrow{AC}$ **[d]** $\overrightarrow{DB}$

(3) Is the figure ACDB a parallelogram ? Explain your answer.

7 In a coordinate orthogonal plane , if A (4 , – 3) , B (4 , 4) and C (– 3 , – 1) , $\overrightarrow{BA}$, $\overrightarrow{CD}$, $\overrightarrow{OM}$ and $\overrightarrow{NO}$ are equivalent directed line segments where O is the origin point.

Find the coordinates of each of : D , M and N

8 **On the lattice , if A (3 , – 2) , B (6 , 2) , C (1 , 3) and D (4 , 7)**

(1) **Find :** $\| \overrightarrow{AB} \|$ and $\| \overrightarrow{CD} \|$

(2) **Prove that :** $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$

(3) If the directed line segments $\overrightarrow{BC}$, $\overrightarrow{AM}$, $\overrightarrow{ND}$ and $\overrightarrow{OR}$ are equivalent.

Find the coordinates of : M , N and R where O is the origin point.

9 Construct a coordinate orthogonal plane where O is the origin point and plot the points A (– 2 , 3) , B (1 , 0) , C (2 , – 3) , N (– 3 , 1) and T (4 , 1) , then draw the directed line segments $\overrightarrow{CD}$, $\overrightarrow{OE}$, $\overrightarrow{NL}$ and $\overrightarrow{KT}$ where each one of them is equivalent to $\overrightarrow{AB}$ **, then from the graph determine the coordinates of :** D , E , L and K

Exercise Two Vectors

📖 From the school book

Test yourself

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) If $\overrightarrow{A} = (5 , -12)$, then $\|\overrightarrow{A}\| = \dots\dots$

(a) 13 (b) -7 (c) 17 (d) 7

(2) All the following vectors are unit vectors except $\dots\dots$

(a) $(1 , 0)$ (b) $(0.6 , 0.8)$ (c) $(0 , -1)$ (d) $(1 , 1)$

(3) If $(6 , 4)$ and $(3 , m)$ are two perpendicular vectors , then $m = \dots\dots$

(a) 2 (b) -2 (c) 8 (d) -4.5

(4) 📖 If $\overrightarrow{A} = (-2 , 1)$ and $\overrightarrow{C} = (-3 , k)$ are parallel , then $k = \dots\dots$

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{3}{2}$ (d) $\frac{3}{2}$

(5) If $\overrightarrow{A} = (4 , 5)$ and $\overrightarrow{B} = (-20 , 16)$, then the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ are $\dots\dots$

(a) perpendicular. (b) parallel. (c) equivalent. (d) otherwise.

(6) If : $\overrightarrow{A} = (k , 2)$, $\overrightarrow{B} = 2\hat{i} - \hat{j}$ and $\overrightarrow{A} \perp \overrightarrow{B}$, then $k = \dots\dots$

(a) 1 (b) -1 (c) ± 1 (d) zero

(7) If : $\overrightarrow{A} = (k , 9)$ is parallel to $\overrightarrow{B} = (-4 , 3)$, then $k = \dots\dots$

(a) -4 (b) -3 (c) -12 (d) 12

(8) 📖 If $\overrightarrow{A} = (4 , 2)$, $\overrightarrow{B} = (1 , -2)$, then $\|\overrightarrow{A} - \overrightarrow{B}\| = \dots\dots$

(a) 3 (b) 4 (c) 5 (d) 7

(9) If : $\overrightarrow{A} = (3 , 5)$, $\overrightarrow{B} = (4 , 6)$, then : $\|-2\overrightarrow{A} + 3\overrightarrow{B}\| = \dots\dots$

(a) 6 (b) 8 (c) 10 (d) 14

(10) If A = (– 3 , 4) , B = (2 , 1) and $\overrightarrow{C} = \overrightarrow{A} + 2\overrightarrow{B}$, then : $\overrightarrow{C}$ = ……………

(a) (– 1 , 5) (b) (– 4 , 9) (c) (1 , 6) (d) (1 , 5)

(11) If $\overrightarrow{A} + \overrightarrow{B} = (8, 16)$ and $\overrightarrow{A} = (5, 12)$, then $\|\overrightarrow{B}\|$ = ……………

(a) 7 (b) 5 (c) 13 (d) $8\sqrt{5}$

(12) If 2 (3 , y) + 3 (x , – 2) = (3 , 2) , then : x + y = ……………

(a) 3 (b) 4 (c) 5 (d) 6

(13) If $\overrightarrow{B} = (-2, 4)$, $\overrightarrow{C} = (4, 3)$ and $2\overrightarrow{A} - 3\overrightarrow{B} = 4\overrightarrow{C}$, then : $\|\overrightarrow{A}\|$ = ……………

(a) 5 (b) 12 (c) 13 (d) 7

(14) If $\overrightarrow{A} = (4, -6)$ and $3(\overrightarrow{A} - \overrightarrow{B}) = 7\overrightarrow{B} - 2\overrightarrow{A}$, then $\overrightarrow{B}$ = ……………

(a) (2 , – 3) (b) (– 2 , 3) (c) (– 2 , – 3) (d) (2 , 3)

(15) If $\overrightarrow{A} = 3\overrightarrow{i} + k\overrightarrow{j}$ and $\|\overrightarrow{A}\| = 5$, then k = ……………

(a) 4 (b) – 4 (c) ± 4 (d) 2

(16) 🕮 If $\|3k\overrightarrow{A}\| = \|-15\overrightarrow{A}\|$, then k = ……………

(a) 5 (b) – 5 (c) ± 5 (d) 15

(17) 🕮 If $\|-8\overrightarrow{A}\| = 5\|k\overrightarrow{A}\|$, then k = ……………

(a) $\frac{8}{5}$ (b) $\frac{5}{8}$ (c) $\pm\frac{8}{5}$ (d) $\pm\frac{5}{8}$

(18) If $\overrightarrow{A} = (k, 2)$, $\overrightarrow{B} = (2, m)$ and $\overrightarrow{A} = 2\overrightarrow{B}$, then k + m = ……………

(a) 4 (b) 5 (c) 1 (d) 3

(19) If $k\|4\overrightarrow{A}\| = \|-3\overrightarrow{A}\|$, then k = ……………

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{-3}{4}$ (d) $\pm\frac{3}{4}$

(20) If $\|k(3, 4)\| = 1$, then k = ……………

(a) $\frac{1}{7}$ (b) $\frac{1}{25}$ (c) $\pm\frac{1}{5}$ (d) ± 5

(21) If $\overrightarrow{A} = \left(6\sqrt{2}, \frac{3\pi}{4}\right)$ is a position vector of the point A , then the coordinates of A are ……………

(a) (6 , – 6) (b) (– 6 , 6) (c) (6 , 6) (d) (– 6 , – 6)

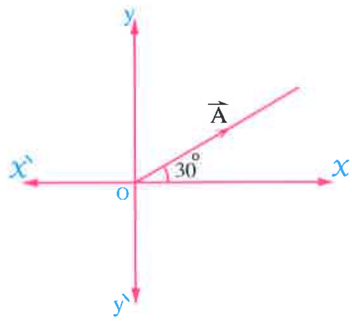
(22) The vector $\overrightarrow{M} = \left(8\sqrt{2}, \frac{\pi}{4}\right)$ is expressed in terms of the fundamental unit vectors by the form ……………

(a) $4\overrightarrow{i} + 4\overrightarrow{j}$ (b) $8\overrightarrow{i} - 8\overrightarrow{j}$ (c) $-4\overrightarrow{i} - 8\overrightarrow{j}$ (d) $8\overrightarrow{i} + 8\overrightarrow{j}$

(23) The polar form of the vector $\overrightarrow{A} = -3\overrightarrow{j}$ is ……………

(a) $\left(-3, \frac{\pi}{2}\right)$ (b) $\left(3, \frac{\pi}{2}\right)$ (c) $\left(-3, \frac{3\pi}{2}\right)$ (d) $\left(3, \frac{3\pi}{2}\right)$

(24) In the opposite figure :



$\|\overrightarrow{A}\| = 4$ length unit

, then : $\overrightarrow{A} =$ ....

(a) $\left(2, 2\sqrt{3}\right)$ (b) $\left(2\sqrt{3}, 2\right)$

(c) $\left(4, \sqrt{3}\right)$ (d) $\left(\sqrt{3}, 2\right)$

(25) If the magnitude of the vector $\overrightarrow{A}$ equals 7 , then the magnitude of the vector $-2\overrightarrow{A}$ equals ....

(a) 7 (b) -2 (c) 14 (d) -14

(26) If $\overrightarrow{A} = \left(3, \frac{\pi}{4}\right)$, then $2\overrightarrow{A} =$ ....

(a) $\left(6, \frac{\pi}{2}\right)$ (b) $\left(6, \frac{\pi}{4}\right)$ (c) $\left(3, \frac{\pi}{2}\right)$ (d) $\left(3, \frac{\pi}{4}\right)$

(27) If $\overrightarrow{L} = (2, -3)$ and $\overrightarrow{K} = (3, 1 - x)$ are parallel , then $x =$ ....

(a) 4 (b) $\frac{11}{2}$ (c) -1 (d) -9

(28) If $\overrightarrow{A} = (x, 4)$, $\overrightarrow{B} = (2, y)$ and $\overrightarrow{A} // \overrightarrow{B}$, then ....

(a) $x + 2y = 0$ (b) $x = 2y$ (c) $xy = 8$ (d) $\frac{x}{y} = 2$

(29) If $\overrightarrow{A} = (2, -1)$, $\overrightarrow{B} = (3, 4)$, $\overrightarrow{C} = (k, 15)$ and the two vectors $\overrightarrow{C}$, $\overrightarrow{B} - \overrightarrow{A}$ are parallel , then k = ...

(a) 1 (b) 2 (c) 3 (d) 4

(30) If the two vectors $\overrightarrow{A} = (n, 1)$, $\overrightarrow{B} = n\overrightarrow{i} - 4\overrightarrow{j}$ are perpendicular , then n = ...

(a) 2 (b) -2 (c) ± 2 (d) -4

(31) If $\overrightarrow{A}$ is a non zero vector , then ....

(a) $-\overrightarrow{A} \perp \overrightarrow{A}$ (b) $-\overrightarrow{A}$, $\overrightarrow{A}$ have the same direction.

(c) $\|-\overrightarrow{A}\| < \|\overrightarrow{A}\|$ (d) $-\overrightarrow{A}$, $\overrightarrow{A}$ have opposite directions.

(32) Which of the following vectors are perpendicular ?

(a) $(3, 0)$, $(2, -1)$ (b) $(-2, 5)$, $(4, -10)$

(c) $(2, 0)$, $(0, 2)$ (d) $(1, -4)$, $(2, -8)$

(33) If $\overrightarrow{A} = (-2, 1)$, $\overrightarrow{B} = (k, 3)$, $\overrightarrow{C} = (m, -4)$ and $\overrightarrow{A} // \overrightarrow{B}$, $\overrightarrow{A} \perp \overrightarrow{C}$, then $\frac{k}{m} =$ ....

(a) -6 (b) -2 (c) 3 (d) -3

(34) If $\overrightarrow{A} = 3\overrightarrow{i} - 4\overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{j}$, $\overrightarrow{C} = \left(5 , \frac{\pi}{18}\right)$, then : $\|\overrightarrow{A}\| + \|\overrightarrow{B}\| + \|\overrightarrow{C}\| =$ ....

(a) 9 (b) 10 (c) 11 (d) 12

(35) The vector represents a uniform velocity of 6 km./h. of a car moves towards western north = ...

(a) $3\sqrt{2}\,\overrightarrow{i} + 3\sqrt{2}\,\overrightarrow{j}$ (b) $3\sqrt{2}\,\overrightarrow{i} - 3\sqrt{2}\,\overrightarrow{j}$

(c) $-3\sqrt{2}\,\overrightarrow{i} + 3\sqrt{2}\,\overrightarrow{j}$ (d) $-6\sqrt{2}\,\overrightarrow{i} + 6\sqrt{2}\,\overrightarrow{j}$

(36) The vector represents a displacement of 40 cm. of a body in direction of eastern south ....

(a) $20\sqrt{2}\,\overrightarrow{i} + 20\sqrt{2}\,\overrightarrow{j}$ (b) $-20\sqrt{2}\,\overrightarrow{i} + 20\sqrt{2}\,\overrightarrow{j}$

(c) $-20\sqrt{2}\,\overrightarrow{i} - 20\sqrt{2}\,\overrightarrow{j}$ (d) $20\sqrt{2}\,\overrightarrow{i} - 20\sqrt{2}\,\overrightarrow{j}$

(37) If the magnitude of force $\overrightarrow{F} = 10$ newton and it acts in direction 30° north of the east , then $\overrightarrow{F} =$ ....

(a) $5\sqrt{3}\,\overrightarrow{i} - 5\,\overrightarrow{j}$ (b) $5\,\overrightarrow{i} + 5\sqrt{3}\,\overrightarrow{j}$ (c) $5\sqrt{3}\,\overrightarrow{i} + 5\,\overrightarrow{j}$ (d) $-5\sqrt{3}\,\overrightarrow{i} + 5\,\overrightarrow{j}$

(38) If $\overrightarrow{A} = (k , k + 3)$, $\|\overrightarrow{A}\| = 3\sqrt{5}$ length units , then one of the values of k is ....

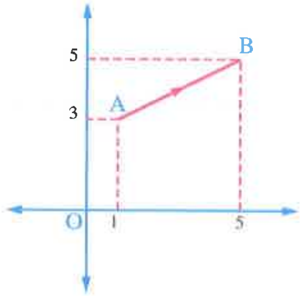
(a) – 3 (b) zero (c) 3 (d) 6

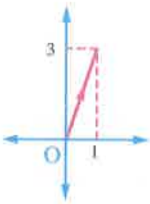
(39) **In the opposite figure :**

$A = (1 , 3)$, $B = (5 , 5)$

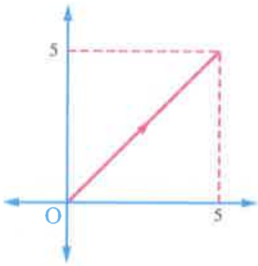
, then the figure represents

$\overrightarrow{AB}$ is ....

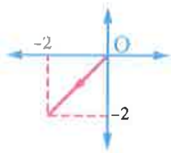




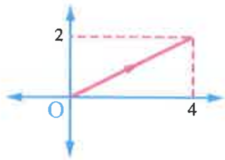
(a)



(b)



(c)



(d)

(40) If $\overrightarrow{A} = (x_1 , y_1)$ and $\overrightarrow{B} = (x_2 , y_2)$, then the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ are perpendicular if ....

(a) $x_1 y_2 - x_2 y_1 = 0$ (b) $x_1 x_2 - y_1 y_2 = 0$

(c) $\frac{x_1 x_2}{y_1 y_2} = -1$ (d) $\frac{x_1 y_2}{x_2 y_1} = 1$

(41) If $\overrightarrow{A} = (-1 , 2)$, $\overrightarrow{B} = (3 , 7)$, $\overrightarrow{C} = (7 , 12)$, then $\overrightarrow{C} =$ ....

(a) $2\overrightarrow{A} - \overrightarrow{B}$ (b) $\overrightarrow{A} + 2\overrightarrow{B}$ (c) $2\overrightarrow{B} - \overrightarrow{A}$ (d) $3\overrightarrow{A} + 2\overrightarrow{B}$

Second Essay questions

1 If $\overrightarrow{A} = (3, -2)$ and $\overrightarrow{B} = (2, 7)$, **find :**

(1) $\overrightarrow{A} + \overrightarrow{B}$ « (5, 5) »
(2) $\overrightarrow{B} - \overrightarrow{A}$ « (−1, 9) »
(3) $2\overrightarrow{A} - 3\overrightarrow{B}$ « (0, −25) »

---

2 If $\overrightarrow{A} = (3, -2)$, $\overrightarrow{B} = (-2, 4)$ and $\overrightarrow{C} = (7, 1)$, **then find :**

(1) $\|\overrightarrow{A} + 2\overrightarrow{C}\|$ « 17 »
(2) $\|5\overrightarrow{B} + 3\overrightarrow{A}\|$ « $\sqrt{197}$ »
(3) $\|\overrightarrow{A} - \frac{1}{2}\overrightarrow{B}\|$ « $4\sqrt{2}$ »
(4) $\|\overrightarrow{A} + \overrightarrow{B} - 2\overrightarrow{C}\|$ « 13 »

---

3 If $\overrightarrow{A} = 3\overrightarrow{i} - 2\overrightarrow{j}$ and $\overrightarrow{B} = -\overrightarrow{i} - 4\overrightarrow{j}$, **find :**

(1) $\overrightarrow{A} + \overrightarrow{B}$ « $2\overrightarrow{i} - 6\overrightarrow{j}$ »
(2) $\overrightarrow{A} - \overrightarrow{B}$ « $4\overrightarrow{i} + 2\overrightarrow{j}$ »
(3) $\|\overrightarrow{A} + \overrightarrow{B}\|$ « $2\sqrt{10}$ »
(4) $2\overrightarrow{A} + 3\overrightarrow{B}$ « $3\overrightarrow{i} - 16\overrightarrow{j}$ »
(5) $\overrightarrow{A} - 3\overrightarrow{B}$ « $6\overrightarrow{i} + 10\overrightarrow{j}$ »
(6) $-3\overrightarrow{A}$ « $-9\overrightarrow{i} + 6\overrightarrow{j}$ »

---

4 **Express each of the following vectors in terms of the fundamental unit vectors, then find the norm of each of them :**

(1) $\overrightarrow{M} = (-4, -3)$ « 5 »
(2) $\overrightarrow{N} = (8, -6)$ « 10 »
(3) $\overrightarrow{F} = (-5, -12)$ « 13 »
(4) $\overrightarrow{A} = \left(0, 2\sqrt{2}\right)$ « $2\sqrt{2}$ »
(5) $\overrightarrow{B} = \left(-3\sqrt{3}, 0\right)$ « $3\sqrt{3}$ »
(6) $\overrightarrow{C} = \left(\sqrt{2}, -3\sqrt{2}\right)$ « $2\sqrt{5}$ »

---

5 **Find the polar form of each of the following vectors :**

(1) $\overrightarrow{M} = 8\sqrt{3}\,\overrightarrow{i} + 8\overrightarrow{j}$
(2) $\overrightarrow{N} = 3\sqrt{2}\,\overrightarrow{i} + 3\sqrt{2}\,\overrightarrow{j}$
(3) $\overrightarrow{OA} = \left(5, 5\sqrt{3}\right)$
(4) $\overrightarrow{B} = \left(7\sqrt{3}, -7\right)$
(5) $\overrightarrow{C} = -4\overrightarrow{i} - 4\overrightarrow{j}$

---

6 If $\overrightarrow{OA}$ is the position vector of the point A with respect to the origin point, **find the coordinates of A in each of the following :**

(1) $\overrightarrow{OA} = \left(12\sqrt{3}, 60°\right)$
(2) $\overrightarrow{OA} = \left(5\sqrt{2}, \frac{3\pi}{4}\right)$
(3) $\overrightarrow{OA} = (24, 150°)$
(4) $\overrightarrow{OA} = \left(6, \frac{5\pi}{3}\right)$

---

7 **Find the value of each of X and y in each of the following :**

(1) $X(-6, y) = (3, 5)$ « $-\frac{1}{2}, -10$ »
(2) $(X, -3) - (5, 0) = (2, y)$ « 7, −3 »
(3) $(3, y) + X(1, -2) - (-4, 1) = \overrightarrow{0}$ « −7, −13 »
(4) $X(2, 3) + y(-3, 1) = (-4, 5)$ « 1, 2 »

8 If $\overrightarrow{A} = (6, -8)$, $\overrightarrow{B} = (-9, 12)$ and $\overrightarrow{C} = (-4, -3)$

(1) **Prove that :** $\overrightarrow{A} \parallel \overrightarrow{B}$, $\overrightarrow{B} \perp \overrightarrow{C}$, $\overrightarrow{C} \perp \overrightarrow{A}$

(2) **Find :** $2\overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{B} - 2\overrightarrow{C}$, $\frac{1}{2}\overrightarrow{A} + \overrightarrow{B} - 3\overrightarrow{C}$ « (3 , – 4) , (– 1 , 18) , (6 , 17) »

---

9 If $\overrightarrow{A} = \left(-2\sqrt{3}, -3\right)$ and $\overrightarrow{B} = \left(2, \sqrt{3}\right)$

, write the relation between the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ with showing the reason.

---

10 If $\overrightarrow{M} = 2\overrightarrow{i} + 3\overrightarrow{j}$, $\overrightarrow{N} = -8\overrightarrow{i} - 12\overrightarrow{j}$, $\overrightarrow{L} = a\overrightarrow{i} + 15\overrightarrow{j}$ and $\overrightarrow{F} = 6\overrightarrow{i} + b\overrightarrow{j}$:

(1) **Prove that :** $\overrightarrow{M} \parallel \overrightarrow{N}$

(2) **Find :** $a \in \mathbb{R}$ if $\overrightarrow{N} \parallel \overrightarrow{L}$

(3) **Find the value of each of :** $4\overrightarrow{M} + \overrightarrow{N}$, $4\left(\overrightarrow{M} + \overrightarrow{N}\right)$

(4) **Find :** $b \in \mathbb{R}$ if $\overrightarrow{F} \perp \overrightarrow{N}$

(5) Is $\overrightarrow{F} \perp \overrightarrow{M}$? Explain your answer.

---

11 If $\overrightarrow{A} = (3, -2)$, $\overrightarrow{B} = (-2, 5)$ and $\overrightarrow{C} = (0, 11)$:

(1) **Write each of the following vectors in terms of the fundamental unit vectors :**

$2\overrightarrow{B}$, $3\overrightarrow{C}$, $\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$, $\frac{1}{2}\left(\overrightarrow{B} + \overrightarrow{C}\right)$

(2) Express $\overrightarrow{C}$ in terms of $\overrightarrow{A}$ and $\overrightarrow{B}$

---

12 If $\overrightarrow{A} = (4, 3)$, $\overrightarrow{B} = (-2, 5)$ and $\overrightarrow{C} = (2, 21)$:

(1) **Find each of the following vectors in terms of the fundamental unit vectors :**

$\overrightarrow{B} - \overrightarrow{C}$, $\frac{1}{2}\left(\overrightarrow{B} + \overrightarrow{C}\right)$

(2) Express $\overrightarrow{C}$ in terms of $\overrightarrow{A}$ and $\overrightarrow{B}$

---

13 If $\overrightarrow{C} = (4, -1)$, $\overrightarrow{D} = (-3, 2)$ and $\overrightarrow{E} = (-5, 1)$

, **find the vector $\overrightarrow{A}$ which satisfies the equation :** $2\overrightarrow{A} = 2\overrightarrow{C} - 3\overrightarrow{D} + 2\overrightarrow{E}$ « $\left(\frac{7}{2}, -3\right)$ »

---

14 If $\overrightarrow{A} = (7, -3)$, $\overrightarrow{B} = (-2, 5)$ and $\overrightarrow{C} = 2\overrightarrow{i} + 3\overrightarrow{j}$

(1) **Prove that :** The vector $\overrightarrow{L} = 2\overrightarrow{C} + 3\overrightarrow{B} - \overrightarrow{A}$ is parallel to the vector $\overrightarrow{M} = 3\overrightarrow{i} - 8\overrightarrow{j}$

(2) If $\overrightarrow{L} = k\overrightarrow{M}$, **find :** k « – 3 »

---

15 On an orthogonal coordinate plane , if $\overrightarrow{L} = 5\overrightarrow{i} - 3\sqrt{3}\,\overrightarrow{j}$, $\overrightarrow{M} = -\overrightarrow{i} - 2\sqrt{3}\,\overrightarrow{j}$

and $\overrightarrow{N} = -2\overrightarrow{i} + \sqrt{3}\,\overrightarrow{j}$, **find $\overrightarrow{C}$ in the polar form where :** $\overrightarrow{C} = \overrightarrow{L} + \overrightarrow{M} + 3\overrightarrow{N}$ « (4 , 240°) »

16 **Find in terms of the fundamental unit vectors , the vector which expresses :**

(1) A force of magnitude 37 newtons acts at a particle in the north direction.

(2) A uniform speed of magnitude 60 km./h. in west direction.

(3) A uniform speed of a car covered 75 km. per hour in east direction.

(4) A displacement of a body a distance 25 m. in south direction.

(5) A vector , its norm is 6 units and makes an angle of measure $\frac{\pi}{4}$ with the positive direction of X-axis.

(6) A displacement of a body a distance 150 cm. in the direction 30° north of west.

(7) A force of magnitude 20 kg.wt. acts on a body in the direction 30° south of east.

(8) A displacement of a body a distance 40 cm. in the direction north west.

---

17 A , B , C and D are four collinear points are ordered from left to right where AB : BC : CD = 2 : 3 : 5

Complete each of the following with a suitable number given that the symbol "=" means "is equivalent to" :

(1) $\overrightarrow{AB} = \dots\dots \overrightarrow{BC}$

(2) $\overrightarrow{CB} = \dots\dots \overrightarrow{AB}$

(3) $\overrightarrow{CD} = \dots\dots \overrightarrow{AB}$

(4) $\overrightarrow{AC} = \dots\dots \overrightarrow{DC}$

(5) $\overrightarrow{BC} = \dots\dots \overrightarrow{CD}$

(6) $\overrightarrow{CD} = \dots\dots \overrightarrow{AC}$

(7) $\overrightarrow{BD} = \dots\dots \overrightarrow{BA}$

(8) $\overrightarrow{CD} = \dots\dots \overrightarrow{CB}$

(9) $\overrightarrow{AD} = \dots\dots \overrightarrow{CA}$

---

18 If $\overrightarrow{A} = 2\overrightarrow{i} + \overrightarrow{j}$ and $\overrightarrow{B} = \overrightarrow{i} + 3\overrightarrow{j}$ **, find :**

(1) The value of k which makes the vector $\left(\overrightarrow{A} + k\overrightarrow{B}\right)$ parallel to the vector $\overrightarrow{i}$ «$-\frac{1}{3}$»

(2) The value of ℓ which makes the vector $\left(\ell\overrightarrow{A} + \overrightarrow{B}\right)$ parallel to the vector $\overrightarrow{j}$ «$-\frac{1}{2}$»

---

19 **The lattice opposite of congruent parallelograms.**

Express each of the following directed line segments in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$:

(1) $\overrightarrow{AB}$

(2) $\overrightarrow{BY}$

(3) $\overrightarrow{EC}$

(4) $\overrightarrow{DE}$

(5) $\overrightarrow{XE}$

(6) $\overrightarrow{XY}$

(7) $\overrightarrow{YM}$

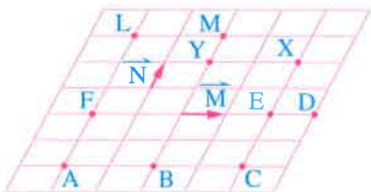
(8) $\overrightarrow{LM}$

(9) $\overrightarrow{BM}$

(10) $\overrightarrow{EF}$

(11) $\overrightarrow{FL}$

(12) $\overrightarrow{FD}$



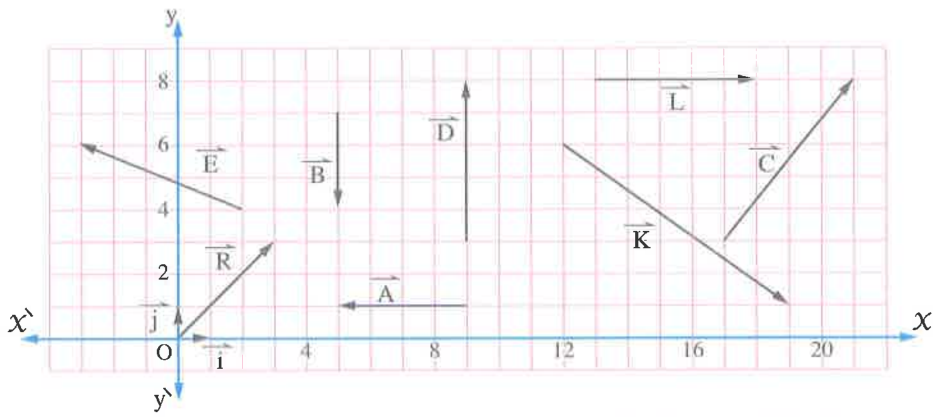
20 **Construct an orthogonal coordinate plane where O is the origin point and plot the position vector of the vector $\overrightarrow{M} = (2 , 3)$, then draw :**

(1) A directed line segment , its starting point is $A = (-3 , -2)$ which represents the vector $2\overrightarrow{M}$, then find the coordinates of the ending point.

(2) A directed line segment , its starting point is $B = (4 , 5)$ which represents the vector $-\overrightarrow{M}$, then find the coordinates of the ending point.

---

21 The following figure shows a representation of some vectors in the orthogonal coordinate plane. Write each vector in terms of the two fundamental unit vectors.



Third Problems that measure high standard levels of thinking

- **Choose the correct answer from those given :**

(1) If $\overrightarrow{A}$ is a vector , $\|\overrightarrow{A}\| = 4$, which of the following is a unit vector ?

(a) $\frac{1}{2}\overrightarrow{A}$ (b) $-\overrightarrow{A}$ (c) $-\frac{1}{4}\overrightarrow{A}$ (d) $\frac{3}{4}\overrightarrow{A}$

(2) If $\overrightarrow{A} = 2\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + 2\overrightarrow{j}$, then ....

(a) $\overrightarrow{A} = \overrightarrow{B}$ (b) $\overrightarrow{A} // \overrightarrow{B}$ (c) $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$ (d) $\overrightarrow{A} \perp \overrightarrow{B}$

(3) If $\overrightarrow{A} = 20\overrightarrow{i} - 15\overrightarrow{j}$, $\overrightarrow{B} = 7\overrightarrow{i} + 24\overrightarrow{j}$ and $\overrightarrow{M} = \overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{N} = \overrightarrow{A} - \overrightarrow{B}$, then ....

(a) $\overrightarrow{M} // \overrightarrow{N}$ (b) $\overrightarrow{M} \perp \overrightarrow{N}$ (c) $\overrightarrow{M} = \overrightarrow{N}$ (d) $\|\overrightarrow{M}\| = \|\overrightarrow{N}\|$

(4) If $\overrightarrow{A} = 3\overrightarrow{i} + 4\overrightarrow{j}$, $\overrightarrow{B} = 7\overrightarrow{i} + 24\overrightarrow{j}$, then the vector which has the same magnitude as $\overrightarrow{B}$ and parallel to $\overrightarrow{A}$ is ....

(a) $5\overrightarrow{i} + 20\overrightarrow{j}$ (b) $15\overrightarrow{i} + 10\overrightarrow{j}$ (c) $\overrightarrow{i} + 3\overrightarrow{j}$ (d) $15\overrightarrow{i} + 20\overrightarrow{j}$

(5) If $\overrightarrow{A} = 2\overrightarrow{i} - \overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{C} = \overrightarrow{i} + 3\overrightarrow{j}$ and $\overrightarrow{A} \perp (k\overrightarrow{B} + \overrightarrow{C})$, then $k =$ ....

(a) -1 (b) 1 (c) 3 (d) 4

(6) Which of the following statements is incorrect ?

(a) If $\overrightarrow{A} = \overrightarrow{B}$, then $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$ (b) If $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$, then $\overrightarrow{A} = \overrightarrow{B}$

(c) If $\overrightarrow{A} \,//\, \overrightarrow{B}$, then $\overrightarrow{A} = k\,\overrightarrow{B}$ (d) If $\overrightarrow{A} = k\,\overrightarrow{B}$, then $\overrightarrow{A} \,//\, \overrightarrow{B}$

(7) If $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$, then ....

(a) $\overrightarrow{A} = \overrightarrow{B}$ (b) $\overrightarrow{A} = -\overrightarrow{B}$

(c) $\overrightarrow{A} = \pm\overrightarrow{B}$ (d) We can not determine the relation between $\overrightarrow{A}$ and $\overrightarrow{B}$

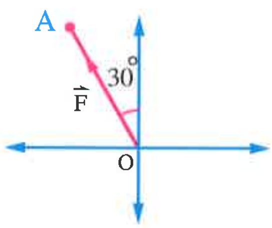
(8) The measure of the angle between two vectors $\overrightarrow{A} = 6\,\overrightarrow{i} - 2\,\overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + 3\,\overrightarrow{j}$ is ....

(a) zero (b) 30° (c) 60° (d) 90°

(9) $\overrightarrow{AB}$ is a directed line segment , C is a point in its plane $C \notin \overline{AB}$, then the number of directed line segments that can be drawn such that its initial point C and equivalent to $\overrightarrow{AB}$ is ....

(a) zero (b) 1 (c) 2 (d) infinite number.

(10) **In the opposite figure :**



If $\overrightarrow{OA}$ represents force $\overrightarrow{F}$, $\|\overrightarrow{F}\| = 12$ units.

Which of the following statements does not represent the force $\overrightarrow{F}$

(a) The Force $\overrightarrow{F}$ of magnitude 12 force unit and acts in direction 60° north of the west.

(b) $\overrightarrow{F} = (12 \text{ force unit} , 120°)$

(c) $\overrightarrow{F} = -6\,\overrightarrow{i} + 6\sqrt{3}\,\overrightarrow{j}$

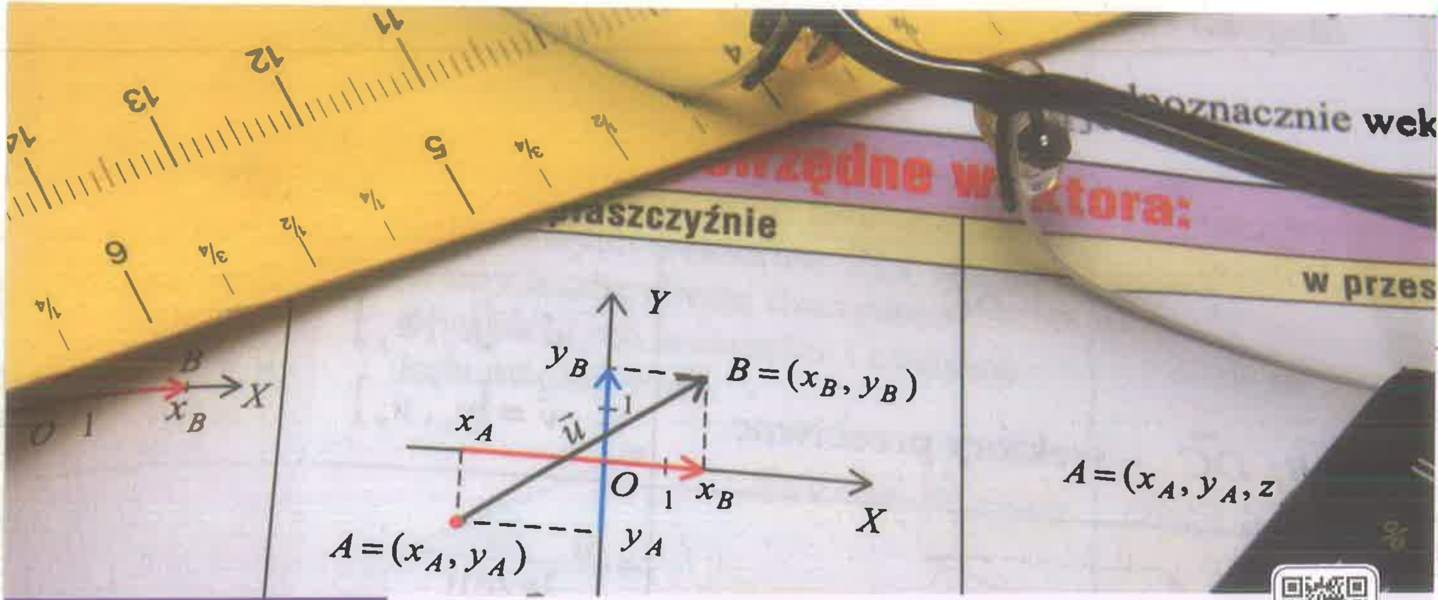
(d) The magnitude of the force is 12 force unit and acts in direction makes 30° with the north.

(11) If the polar form of the vector $\overrightarrow{A}$ is $\left(12 , \frac{2\pi}{3}\right)$, then the polar form of the vector $-\overrightarrow{A}$ is ....

(a) $\left(12 , \frac{\pi}{6}\right)$ (b) $\left(12 , \frac{2\pi}{3}\right)$ (c) $\left(6 , \frac{4\pi}{3}\right)$ (d) $\left(12 , \frac{5\pi}{3}\right)$

(12) If the position vector $\overrightarrow{A} = \left(\sqrt{3} , 1\right)$ rotates about origin with angle of measure 45° anticlockwise , then the polar form of the vector $\overrightarrow{A}$ after rotation is ....

(a) (2 , 30°) (b) (2 , 45°) (c) (2 , 75°) (d) (4 , 75°)



Exercise **Three**

Operations on vectors

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) If $\overrightarrow{A} = (3 , -2)$, $\overrightarrow{B} = (1 , 2)$, then $\overrightarrow{AB} =$ ....

(a) $(3 , -4)$ (b) $(-2 , 4)$ (c) $(4 , 0)$ (d) $(2 , -4)$

(2) If $\overrightarrow{A} = (-1 , 5)$, $\overrightarrow{B} = (2 , 1)$, then $\|\overrightarrow{AB}\| =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(3) If $\overrightarrow{A} = (4 , -2)$, $\overrightarrow{AB} = (3 , 5)$, then $\overrightarrow{B} =$ ....

(a) $(1 , -7)$ (b) $(7 , 3)$ (c) $(-1 , 7)$ (d) $(3 , 7)$

(4) If $\overrightarrow{A} = 5\,\overrightarrow{i} - 6\,\overrightarrow{j}$, $\overrightarrow{B} = (1 , 2)$, then $\overrightarrow{BA} =$ ....

(a) $4\,\overrightarrow{i} - 8\,\overrightarrow{j}$ (b) $-4\,\overrightarrow{i} + 8\,\overrightarrow{j}$

(c) $5\,\overrightarrow{i} - 4\,\overrightarrow{j}$ (d) $8\,\overrightarrow{i} - 4\,\overrightarrow{j}$

(5) If $\overrightarrow{A} = (7 , 0)$, $\overrightarrow{B} = \left(5\sqrt{2} , \frac{3\pi}{4}\right)$, then $\|\overrightarrow{AB}\| =$ .... length unit.

(a) 12 (b) 13 (c) 14 (d) $\sqrt{29}$

(6) $\overrightarrow{AB} - \overrightarrow{BA} =$ ....

(a) zero (b) $2\,\overrightarrow{AB}$ (c) $2\,\overrightarrow{BA}$ (d) $\overrightarrow{O}$

(7) If $\overrightarrow{AB} = 3\,\overrightarrow{i} + 3\,\overrightarrow{j}$, $\overrightarrow{BC} = \overrightarrow{j}$, then : $\|\overrightarrow{AC}\| =$ ....

(a) 6 (b) $3\sqrt{2}$ (c) 1 (d) 5

(8) If the two vectors $\overrightarrow{AB} = (2 , 3)$, $\overrightarrow{CB} = (-3 , 5)$, then $\overrightarrow{AC} =$ ....

(a) $(5 , -2)$ (b) $(-8 , 1)$ (c) $(-2 , 5)$ (d) $(2 , 5)$

(9) If $\overrightarrow{AB} = (3 , -4)$, $\overrightarrow{BC} = (2 , 1)$, then $\overrightarrow{CA} =$ ....

(a) $(1 , -5)$ (b) $(5 , -3)$ (c) $(-3 , 5)$ (d) $(-5 , 3)$

(10) If $3\overrightarrow{A} + \overrightarrow{B} = (5 , -2)$, $\overrightarrow{AB} = (-3 , 10)$, then $\overrightarrow{A} =$ ....

(a) $(2 , -1)$ (b) $(-1 , 2)$ (c) $(2 , -3)$ (d) $(-2 , 3)$

(11) If $A = (3 , -5)$, $B = (-1 , 5)$ and $\overrightarrow{M} = (6 , k)$ and $\overrightarrow{AB} \parallel \overrightarrow{M}$, then $k =$ ....

(a) -15 (b) -10 (c) -5 (d) 5

(12) If $\overrightarrow{AB} = (2 , 6)$, $\overrightarrow{AC} = (-2 , 9)$, then $\|\overrightarrow{BC}\| =$ ....

(a) 15 (b) 13 (c) 4 (d) 5

(13) If M is the midpoint of $\overline{XY}$, then $\overrightarrow{XM} + \overrightarrow{YM} =$ ....

(a) $2\overrightarrow{XM}$ (b) $\overrightarrow{XY}$ (c) $\overrightarrow{O}$ (d) $\overrightarrow{YX}$

(14) If ABC is a triangle , then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} =$ ....

(a) $\overrightarrow{O}$ (b) $2\overrightarrow{AC}$ (c) $2\overrightarrow{CA}$ (d) $\overrightarrow{AC}$

(15) If ABC is a triangle , then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC} =$ ....

(a) $\overrightarrow{O}$ (b) $2\overrightarrow{AC}$ (c) $2\overrightarrow{CA}$ (d) $\overrightarrow{AC}$

(16) If ABCD is a quadrilateral , then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} =$ ....

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{AD}$ (c) $\overrightarrow{O}$ (d) $2\overrightarrow{AD}$

(17) If ABC is a triangle , then $\overrightarrow{BA} - \overrightarrow{BC} =$ ....

(a) $\overrightarrow{BC}$ (b) $\overrightarrow{CB}$ (c) $\overrightarrow{CA}$ (d) $\overrightarrow{AC}$

(18) Which of the following equivalent to the zero vector ?

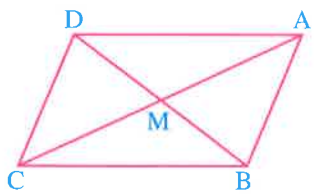
(a) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} - \overrightarrow{DF} - \overrightarrow{FC}$

(c) $\overrightarrow{FE} - \overrightarrow{FD} - \overrightarrow{DE}$ (d) $\overrightarrow{AB} - \overrightarrow{AN} + \overrightarrow{CN}$

(19) ABCD is a parallelogram , $\overline{AC} \cap \overline{BD} = \{M\}$, then : $\overrightarrow{AB} + \overrightarrow{AD} =$ ....

(a) $\overrightarrow{CA}$ (b) $\overrightarrow{BD}$ (c) $2\overrightarrow{MC}$ (d) $2\overrightarrow{DM}$

(20) **In the opposite figure :**



ABCD is a parallelogram

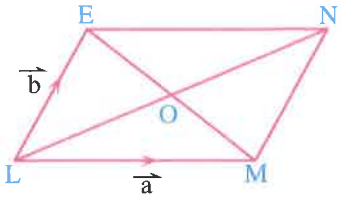
If the diagonals intersect at M , then all the following statements are expressions for $\overrightarrow{AC}$ except ....

(a) $\overrightarrow{AB} + \overrightarrow{BD}$ (b) $2\overrightarrow{AM}$ (c) $\overrightarrow{AD} + \overrightarrow{DC}$ (d) $\overrightarrow{BC} + \overrightarrow{DC}$

(21) In Δ ABC , if D is the midpoint of $\overline{BC}$, then $\overrightarrow{BA} + \overrightarrow{CA} + \overrightarrow{AD} =$ ....

(a) $\overrightarrow{BC}$ (b) $\overrightarrow{DA}$ (c) $2\,\overrightarrow{DA}$ (d) $\overrightarrow{CB}$

(22) **In the given parallelogram :**



$\overrightarrow{OM} + \overrightarrow{ON} =$ ....

(a) $\vec{a}$ (b) $\vec{b}$

(c) $\frac{1}{2}(\vec{a} + \vec{b})$ (d) $\frac{1}{2}(\vec{a} - \vec{b})$

(23) If ABCD is a rectangle , then $\overrightarrow{AC} + \overrightarrow{BD} =$ ....

(a) $\overrightarrow{CD}$ (b) $2\,\overrightarrow{BA}$ (c) $\overrightarrow{BC}$ (d) $2\,\overrightarrow{BC}$

(24) If D is the midpoint of $\overline{BC}$, A is a point $\notin \overline{BC}$, then ....

(a) $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$ (b) $\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$

(c) $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \vec{O}$ (d) $\overrightarrow{AB} + \overrightarrow{AC} + 2\,\overrightarrow{AD} = \vec{O}$

(25) If : $\overrightarrow{AB} = \overrightarrow{CD}$ where $\overrightarrow{AB} = (6 , 4)$, $\vec{C} = (-1 , 3)$, then $\vec{D} =$ ....

(a) (5 , 7) (b) (– 5 , – 7) (c) (– 5 , 7) (d) (7 , 7)

(26) In parallelogram ABCD , A = (7 , – 2) , B = (15 , 4) , C = (9 , 6) , then the coordinates of D are ....

(a) (1 , 0) (b) (0 , 1) (c) (– 1 , 0) (d) (0 , – 1)

(27) If ABCDE is a regular pentagon , then $\overrightarrow{DE} + \overrightarrow{EA} - \overrightarrow{BA} =$ ....

(a) $\overrightarrow{DB}$ (b) $\overrightarrow{AD}$ (c) $\overrightarrow{CE}$ (d) $\overrightarrow{BD}$

(28) If $\overrightarrow{AB} = 2\,\overrightarrow{AC}$, then ....

(a) Δ ABC is a right-angled triangle. (b) B is the midpoint of $\overline{AC}$

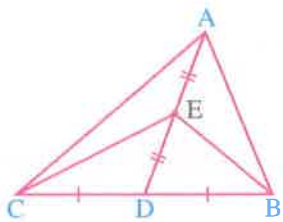
(c) $\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{CB}$ (d) C is the midpoint of $\overline{AB}$

(29) **In the opposite figure :**

ABC is a triangle, if D is the midpoint of $\overline{BC}$,

E is the midpoint of $\overline{AD}$

, then : $\overrightarrow{AB} + \overrightarrow{AC} =$ .... $\overrightarrow{AE}$



(a) 1 (b) 2

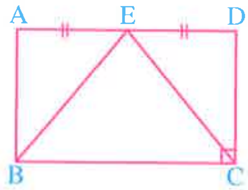
(c) 4 (d) – 4

(30) In the opposite figure :

ABCD is a rectangle ,

E is the midpoint of $\overline{AD}$, then

$\overrightarrow{EB} + \overrightarrow{BA} - \overrightarrow{DC} =$ ....

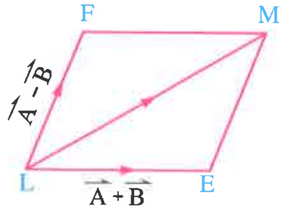


(a) $\overrightarrow{EB}$ (b) $\overrightarrow{BE}$

(c) $\overrightarrow{EC}$ (d) $\overrightarrow{CE}$

(31) In the opposite figure :

$\overrightarrow{LM}$ is a vector represents ....



(a) $2\,\vec{A}$ (b) $2\,\vec{B}$

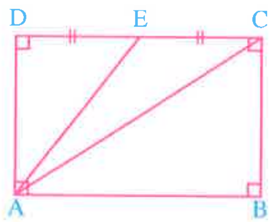
(c) $2\,\vec{A} - \vec{B}$ (d) $2\,\vec{B} - \vec{A}$

(32) In the opposite figure :

ABCD is a rectangle , E is the midpoint

of $\overline{CD}$, then

First : $\overrightarrow{AE} + \overrightarrow{DE} =$ ....



(a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$

(c) $2\,\overrightarrow{AD}$ (d) $\overrightarrow{CA}$

Second : $\overrightarrow{AD} - 2\,\overrightarrow{AE} + \overrightarrow{AB} =$ ....

(a) $\overrightarrow{AD}$ (b) $2\,\overrightarrow{CB}$ (c) $\overrightarrow{CB}$ (d) $2\,\overrightarrow{AD}$

(33) In Δ ABC , if D , E are the midpoints of $\overline{AB}$, $\overline{AC}$ respectively and $\overrightarrow{AB} = \vec{M}$, $\overrightarrow{AC} = \vec{N}$

, then $\overrightarrow{DE} =$ ....

(a) $\vec{M} + \vec{N}$ (b) $\vec{M} - \vec{N}$ (c) $\frac{1}{2}(\vec{M} - \vec{N})$ (d) $-\frac{1}{2}(\vec{M} - \vec{N})$

(34) ABCDEF is a regular hexagon , $\overrightarrow{AB} = \vec{M}$, $\overrightarrow{BC} = \vec{N}$, $\overrightarrow{CD} = \vec{F}$

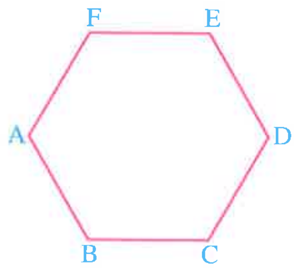
, then $\overrightarrow{AE} =$ .... (in term of $\vec{M}$, $\vec{N}$, $\vec{F}$)

(a) $\vec{M} + \vec{N} + \vec{F}$ (b) $\vec{M} - \vec{N} + \vec{F}$ (c) $\vec{N} + \vec{F}$ (d) $\vec{F} - \vec{M}$

(35) In the opposite figure :

If ABCDEF is a regular hexagon

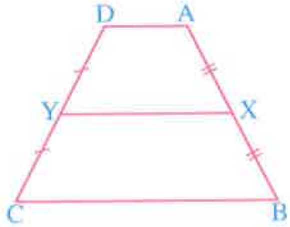
, then $(\overrightarrow{AB} - \overrightarrow{CB}) + \overrightarrow{AF} + \overrightarrow{DE} =$ ....



(a) $\overrightarrow{FE}$ (b) $\overrightarrow{AE}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{AC}$

(36) In the opposite figure :



ABCD is a trapezium

If : $\overrightarrow{AD} + \overrightarrow{BC} = k\ \overrightarrow{YX}$

, then the value of k = ...

Where $k \in \mathbb{R}$

(a) -2 (b) -1 (c) 1 (d) 2

(37) If A = (2 , 2) , B = (4 , $-$ 2) , C = ($-$ 2 , 0) , D = (1 , k) and $\overrightarrow{DA} \perp \overrightarrow{CB}$,

then k = ...

(a) 1 (b) -1 (c) $\frac{7}{3}$ (d) 2

(38) If ABCD is a parallelogram,

A (1 , 2) , B (x , $-$ 3) , C ($-$ 3 , 5) , D ($-$ 7 , y)

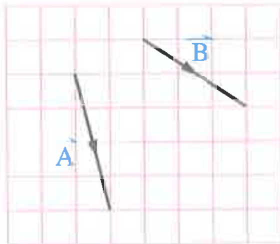
, then : $\overrightarrow{BD}$ = ...

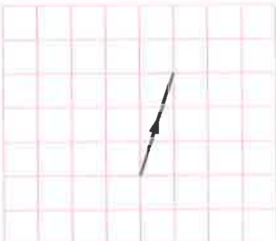
(a) ($-$ 2 , 7) (b) ($-$ 12 , $-$ 13) (c) (4 , $-$ 3) (d) ($-$ 12 , 13)

(39) If A = (3 , 5) , B = ($-$ 1 , m) , $\|\overrightarrow{AB}\|$ = 4 length units , then m = ...

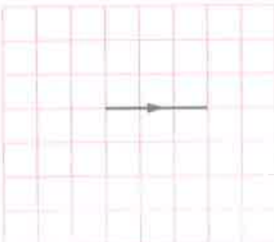
(a) zero (b) 5 (c) -1 (d) -5

(40) The given figure represents the two vectors $\overrightarrow{A}$, $\overrightarrow{B}$ Which of the following figures represents the vector $\overrightarrow{A}\ \ \overrightarrow{B}$?



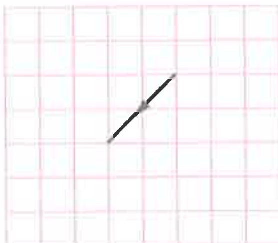


(a)



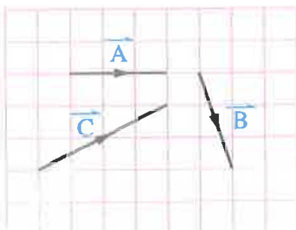
(b)

(c)



(d)

(41) In the opposite figure :

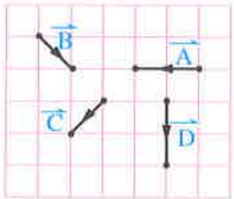


$\|\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}\|$ = ... (where the side length of each square in the lattice represents to length unit).

(a) 1 (b) 2

(c) 5 (d) 4

(42) The opposite figure represents the vectors $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$ which of the following is true ?



(a) $\overrightarrow{B} = -\overrightarrow{C}$ (b) $\overrightarrow{A} = \overrightarrow{B} + \overrightarrow{C}$

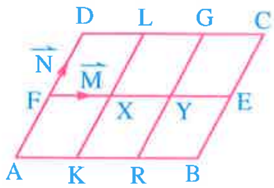
(c) $\overrightarrow{D} = \overrightarrow{B} + \overrightarrow{C}$ (d) $\overrightarrow{A} + \overrightarrow{D} = 2\,\overrightarrow{B}$

(43) In the opposite figure :

6 congruent parallelograms.

If $\overrightarrow{FX} = \overrightarrow{M}$, $\overrightarrow{FD} = \overrightarrow{N}$

, then $\overrightarrow{DR} =$ .... (in terms of $\overrightarrow{M}$, $\overrightarrow{N}$)



(a) $\overrightarrow{M} + \overrightarrow{N}$ (b) $2\,\overrightarrow{M} + 2\,\overrightarrow{N}$

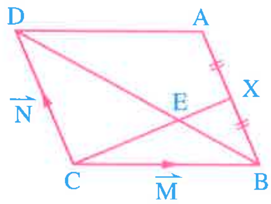
(c) $2\,\overrightarrow{M} - 2\,\overrightarrow{N}$ (d) $2\,\overrightarrow{N} - 2\,\overrightarrow{M}$

(44) In the opposite figure :

ABCD is a parallelogram

, X is the midpoint of $\overline{AB}$

, then $\overrightarrow{EX} =$ ....



(a) $\frac{1}{3}(\overrightarrow{N} + 2\,\overrightarrow{M})$ (b) $\frac{1}{3}(2\,\overrightarrow{N} - \overrightarrow{M})$

(c) $\frac{1}{3}(\overrightarrow{M} + \frac{1}{2}\,\overrightarrow{N})$ (d) $\frac{2}{3}(\overrightarrow{N} - 2\,\overrightarrow{M})$

Second Essay questions

1 ABCD is a parallelogram , in which A (3 , 0) , B (0 , 4) , D (– 2 , – 1)
Find the coordinates of the point C « (– 5 , 3) »

---

2 ABCD is a parallelogram in which A = (x , 2) , B = (3 , 8) , C = (9 , 1) , D = (7 , y)
Find the values of x , y , then find : $\|\overrightarrow{AB}\|$, $\|\overrightarrow{AD}\|$ « 1 , – 5 , $2\sqrt{10}$, $\sqrt{85}$ »

---

3 In an orthogonal cartesian coordinates plane if A = (– 1 , – 4) , B = (1 , 1) , C = (6 , – 1)
, find each of $\overrightarrow{AB}$, $\overrightarrow{BC}$ in terms of the two fundamental unit vectors ,
then prove that : $\overrightarrow{AB} \perp \overrightarrow{BC}$

---

4 If $2\,\overrightarrow{M} + 3\,\overrightarrow{AB} = 2\,\overrightarrow{CB} - \overrightarrow{BA}$, **prove that :** $\overrightarrow{M} = \overrightarrow{CA}$

---

5 In any triangle XYZ , **prove that :** $\overrightarrow{XY} + \overrightarrow{YZ} + \overrightarrow{ZX} = \overrightarrow{O}$

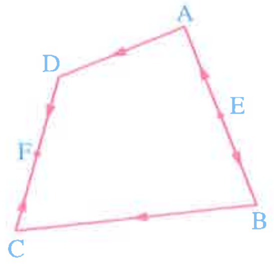
6 **In the opposite figure :**

ABCD is a quadrilateral ,

$E \in \overline{AB}$, $F \in \overline{CD}$

Prove that :

$\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CF} = \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DF}$



7 **In the quadrilateral ABCD , prove that :**

(1) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$

(2) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{DC} + \overrightarrow{AD}$

(3) $\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{AD}$

(4) $\overrightarrow{AB} + \overrightarrow{CD} - \overrightarrow{CB} - \overrightarrow{AD} = \overrightarrow{O}$

(5) $\overrightarrow{DB} - \overrightarrow{AC} = \overrightarrow{DA} - \overrightarrow{BC}$

8 ABCD is a trapezium in which $\overline{AD} \mathbin{//} \overline{BC}$, E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{DC}$ **Prove that :** $\overrightarrow{AD} + \overrightarrow{BC} = 2\,\overrightarrow{EF}$

9 ABCD is a quadrilateral in which $\overrightarrow{BC} = 3\,\overrightarrow{AD}$ **Prove that :**

(1) ABCD is a trapezium.

(2) $\overrightarrow{AC} + \overrightarrow{BD} = 4\,\overrightarrow{AD}$

10 ABCD is a trapezium in which $\overline{AD} \mathbin{//} \overline{BC}$, $\frac{AD}{BC} = \frac{2}{3}$ **Prove that :** $\overrightarrow{AC} + \overrightarrow{BD} = \frac{5}{2}\,\overrightarrow{AD}$

11 ABCD is a quadrilateral in which $2\,\overrightarrow{BC} = 5\,\overrightarrow{AD}$ **Prove that :**

(1) $2\,\overrightarrow{AC} + 2\,\overrightarrow{BD} = 7\,\overrightarrow{AD}$

(2) $2\,\overrightarrow{AB} - 2\,\overrightarrow{DC} = 3\,\overrightarrow{DA}$

12 ABCD is a parallelogram , its diagonals intersect at M , N is a point outside the parallelogram. **Prove that :**

(1) $\overrightarrow{AC} + \overrightarrow{DB} = 2\,\overrightarrow{AB}$

(2) $\overrightarrow{AC} - \overrightarrow{DB} = 2\,\overrightarrow{BC}$

(3) $\overrightarrow{AB} + \overrightarrow{AD} + 2\,\overrightarrow{CM} = \overrightarrow{O}$

(4) $\overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND}$

(5) $\overrightarrow{NA} + \overrightarrow{NB} + \overrightarrow{NC} + \overrightarrow{ND} = 4\,\overrightarrow{NM}$

(6) $\overrightarrow{AN} + \overrightarrow{BN} + \overrightarrow{ND} + \overrightarrow{NC} = 2\,\overrightarrow{AD}$

13 ABCD is a parallelogram in which E is the midpoint of $\overline{BC}$

Prove that : $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DC} = 2\,\overrightarrow{AE}$

14 ABCD is a quadrilateral in which X is the midpoint of $\overline{AC}$, Y is the midpoint of $\overline{BD}$

Prove that : $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\,\overrightarrow{XY}$

15 XYZ is a triangle , $L \in \overline{YZ}$ such that $YL : LZ = 5 : 3$

Prove that : $5\,\overrightarrow{XZ} + 3\,\overrightarrow{XY} = 8\,\overrightarrow{XL}$

---

16 If M is the point of intersection of the medians of the triangle ABC , P is a point outside the triangle , **prove that :** $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = 3\,\overrightarrow{PM}$

---

17 ABC is a triangle , D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AC}$ and F is the midpoint of $\overline{AB}$ **Prove that :**

(1) $\overrightarrow{AF} + \overrightarrow{AE} = \overrightarrow{AD}$

(2) $\overrightarrow{AF} - \overrightarrow{AE} = \frac{1}{2}\,\overrightarrow{CB}$

(3) $\overrightarrow{CF} + \overrightarrow{EB} = 3\,\overrightarrow{EF}$

---

18 ABC is a triangle in which D , E , F are the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively. M is the point of intersection of its medians. **Prove that :**

(1) 📖 $\overrightarrow{AE} + \overrightarrow{BF} = \overrightarrow{DC}$

(2) $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \overrightarrow{MD} + \overrightarrow{ME} + \overrightarrow{MF}$

---

19 📖 ABC is a triangle in which D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$

Prove that : $\overrightarrow{AE} + \overrightarrow{CD} = \overrightarrow{EB} + \overrightarrow{DA}$

---

20 ABC is a triangle , $D \in \overline{BC}$, $E \in \overline{BC}$ such that $BD = CE$

Prove that : $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{AE}$

---

21 If $A = (3 , 1)$, $B = (1 , 4)$, $C = (-1 , -5)$, **find N where :** $\overrightarrow{NA} + \overrightarrow{NB} + \overrightarrow{NC} = \overrightarrow{O}$

« (1 , 0) »

---

22 ABCD is a quadrilateral. If $\overrightarrow{AB} = (2 , 1)$, $\overrightarrow{BC} = (-3 , 4)$, $\overrightarrow{DC} = (5 , -1)$ **, find the vector which is represented by $\overrightarrow{AD}$ and if $B = (-1 , 3)$, find the coordinates of A , C , D**

---

23 If $\overrightarrow{AB} = (1 , -4)$, $A = (2 , 3)$, $C = (-1 , 15)$ **, find the values of ℓ and m such that :** $\ell\,\overrightarrow{A} - m\,\overrightarrow{B} = \overrightarrow{C}$

« 4 , 3 »

---

24 If ABC is a right-angled triangle at B where $A = (2 , 0)$, $B = (3 , 2)$, $C = (-5 , 5 - x)$ **, find the value of :** x

« – 1 »

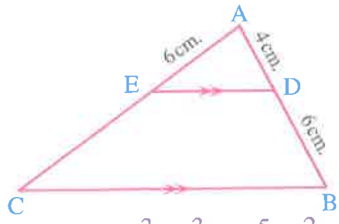
---

25 If ABCD is a rectangle in which $A = (5 , 1)$, $B = (2 , -2)$, $C = (-3 , k)$, then find the value of k and find the coordinates of the point D

« 3 , (0 , 6) »

26 **In the opposite figure :**

If $\overline{DE} \,//\, \overline{BC}$ **, find the numerical values of k , ℓ , m , n :**



(1) $\overrightarrow{BD} = k\,\overrightarrow{DA}$

(2) $\overrightarrow{CE} = \ell\,\overrightarrow{CA}$

(3) $\overrightarrow{BC} = m\,\overrightarrow{ED}$

(4) $\overrightarrow{AD} + \overrightarrow{DE} = n\,\overrightarrow{AC}$

« $\frac{3}{2}$, $\frac{3}{5}$, $\frac{-5}{2}$, $\frac{2}{5}$ »

---

27 If ABCDEF is a regular hexagon whose centre is M

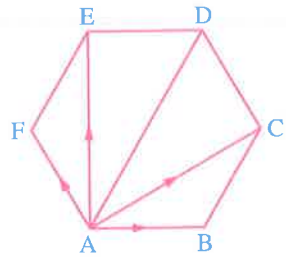
, prove that : $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} + \overrightarrow{ME} + \overrightarrow{MF} = \overrightarrow{O}$

---

28 📖 **In the opposite figure :**

ABCDEF is a regular hexagon.

Prove that : $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AE} + \overrightarrow{AF} = 2\,\overrightarrow{AD}$



---

29 In the orthogonal cartesian coordinate plane , $\overrightarrow{AB} = (-2 , 3)$, $\overrightarrow{CB} = (-6 , -4)$,

$2\,\overrightarrow{B} + \overrightarrow{AC} = (6 , 11)$ **Find :**

(1) The coordinates of each of the points A , B , C

(2) The area of Δ ABC using vectors. « 13 »

---

30 📖 ABCD is a trapezium in which A = (– 2 , – 3) , B = (4 , – 1) , C = (2 , 5) , D = (– 1 , k)

(1) If $\overrightarrow{AB} \,//\, \overrightarrow{DC}$ **, find the value of :** k

(2) **Prove that :** $\overrightarrow{CB} \perp \overrightarrow{AB}$

(3) **Find :** the area of the trapezium ABCD « 4 , 30 »

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) If $\overrightarrow{A}$, $\overrightarrow{B}$ are two non-zeroes vectors , then $\|\overrightarrow{A}\| + \|\overrightarrow{B}\|$ .... $\|\overrightarrow{A} + \overrightarrow{B}\|$

(a) > (b) < (c) ≥ (d) ≤

(2) If $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{C}$, $\|\overrightarrow{A}\| + \|\overrightarrow{B}\| = \|\overrightarrow{C}\|$, then ....

(a) $\overrightarrow{A}$, $\overrightarrow{B}$ are perpendicular.

(b) $\overrightarrow{A}$, $\overrightarrow{B}$ are equivalent.

(c) $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel

(d) $\overrightarrow{C}$ is perpendicular to $\overrightarrow{A}$ and $\overrightarrow{B}$

(3) If $\overrightarrow{A}$, $\overrightarrow{B}$ are two non-zeroes vectors and $\|\overrightarrow{A} + \overrightarrow{B}\| = \|\overrightarrow{A} - \overrightarrow{B}\|$, then ....

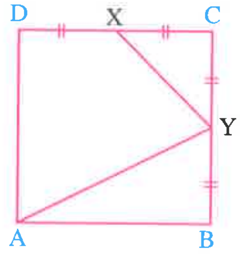
(a) $\overrightarrow{A} = -\,\overrightarrow{B}$

(b) $\overrightarrow{A}$, $\overrightarrow{B}$ are equivalent.

(c) $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel.

(d) $\overrightarrow{A}$, $\overrightarrow{B}$ are perependicular.

(4) In the opposite figure :



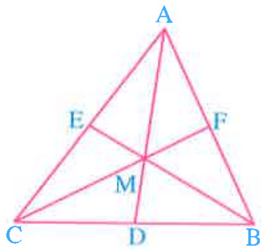
ABCD is a square and

$\overrightarrow{AY} + \overrightarrow{XY} = k\ \overrightarrow{XC}$, then k = ...

(a) 1 (b) 2

(c) 3 (d) 4

(5) In the opposite figure :



If M is the point of intersection of the medians of Δ ABC , then

First : $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} =$ ....

(a) $\overrightarrow{BC}$ (b) $\overrightarrow{\text{zero}}$

(c) $2\ \overrightarrow{BC}$ (d) $\overrightarrow{AB} + \overrightarrow{AC}$

Second : $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} =$ ....

(a) $\overrightarrow{MD} + \overrightarrow{ME}$ (b) $3\ \overrightarrow{MA}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ (d) $\frac{1}{3}\ (\overrightarrow{AB} + \overrightarrow{AC})$

Third : If $\overrightarrow{AB} + \overrightarrow{AC} = k\ \overrightarrow{AM}$, then k = ...

(a) 1 (b) 2 (c) 3 (d) 4

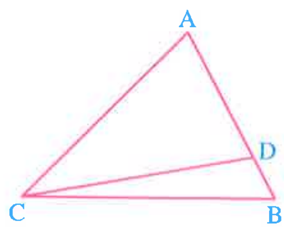
(6) M is the centre of regular hexagon ABCDEF and $\overrightarrow{DB} + \overrightarrow{DA} + \overrightarrow{DF} = k\ \overrightarrow{DM}$, then k = ...

(a) 2 (b) 3 (c) 4 (d) 5

(7) If the sum of two unit vectors $\vec{A}$, $\vec{B}$ is also unit vector $\vec{C}$ i.e. $\vec{A} + \vec{B} = \vec{C}$, then the magnitude of their difference $\| \vec{A} - \vec{B} \| =$ ....

(a) zero (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

(8) In the opposite figure :



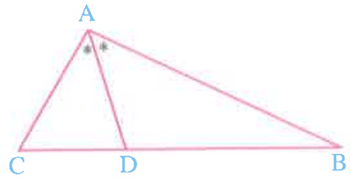
ABC is a triangle , D $\in \overline{AB}$

If AD = 3 DB and $\overrightarrow{CD} = k\ \overrightarrow{AC} + m\ \overrightarrow{CB}$, then k + m = ...

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) 2 (d) 3

(9) **In the opposite figure :**



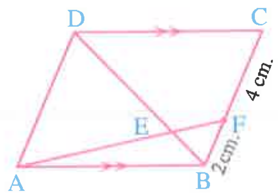
$\overrightarrow{AD}$ bisects $\angle$ BAC and AB = 2 AC

, then $\overrightarrow{AD}$ = ...

(a) $\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$ (b) $\frac{1}{3}(\overrightarrow{AC}+2\,\overrightarrow{AB})$

(c) $\frac{1}{2}(3\,\overrightarrow{AC}+2\,\overrightarrow{AB})$ (d) $\frac{1}{3}(2\,\overrightarrow{AC}+\overrightarrow{AB})$

(10) **In the opposite figure :**



ABCD is a parallelogram in which

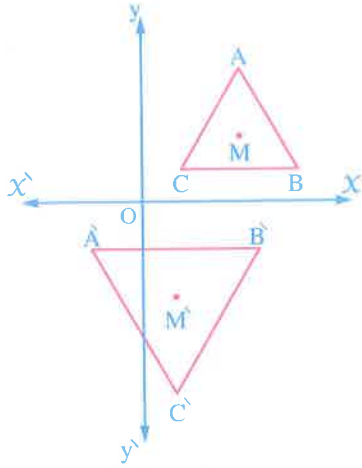
BF = 2 cm. , FC = 4 cm.

, then $\overrightarrow{AE}$ = ...

(a) $\frac{1}{2}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AD}$ (b) $\frac{1}{4}\overrightarrow{AB}+\frac{3}{4}\overrightarrow{AD}$

(c) $\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AD}$ (d) $\frac{3}{4}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{AD}$

(11) **In the opposite figure :**



If M (3 , 2) is the point of intersection

of medians of Δ ABC , M` (1 , – 3)

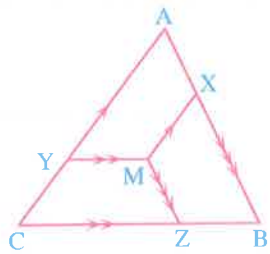
is the point of intersection of medians

of Δ A` B` C` , then $\overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}$ = ...

(a) (2 , 5) (b) (6 , 15)

(c) (– 2 , – 5) (d) (– 6 , – 15)

(12) **In the opposite figure :**



If M is the point of intersection

of medians of Δ ABC

, then $\overrightarrow{MX}+\overrightarrow{MY}+\overrightarrow{MZ}$ = ...

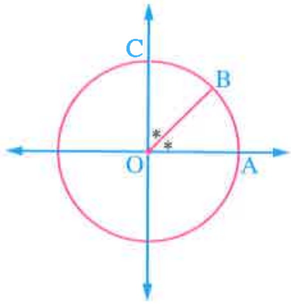
(a) 3 $\overrightarrow{ZB}$ (b) 2 $\overrightarrow{YC}$

(c) $\overrightarrow{\text{zero}}$ (d) 5 $\overrightarrow{AX}$

(13) **In the opposite figure :**

$\overrightarrow{OB}$ bisects $\angle$ AOC where O is the centre of the circle

, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} =$ ....



(a) $\sqrt{2}\ \overrightarrow{OB}$ (b) $2\ \overrightarrow{OB}$

(c) $\left(\sqrt{2}+1\right)\overrightarrow{OB}$ (d) $3\ \overrightarrow{OB}$

2 ABCD is a quadrilateral , X , Y , Z and N are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. **Prove that :** $\overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{DA} = 2\left(\overrightarrow{ZX} + \overrightarrow{NY}\right)$

3 ABC is a triangle , $D \in \overline{BC}$, if $\overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DB} = k\ \overrightarrow{AB}$

, then find the value of : k « 2 »

Exercise **Four**

Applications on vectors

Test yourself

From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

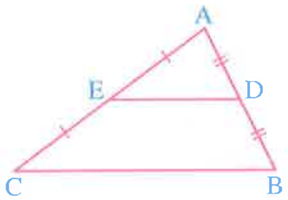
Problems on geometric applications

(1) ABCD is a trapezium, $\overline{AB} \parallel \overline{CD}$, A (2, 1), B (3, 2), C (4, 0). If AB = 2 CD, then the coordinates of D are ....

(a) $\left(3\frac{1}{2}, \frac{-1}{2}\right)$ (b) $\left(3\frac{1}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, 3\frac{1}{2}\right)$ (d) $\left(\frac{-1}{2}, 3\frac{1}{2}\right)$

(2) **In the given figure :**

D, E are the midpoints of $\overline{AB}$, $\overline{AC}$, $\overrightarrow{AB} = \overrightarrow{m}$, $\overrightarrow{AC} = \overrightarrow{n}$, then : $\overrightarrow{DE} =$ ....

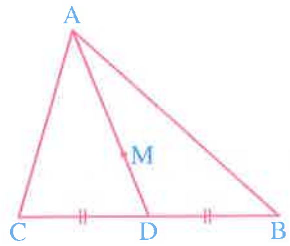


(a) $\overrightarrow{m} + \overrightarrow{n}$ (b) $\overrightarrow{m} - \overrightarrow{n}$

(c) $\frac{1}{2}(\overrightarrow{m} - \overrightarrow{n})$ (d) $-\frac{1}{2}(\overrightarrow{m} - \overrightarrow{n})$

(3) **In the given figure :**

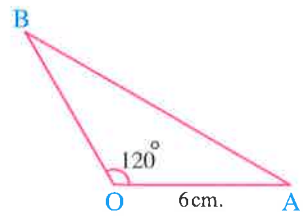
M is the point of intersection of the medians of Δ ABC, $\overrightarrow{AB} + \overrightarrow{AC} = k\,\overrightarrow{AM}$, then : k = ...



(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) 2 (d) 3

(4) **In the opposite figure :**



OA = OB = 6 cm. , m (∠ O) = 120°

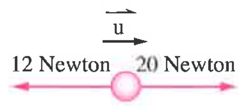
, then $\|\overrightarrow{AB}\|$ = ... cm.

(a) 6 (b) 12

(c) $6\sqrt{3}$ (d) $6\sqrt{2}$

Problems on physical applications

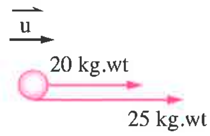
(1) **In the opposite figure :**



The resultant of the two forces in terms of unit vector $\vec{u}$ = ...

(a) $8\,\vec{u}$ (b) $-8\,\vec{u}$ (c) $32\,\vec{u}$ (d) $-32\,\vec{u}$

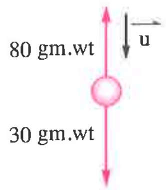
(2) **In the opposite figure :**



The resultant of the two forces in terms of unit vector $\vec{u}$ = ...

(a) $5\,\vec{u}$ (b) $-5\,\vec{u}$ (c) $45\,\vec{u}$ (d) $-45\,\vec{u}$

(3) **In the opposite figure :**



The resultant of the two forces in terms of unit vector $\vec{u}$ = ...

(a) $50\,\vec{u}$ (b) $-50\,\vec{u}$

(c) $110\,\vec{u}$ (d) $-110\,\vec{u}$

(4) The magnitude of resultant of the forces acting on a body as trying to move it with a force of magnitude 70 newton and the magnitude of the force of friction is 55 newtons equals .... newtons.

(a) 70 (b) 55 (c) 125 (d) 15

(5) If $\vec{F}_1 = \vec{i} - 3\,\vec{j}$, $\vec{F}_2 = 3\,\vec{i} - \vec{j}$ act on a particle , then the norm of the resultant equals .... force unit.

(a) $2\sqrt{10}$ (b) 8 (c) $4\sqrt{2}$ (d) 4

(6) If $\vec{F}_1 = (a , b)$, $\vec{F}_2 = -3\,\vec{i} + 4\,\vec{j}$ act on a particle and the system is in equilibrium , then a + b = ...

(a) zero (b) – 1 (c) 1 (d) 7

(7) If $\vec{v}_A = 12\,\vec{e}$, $\vec{v}_B = 8\,\vec{e}$, then $\vec{v}_{BA}$ = ...

(a) $20\,\vec{e}$ (b) $-20\,\vec{e}$ (c) $4\,\vec{e}$ (d) $-4\,\vec{e}$

(8) If $\vec{v}_A = 120\,\vec{e}$, $\vec{v}_B = -80\,\vec{e}$, then $\vec{v}_{AB} =$ ....

(a) $40\,\vec{e}$ (b) $200\,\vec{e}$ (c) $-200\,\vec{e}$ (d) $-40\,\vec{e}$

(9) If $\vec{v}_{AB} = 75\,\vec{e}$, $\vec{v}_A = -60\,\vec{e}$, then $\vec{v}_B =$ ....

(a) $135\,\vec{e}$ (b) $-135\,\vec{e}$ (c) $15\,\vec{e}$ (d) $-15\,\vec{e}$

(10) A cyclist A moves on a horizontal straight road with velocity 14 km./h. If he met another cyclist B moving with velocity 20 km./h. in the opposite direction. , then magnitude of the relative velocity between them = ... km./h.

(a) 20 (b) 14 (c) 34 (d) 6

(11) 🕮 A car moves on a straight road with a speed 90 km./h. If a motorcycle moves with speed 40 km./h. on the same road , then the magnitude of velocity of the motorcycle with respect to the car when they move in the same direction = ... km./h.

(a) 50 (b) 30 (c) 90 (d) 40

(12) If two forces $\overrightarrow{F_1} = 4\,\vec{i} - 6\,\vec{j}$, $\overrightarrow{F_2} = -6\,\vec{i} + 8\,\vec{j}$ act at a point , then their resultant $\vec{R} =$ ....

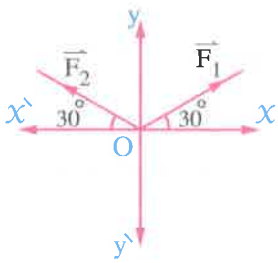
(a) $(8 , 135^\circ)$ (b) $\left(2\sqrt{2} , 45^\circ\right)$ (c) $\left(2\sqrt{2} , 135^\circ\right)$ (d) $(8 , 45^\circ)$

(13) If the forces $\overrightarrow{F_1} = (7 , -2)$, $\overrightarrow{F_2} = a\,\vec{i} + 3\,\vec{j}$, $\overrightarrow{F_3} = (-4 , b)$ act at a point are in equilibrium , then $a + b =$ ....

(a) 4 (b) -4 (c) -3 (d) -1

(14) **In the opposite figure :**

If $F_1 = F_2 = 3$ newton , then the resultant of the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ is $\vec{R} =$ ....



(a) $(3 , 180^\circ)$ (b) $(6 , 180^\circ)$

(c) $(3 , 90^\circ)$ (d) $(6 , 90^\circ)$

(15) If the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ act at a point. $F_1 = 34$ gm.wt. and acts in the eastern north direction , $F_2 = 34$ gm.wt. and acts in the western south direction , then the resultant of the two forces = ...

(a) 68 gm.wt. in the north direction.

(b) $34\sqrt{2}$ gm.wt. in the western north direction

(c) 68 gm.wt. in the western north direction.

(d) $\overrightarrow{\text{zero}}$

Second Essay questions

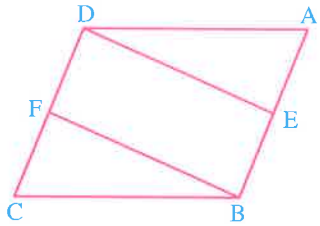
Problems on Geometric applications

1 **In the opposite figure :**

ABCD is a parallelogram , E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{DC}$

Prove using vectors that :

The figure DEBF is a parallelogram.



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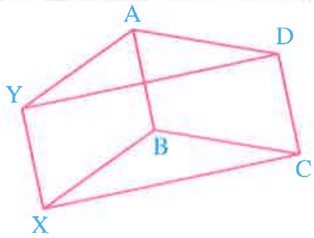
2 **In the opposite figure :**

ABCD and ABXY are two parallelograms.

Prove using vectors that :

The figure CXYD is a parallelogram.



---

3 If XYZL is a parallelogram , $E \in \overline{XL}$, $F \in \overline{YZ}$ such that EX = ZF , **prove using vectors that :** $\overline{EF}$ and $\overline{YL}$ bisect each other.

---

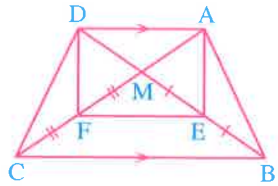
4 **In the opposite figure :**

ABCD is a trapezium in which $\overline{AD} \,//\, \overline{BC}$

, $AD = \frac{1}{2}$ BC , its diagonals intersect at M

If E and F are the midpoints of $\overline{MB}$ and $\overline{MC}$ respectively

, **prove using vectors that :** AEFD is a parallelogram.



---

5 ABCD is a quadrilateral. If $\overrightarrow{AC} + \overrightarrow{BD} = 2\,\overrightarrow{DC}$

, **prove that :** ABCD is a parallelogram.

---

6 **Use vectors to prove that :** The line segment drawn between the two midpoints of any two opposite sides of a parallelogram is parallel to the other two sides and its length equals the length of each of them.

---

7 **Using vectors , prove that :** If two opposite sides in a quadrilateral are parallel and equal in length , then the two other sides are parallel and equal in length also i.e. the quadrilateral is a parallelogram.

---

8 ABC is a triangle in which D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$

, **prove using vectors that :** $\overline{DE} \,//\, \overline{BC}$, $DE = \frac{1}{2}$ BC

9 If A (6 , 5) , B (8 , – 3) and C (– 2 , – 5) are the vertices of the triangle ABC , find using vectors the coordinates of the point of intersection of its medians. « (4 , – 1) »

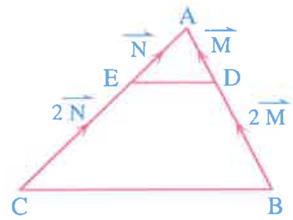
10 If A = (5 , 1) , B = (2 , 5) , C = (– 2 , 3) and D = (– 5 , – 4) **, prove using vectors that :** The figure ABCD is a trapezium.

11 **Using vectors , prove that the points :** A = (3 , 4) , B = (1 , – 1) , C = (– 4 , – 3) and D = (– 2 , 2) are vertices of a rhombus.

12 If ABCD is a quadrilateral in which A = (1 , – 2) , B = (9 , 0) , C = (8 , 4) and D = (0 , 2) **, prove using vectors that :** The figure ABCD is a rectangle , then find its perimeter and its area. « $6\sqrt{17}$, 34 »

13 **Using vectors , prove that the points :** A = (1 , 3) , B = (6 , 1) , C = (4 , – 4) and D = (– 1 , – 2) are vertices of a square and find its area. « 29 »

14 **In the opposite figure :**

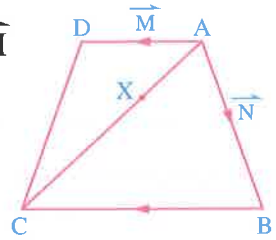


ABC is a triangle in which $D \in \overline{AB}$, $E \in \overline{AC}$, $\overrightarrow{DA} = \overrightarrow{M}$, $\overrightarrow{EA} = \overrightarrow{N}$, $\overrightarrow{BD} = 2\overrightarrow{M}$, $\overrightarrow{CE} = 2\overrightarrow{N}$

Find : $\overrightarrow{BC}$ in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$

, then prove that : $\overline{BC} \parallel \overline{DE}$ « $3(\overrightarrow{M} - \overrightarrow{N})$ »

15 **In the opposite figure :**

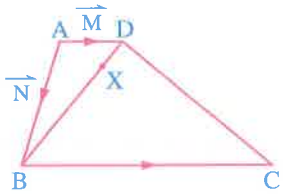


ABCD is a trapezium , $\overline{AD} \parallel \overline{BC}$, AD = $\frac{1}{2}$ BC , $\overrightarrow{AB} = \overrightarrow{N}$, $\overrightarrow{AD} = \overrightarrow{M}$

(1) **Express in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$ each of the following :**
$\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{DC}$, $\overrightarrow{DB}$

(2) If $X \in \overline{AC}$ where AX = $\frac{1}{3}$ AC **, prove that :** The points D , X and B are collinear.

16 **In the opposite figure :**



ABCD is a trapezium in which $\overline{AD} \parallel \overline{BC}$, BC = 4 AD , $\overrightarrow{AD} = \overrightarrow{M}$, $\overrightarrow{AB} = \overrightarrow{N}$

(1) **Express in terms of $\overrightarrow{M}$ and $\overrightarrow{N}$ each of :** $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$, $\overrightarrow{DC}$

(2) If $X \in \overline{DB}$ where DX = $\frac{1}{4}$ XB **, prove that :** The points A , X , C are collinear.

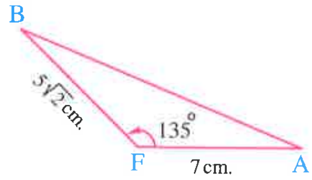
17 **In the opposite figure :**

FAB is a triangle in which

FA = 7 cm. , FB = $5\sqrt{2}$ cm.

, m (∠ AFB) = 135°

Find using vectors : The length of $\overline{AB}$



« 13 cm. »

18 ABCD is a quadrilateral in which X , Y , Z and L are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. **Use vectors to prove that :**

(1) The quadrilateral XYZL is a parallelogram.

(2) The perimeter of the quadrilateral XYZL equals the sum of lengths of the two diagonals of the quadrilateral ABCD

Problems on physical applications

1 If the forces $\overrightarrow{F}_1 = 2\,\overrightarrow{i} - 3\,\overrightarrow{j}$, $\overrightarrow{F}_2 = 4\,\overrightarrow{i} + 7\,\overrightarrow{j}$, $\overrightarrow{F}_3 = 3\,\overrightarrow{i} + 8\,\overrightarrow{j}$ act on a particle , find the magnitude and the direction of the resultant of these forces (given that forces are measured in newton).

« 15 newton , 53° 7′ 48″ »

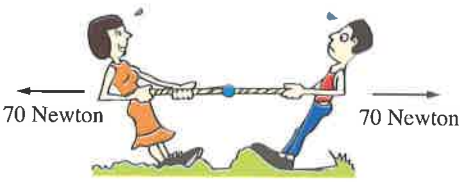
2 If the forces $\overrightarrow{F}_1 = (-6 , 6)$, $\overrightarrow{F}_2 = 9\,\overrightarrow{i} + 13\,\overrightarrow{j}$, $\overrightarrow{F}_3 = (2 , -7)$ act on a particle where the forces are measured in dyne , find the magnitude and direction of the resultant of these forces.

« 13 dyne , 67° 22′ 48″ »

3 **Find the resultant force $\overrightarrow{F}$ acting in each of the following :**

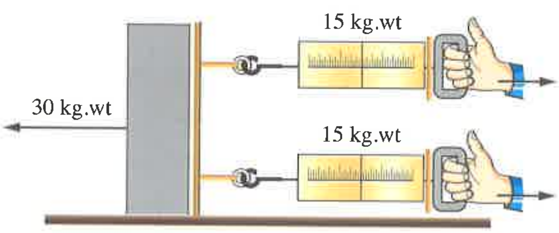
(1)

(2)



(3)

(4)



4 **In each of the following , the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ act at a particle.**

Show the magnitude and the direction of the resultant of each two forces :

(1) F_1 = 15 newtons acts in the east direction ,

F_2 = 40 newtons acts in the west direction.

(2) F_1 = 50 dyne acts in 60° west of the north direction ,

F_2 = 50 dyne acts in 30° south of the east direction.

(3) F_1 = 30 newtons acts in 20° east of the north direction ,

F_2 = 30 newtons acts in 70° north of the east direction.

---

5 The forces $\overrightarrow{F_1} = 2\vec{i} + 3\vec{j}$, $\overrightarrow{F_2} = a\vec{i} + \vec{j}$, $\overrightarrow{F_3} = 5\vec{i} + b\vec{j}$ act at a particle.

Find the values of a and b if their resultant force $\vec{F}$ is as follows :

(1) $\vec{F} = 5\vec{i} - 2\vec{j}$ (2) $\vec{F} = \vec{O}$ « −2 , −6 , −7 , −4 »

---

6 The forces $\overrightarrow{F_1} = 7\vec{i} - 5\vec{j}$, $\overrightarrow{F_2} = a\vec{i} + 3\vec{j}$, $\overrightarrow{F_3} = -4\vec{i} + (b-3)\vec{j}$ act at a particle.

Find the values of a and b if :

(1) The resultant of the set of forces equals $4\vec{i} - 7\vec{j}$

(2) The set of forces is in equilibrium. « 1 , −2 , −3 , 5 »

---

7 A car "A" moves on a straight road with velocity 140 km./h. , another car "B" moves on the same road with velocity 110 km./h.

Find the velocity of the car "A" relative to the car "B" when :

(1) The two cars move in the same direction.

(2) The two cars move in two opposite directions. « 30 km./h. , 250 km./h. »

---

8 A car moves on a straight road with velocity 75 km./h. If a motorcycle moves on the same road with velocity 45 km./h.

, find its velocity relative to the car in each of the following cases :

(1) The motorcycle moves in thc opposite direction of the car.

(2) The motorcycle moves in the same direction of the car. « 120 km./h. , 30 km./h. »

---

9 A car for watching the velocity on the desert road (Cairo - Alex.) moves with velocity 30 km./h. This car watched a truck coming in the opposite direction , it seems that the truck moves with velocity 110 km./h. What is the actual velocity of the truck ? « 80 km./h. »

10 A controlling speed car (Radar) moves on the desert road at 40 km./h. It watched a car coming from the other opposite road which seemed to be moving at 135 km./h. If the maximum available speed on the road is 100 km./h. , is the coming car in violation of the prescribed speed ? Explain your answer. « It is no in violation »

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) If $\overrightarrow{F_1} = \left(6 , \frac{2\pi}{3}\right)$, $\overrightarrow{F_2} = \left(6 , \frac{4\pi}{3}\right)$, $\overrightarrow{F_3} = = 9\,\vec{i} + 4\,\vec{j}$ measured in dyne , then the magnitude of their resultant = ... dynes.

(a) 13 (b) 10 (c) 5 (d) $6\sqrt{3}$

(2) If the forces $\overrightarrow{F_1} = \left(8\sqrt{2} , \frac{3\pi}{4}\right)$, $\overrightarrow{F_2} = a\,\vec{i} + 3\,\vec{j}$, $\overrightarrow{F_3} = -5\,\vec{i} + (b+2)\,\vec{j}$ act at one point and the system in equilibrium , then $\frac{a}{b}$ = ...

(a) 13 (b) – 13 (c) 1 (d) – 1

(3) If $\overrightarrow{F_1} = \vec{i} - 3\,\vec{j}$, $\overrightarrow{F_2} = 3\,\vec{i} + 6\,\vec{j}$, then the force $\overrightarrow{F_3}$ which makes the resultant of the three forces is unit vector acts due to y-axis equals ....

(a) $-3\,\vec{i} - 3\,\vec{j}$ (b) $-4\,\vec{i} - 2\,\vec{j}$

(c) $-5\,\vec{i} - 3\,\vec{j}$ (d) $-4\,\vec{i} - 3\,\vec{j}$

(4) A system of 100 forces , magnitude of each is 10 newtons act at one point and the maesure between any two consecutive forces is $\frac{\pi}{50}$, then the magnitude of their resultant is ....

(a) 100 (b) 500 (c) 10 (d) zero

---

2 If $\overrightarrow{F_1} = 5\,\vec{i} + 3\,\vec{j}$, $\overrightarrow{F_2} = (7+a)\,\vec{i} + 6\,\vec{j}$, $\overrightarrow{F_3} = -14\,\vec{i} + (3-b)\,\vec{j}$ are three coplanar forces meeting at a point , the polar form of their resultant $\vec{F} = \left(10\sqrt{2} , 135^\circ\right)$, **find the values of :** a and b « – 8 , 2 »

---

3 A car "A" moved on a straight road measured the relative velocity of a car "B" in front of it in the same direction it found it 20 km./h. , then when the car "A" decreased its speed to the half and remeasured the speed , it found that the relative velocity of the car "B" became 50 km./h. What is the actual velocity of each car ? « 60 km./h. , 80 km./h. »

Unit 5

STRAIGHT LINE

Exercise Five	:	Division of a line segment.
Exercise Six	:	Equation of the straight line.
Exercise Seven	:	Measure of the angle between two straight lines.
Exercise Eight	:	The length of the perpendicular from a point to a straight line.
Exercise Nine	:	General equation of the straight line passing through the point of intersection of two lines.

Exercise Five

Division of a line segment

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) If A = (3 , 6) , B = (– 7 , 4) , then the midpoint of $\overline{AB}$ = ...

(a) (– 4 , 10) (b) (– 4 , 5) (c) (5 , 1) (d) (– 2 , 5)

(2) If M is the point of intersection of the two diagonals of the parallelogram ABCD where A = (3 , 7) , C = (– 3 , 1) , then M = ...

(a) (0 , 4) (b) (3 , 3) (c) (0 , 8) (d) (6 , 6)

(3) If the point (3 , 6) is the midpoint of $\overline{AB}$ where A = (– 3 , 7) , then the point B = ...

(a) (6 , – 1) (b) (– 6 , 1) (c) (9 , 5) (d) (0 , 6.5)

(4) If C (2 , 4) is the midpoint of $\overline{AB}$ where A (x , 4) , B(1 , y) , then x + y = ...

(a) 7 (b) 1 (c) – 1 (d) – 7

(5) A circle of center (2 , – 2) , if one of the two ends of its diameter is (4 , 2) , then the other end of this diameter is ....

(a) (– 4 , 2) (b) (0 , – 6) (c) (– 3 , – 3) (d) (8 , 4)

(6) If A (– 3 , – 7) , B (4 , 0) , then the coordinates of C which divides $\overrightarrow{AB}$ internally in the ratio 5 : 2 is ....

(a) (– 2 , 2) (b) (2 , – 2) (c) (2 , 2) (d) (– 2 , – 2)

(7) If A (2 , 5) , B (7 , – 1) , then the coordinates of C which divides $\overrightarrow{AB}$ externally in the ratio 3 : 2 is ....

(a) (– 25 , – 7) (b) (25 , 7) (c) (17 , 13) (d) (17 , – 13)

(8) If A = (– 4 , 4) , B = (5 , – 8) , C ∈ $\overline{AB}$ such that CB : AC = 1 : 2 , then C = ...

(a) (4 , – 8) (b) (2 , – 4) (c) (– 8 , 4) (d) (– 4 , 2)

(9) If C ∈ $\overline{AB}$, AB = 4 BC and A (– 1 , 4) , B (3 , 4) , then the coordinates of the point C is ....

(a) (0 , 4) (b) (4 , 2) (c) (4 , 0) (d) (2 , 4)

(10) If A (– 3 , – 4) , B (– 8 , 7) and C ∈ $\overrightarrow{AB}$, C ∉ $\overline{AB}$ where AC = 2 CB , then C is ....

(a) (13 , 18) (b) (– 13 , 18) (c) (– 13 , – 18) (d) (13 , – 18)

(11) If B (0 , 3) , C (3 , 0) and A lies at the third distance between B and C , then A is ....

(a) (1 , 2) (b) (2 , 1) (c) (– 1 , – 2) (d) (– 2 , – 1)

(12) If A (2 , 3) , B (6 , – 1) , then the coordinates of C which lies at quarter distance from A to B is ....

(a) (2 , 3) (b) (2 , – 3) (c) (3 , 2) (d) (– 3 , 2)

(13) The coordinates of the point lies at $\frac{2}{5}$ the distance from A to B to the directed segment $\overrightarrow{AB}$ where A (3 , – 2) , B (– 1 , 5) is ....

(a) (– 1 , 3) (b) $\left(\frac{7}{5} , \frac{4}{5}\right)$ (c) (3 , – 1) (d) $\left(\frac{4}{5} , \frac{7}{5}\right)$

(14) If C (4 , 4) divides $\overrightarrow{AB}$ internally in the ratio 1 : 2 and A (7 , 8) , then B is ....

(a) (– 2 , – 4) (b) (1 , 2) (c) (– 1 , – 2) (d) (2 , 4)

(15) If $\overrightarrow{AB}$ = (3 , 4) , A (– 2 , 5) , C divides $\overrightarrow{AB}$ by the ratio 3 : 2 externally , then C = ...

(a) (7 , 17) (b) (8 , 3) (c) (– 8 , 3) (d) (– 7 , – 17)

(16) The ratio of division that the x-axis divides the line segment $\overrightarrow{AB}$ where A (2 , 5) , B (7 , – 2) is ....

(a) 5 : 2 internally (b) 2 : 3 internally

(c) 3 : 2 externally (d) 2 : 5 externally

(17) The ratio by which the y-axis divides $\overrightarrow{AB}$ where A (2 , 5) , B (6 , 7) equals ....

(a) 1 : 3 externally (b) 3 : 1 internally

(c) 1 : 2 externally (d) 3 : 2 internally

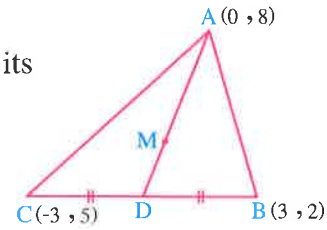
(18) If A (2 , 5) , B (5 , 2) , C (4 , y) are three collinear points , then C divides $\overrightarrow{AB}$ in the ratio ....

(a) 1 : 2 internally (b) 2 : 1 internally

(c) 2 : 1 externally (d) 1 : 2 externally

(19) In the opposite figure :

$\overline{AD}$ is a median in ΔABC , M is the point of intersection of its medians where A = (0 , 8) , B = (3 , 2) , C = (– 3 , 5) , then the point M = ...



(a) (0 , 7.5) (b) (0 , 5)

(c) (– 3 , 5) (d) (5 , 0)

(20) If $\overline{AD}$ is a median in ΔABC , where A = (1 , 2) , D = (4 , – 4) , then the point of intersection of the medians of the triangle ABC is ....

(a) (3 , – 2) (b) (– 3 , 2) (c) (2 , – 3) (d) (– 2 , – 3)

(21) ABC is a triangle in which A (– 3 , 1) , B (1 , 7) and M is the point of intersection of its medians where M = (1 , 2) , then the point C = ...

(a) (5 , 2) (b) (5 , – 2) (c) (– 5 , 2) (d) (– 5 , – 2)

(22) ABC is a triangle in which A (8 , 7) , M is the point of intersection of its medians where M = (2 , 1) and D is the midpoint of $\overline{BC}$, then D = ...

(a) (1 , 2) (b) (2 , 1) (c) (– 1 , – 2) (d) (– 1 , 2)

(23) If $\overline{AE}$ is a median of ΔABC , M is the centriod of the triangle ABC , A (5 , 4) , M (7 , 8) , then $\overrightarrow{AE}$ = ...

(a) $\left(\frac{4}{3} , \frac{8}{3}\right)$ (b) $\left(\frac{2}{3} , \frac{4}{3}\right)$ (c) (3 , 6) (d) (1 , 2)

(24) If C divides $\overrightarrow{BA}$ by the ratio 2 : 3 internally , then $\frac{AC}{AB}$ = ...

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

(25) If C divides $\overrightarrow{AB}$ by the ratio 5 : 7 externally , then $\frac{AC}{AB}$ = ...

(a) $\frac{2}{7}$ (b) $\frac{7}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5}{2}$

(26) If $C \in \overline{AB}$, $3\,\overrightarrow{AB} = 5\,\overrightarrow{BC}$, then C divides $\overrightarrow{BA}$ by the ratio ....

(a) 2 : 3 (b) 3 : 2 (c) 3 : 5 (d) 5 : 3

(27) If A divides $\overrightarrow{BC}$ by the ratio 2 : 3 externally , then ....

(a) B divides $\overrightarrow{AC}$ by the ratio 2 : 3 internally.

(b) B divides $\overrightarrow{AC}$ by the ratio 2 : 1 internally.

(c) C divides $\overrightarrow{AB}$ by the ratio 3 : 1 internally.

(d) C divides $\overrightarrow{AB}$ by the ratio 3 : 2 externally.

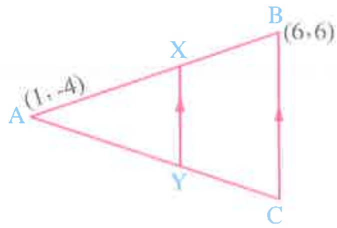
(28) In triangle ABC , B (3 , 5) , C (– 3 , – 7) , D $\in \overline{BC}$ such that the area of Δ ABD = $\frac{1}{3}$ the area of Δ ABC , then D = ...

(a) $\left(3 , \frac{17}{3}\right)$ (b) $\left(\frac{2}{3} , 2\right)$ (c) (0 , – 1) (d) (1 , 1)

(29) If the midpoints of the sides of a triangle are (– 2 , 3) , (7 , – 1) , (4 , 4) , then the point of intersection of the medians of the triangle is ....

(a) (3 , 2) (b) (9 , 6) (c) (5 , 0) (d) $\left(\frac{5}{3} , 0\right)$

(30) **In the opposite figure :**

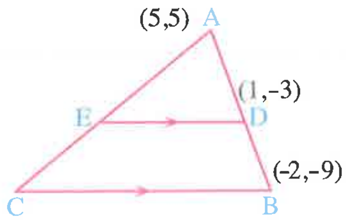


If $\overline{XY}$ // $\overline{BC}$, $\frac{AY}{AC} = \frac{3}{5}$, then X = ...

(a) (2 , 4) (b) (4 , 2)

(c) (– 2 , 4) (d) (– 4 , 2)

(31) **In the opposite figure :**

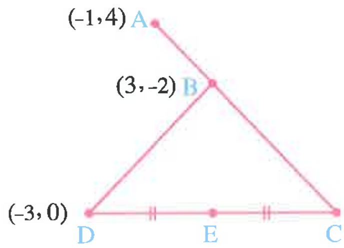


$\frac{AE}{EC}$ = ...

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{2}{3}$

(32) **In the opposite figure :**



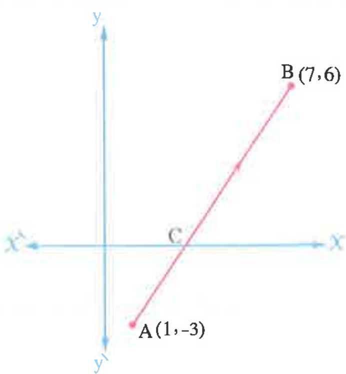
If A $\in \overrightarrow{CB}$ and AC = 3 AB

, then E = ...

(a) (4 , – 2) (b) (– 2 , 0)

(c) (4 , – 7) (d) (8 , – 5)

(33) **In the opposite figure :**



The point C is ....

(a) (5 , 0)

(b) (4 , 0)

(c) (3 , 0)

(d) (2 , 0)

(34) In the opposite figure :

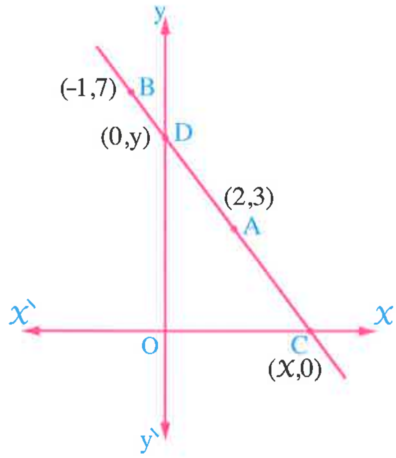
CA : CB = ...

(a) 2 : 1

(b) 7 : 3

(c) 3 : 7

(d) 1 : 2



Second Essay questions

1 If A = (0 , – 3) , B = (3 , 6) , find the coordinates of the point C which divides $\overrightarrow{BA}$ internally by the ratio 1 : 2 « (2 , 3) »

2 If A = (3 , – 2) , B = (– 1 , 5) , **find :**

(1) The coordinates of the point C which divides $\overrightarrow{AB}$ by the ratio 2 : 3 internally.

(2) The coordinates of the point D which divides $\overrightarrow{AB}$ by the ratio 4 : 3 externally.

« $\left(\frac{7}{5}, \frac{4}{5}\right)$, (– 13 , 26) »

3 Find the coordinates of the point C which lies at the fifth of the distance from A to B where A = (– 1 , – 1) , B = (9 , 4) « (1 , 0) »

4 If C $\in \overrightarrow{BA}$, C $\notin \overline{AB}$, and A = (3 , 1) , B = (4 , 2) , AC = 2 AB , find the coordinates of the point C « (1 , – 1) »

5 If A = (1 , 3) , B = (– 4 , – 2) , find the coordinates of the point C if C $\in \overline{AB}$ where 3 AC = 2 CB « (– 1 , 1) »

6 If A = (4 , 3) , B = (– 3 , 5) , find the point C $\in \overleftrightarrow{AB}$ where 3 AC = 5 CB

« $\left(-\frac{3}{8}, \frac{17}{4}\right)$, $\left(-\frac{27}{2}, 8\right)$ »

7 If A = (2 , 1) , B = (– 1 , – 2) , find the coordinates of the point C $\in \overrightarrow{AB}$, C $\notin \overline{AB}$ such that its distance from A equals 4 times its distance from B « (– 2 , – 3) »

8 If the points A = (3 , – 4) , C = (– 1 , ℓ) , B = (k , 1) are collinear , C $\in \overline{AB}$, $\frac{AC}{CB} = \frac{2}{3}$, **find :** ℓ and k « – 2 , – 7 »

9 If $A = (8, -4)$, $B = (-1, 2)$, find the coordinates of the points which divide $\overrightarrow{AB}$ into 3 equal parts in length. « $(5, -2)$, $(2, 0)$ »

10 If $A = (1, -4)$, $B = (5, 4)$, find the coordinates of the points C , D and E which divide $\overrightarrow{AB}$ into 4 equal parts in length. « $(2, -2)$, $(3, 0)$, $(4, 2)$ »

11 If $A \in X$-axis , $B \in$ y-axis , $C = (-4, 3)$ is the midpoint of $\overline{AB}$, **find the coordinates of each of :** A and B « $(-8, 0)$, $(0, 6)$ »

12 If $A = (3, -2)$, $B = (-2, 3)$, find the ratio by which the point $C = (8, y)$ divides $\overline{AB}$ showing the type of division , then find the value of y « 1 : 2 externally , -7 »

13 Find the ratio by which the y-axis divides the line segment $\overrightarrow{AB}$ where $A = (2, 3)$, $B = (-3, 7)$ showing the type of division and find the coordinates of the point of division. « $\frac{2}{3}$ internally , $\left(0, \frac{23}{5}\right)$ »

14 If $A = (-2, 3)$, $B = (4, -2)$, find the ratio by which the X-axis divides the directed line segment $\overrightarrow{AB}$ showing the type of division and find the coordinates of the point of division. « $\frac{3}{2}$ internally , $\left(\frac{8}{5}, 0\right)$ »

15 If $A = (5, 2)$, $B = (2, -1)$, find the ratio by which $\overrightarrow{AB}$ is divided by the points of intersection of $\overleftrightarrow{AB}$ with the two axes , showing the type of division in each case , then find the coordinates of the division point. « 2 : 1 (internally) , 5 : 2 (externally) , $(3, 0)$, $(0, -3)$ »

16 If C and D are the two points of intersection of $\overleftrightarrow{AB}$ with the two axes , find the ratio by which the points C and D divide $\overline{AB}$ showing the type of division.
given that $A = (-5, 7)$, $B = (-3, 2)$ « 7 : 2 externally , 5 : 3 externally »

17 If the points $A = (1, -1)$, $B = (-1, 1)$, $C = \left(\sqrt{3}, \sqrt{3}\right)$ are the vertices of a triangle , find the coordinates of the point of intersection of its medians. « $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ »

18 If $A = (4, 12)$, $B = (-2, 10)$, $C = (1, 3)$, $D = (2, 7)$, E is the midpoint of $\overline{AB}$, M divides $\overline{CD}$ externally by the ratio 3 : 2 , **find :** the length of $\overline{EM}$ « 5 length units »

19 ABCD is a parallelogram. If $A = (7, -2)$, $B = (15, 4)$, $C = (9, 6)$, find the coordinates of the point of intersection of its diagonals $\overline{AC}$, $\overline{BD}$, then find the coordinates of the vertex D « $(8, 2)$, $(1, 0)$ »

20 If ABCD is a quadrilateral , A = (4 , 3) , B = (0 , 2) , C = (– 2 , – 3) , D = (2 , – 2) , find the midpoint of each of $\overline{AC}$, $\overline{BD}$, then determine the kind of the figure ABCD

« (1 , 0) , (1 , 0) , a parallelogram »

---

21 **Prove that the points :** A = (1 , 4) , B = (3 , – 2) , C = (– 3 , 16) are collinear , **then find :**

(1) The ratio by which A divides $\overrightarrow{BC}$ showing the type of division. « 1 : 2 internally »

(2) The ratio by which B divides $\overline{CA}$ showing the type of division. « 3 : 1 externally »

(3) The ratio by which C divides $\overline{AB}$ showing the type of division. « 2 : 3 externally »

---

22 D , E , R are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively in ΔABC

If D = (2 , 3) , E = (– 1 , 4) , R = (4 , 5)

, **find the coordinates of :** A , B , C « (7 , 4) , (– 3 , 2) , (1 , 6) »

---

23 ABC is a triangle , its vertices A = (3 , 5) , B = (6 , – 4) , C = (1 , 1)

If D divides $\overrightarrow{AB}$ by the ratio 1 : 2 , E divides $\overrightarrow{AC}$ by the ratio 1 : 2 also

, **prove that :** $\overline{DE}$ // $\overline{BC}$, DE = $\frac{1}{3}$ BC

---

24 **Distance :** A bus moves from city A to city B where A = (5 , – 6) , B = (1 , 0)

It stopped twice during its movement. **Find the coordinates of the two points at which the bus has been stopped if :**

(1) It stopped at the middle of the road.

(2) It stopped at two thirds of the road from city A « (3 , – 3) , $\left(\frac{7}{3} , -2\right)$ »

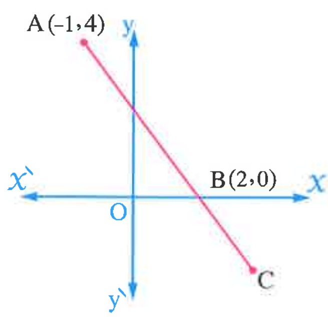
Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from the given ones :**

(1) **In the opposite figure :**

If BC = 2.5 length unit

, then the point C = ...



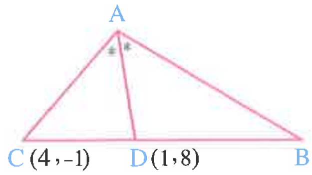
(a) (3 , – 2) (b) (3.5 , – 2)

(c) (1 , – 2) (d) (2.5 , – 1)

(2) **In the opposite figure :**

If 4 AC = 3 AB

, then the point B is ....

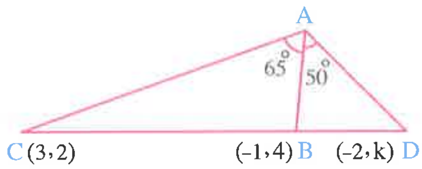


(a) (– 5 , 14) (b) (– 4 , 16)

(c) (– 3 , 20) (d) (– 2 , 21)

(3) **In the opposite figure :**

If B $\in \overline{CD}$, then $\frac{AB}{AD}$ = ...



(a) $\frac{6}{5}$ (b) 1

(c) $\frac{4}{5}$ (d) $\frac{3}{5}$

(4) If A and B are the images of the point (3 , 1) by reflection in the X-axis and y-axis respectively , then the coordinate of the point that divides $\overline{AB}$ by the ratio 2 : 3 internally is ....

(a) (3 , – 1) (b) $\left(\frac{3}{5}, \frac{-1}{5}\right)$ (c) $\left(\frac{-3}{5}, \frac{1}{5}\right)$ (d) (0 , 0)

(5) If the origin point is the point of intersection of the medians of the triangle whose vertices are (a , b) , (b , c) , (c , a) , then $a^3 + b^3 + c^3$ = ...

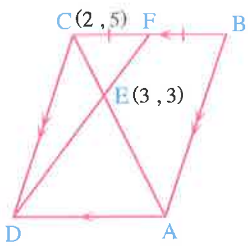
(a) zero (b) abc (c) a + b + c (d) 3 abc

2 **In the opposite figure :**

ABCD is a parallelogram in which

F is the midpoint of $\overline{BC}$, $\overline{AC} \cap \overline{DF} = \{E\}$

, if E = (3 , 3) , C = (2 , 5) , then find the coordinates of the point A



« (5 , – 1) »

3 If A = (2 , 2) , B = (5 , 6) , C = (10 , – 4) are the vertices of a triangle , D $\in \overline{BC}$ such that $\overrightarrow{AD}$ bisects ∠ A internally , find the coordinates of D « $\left(\frac{20}{3}, \frac{8}{3}\right)$ »

Exercise Six

Equation of the straight line

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) If A (3 , – 2) , B (5 , 6) , then the slope of the straight line $\overleftrightarrow{AB}$ = ...

(a) – 1 (b) $\frac{1}{4}$ (c) 4 (d) 1

(2) The straight line whose general equation is $4X + 3y + 5 = 0$, its slope = ...

(a) $\frac{4}{3}$ (b) $\frac{-4}{3}$ (c) $\frac{3}{4}$ (d) $\frac{-3}{4}$

(3) The straight line which passes through the two points (4 , –2) , (5 , 3)

, the slope of the perpendicular straight line to it = ...

(a) 5 (b) $\frac{1}{5}$ (c) – 5 (d) $\frac{-1}{5}$

(4) If the slope of the straight line : $(3a + 1)X - 2ay + 3 = 0$ equals 2 , then a = ...

(a) 1 (b) – 1 (c) $\frac{5}{3}$ (d) $-\frac{1}{2}$

(5) If the straight line : $aX - 4y + 5 = 0$ makes with the positive direction of the X-axis an angle of tangent 0.75 , then a = ...

(a) $\frac{-16}{3}$ (b) – 3 (c) $\frac{16}{3}$ (d) 3

(6) If the points : (1 , 8) , (3 , y) , (9 , – 4) lies on the same straight line then y = ...

(a) 11 (b) 5 (c) – 11 (d) – 5

(7) If the straight line passing through the points (3 , 0) and (0 , 2) is parallel to the straight line whose equation is $y = aX - 3$, then a = ...

(a) $\frac{-2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

(8) If the two straight lines $L_1 : 3x - 2y + 7 = 0$ and $L_2 : ax + 3y + 5 = 0$ are perpendicular , then a = ...

(a) 1 (b) 2 (c) – 2 (d) – 1

(9) The slope of the straight line which makes with the positive direction of the x-axis a positive angle of cosine = $\frac{4}{5}$ is ....

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

(10) The straight line which makes a positive angle of measure $\frac{\pi}{4}$ with the positive direction of the x-axis , its direction vector = ...

(a) (0 , 1) (b) (1 , 0) (c) (– 1 , 1) (d) (1 , 1)

(11) The straight line whose equation is $y = \frac{5}{4}x + 7$, its direction vector = ...

(a) (5 , 4) (b) (4 , 5) (c) (– 5 , 4) (d) (5 , – 4)

(12) The straight line $ax + by + c = 0$, its direction vector is ....

(a) (a , b) (b) (a , –b) (c) (b , a) (d) (b , – a)

(13) The slope of the straight line which passes through the two points (a , a^2) , (b , b^2) is ....

(a) $a^2 - b^2$ (b) a – b (c) a + b (d) ab

(14) If $\vec{u} = (2 , -5)$ is a direction vector of a straight line , then all of the following vectors are direction vectors to the same straight line except the vector ....

(a) (– 2 , 5) (b) (6 , – 15) (c) (2 , 5) (d) (– 1 , 2.5)

(15) If $\vec{u} = \left(\frac{1}{2} , 1\right)$ is a direction vector of a straight line , then all the following vectors are perpendicular to the straight line except the vector ....

(a) $\left(1 , -\frac{1}{2}\right)$ (b) (2 , – 1) (c) $\left(-1 , -\frac{1}{2}\right)$ (d) (4 , – 2)

(16) If the slope of a straight line = $\frac{-2}{3}$, then its direction vector is ....

(a) (3 , – 2) (b) (– 3 , 2)

(c) (6 , – 4) (d) all the previous right.

(17) If (6 , 4) and (3 , m) are direction vectors of two perpendicular straight lines , then m = ...

(a) $\frac{2}{3}$ (b) $\frac{-2}{9}$ (c) $\frac{9}{2}$ (d) $\frac{-9}{2}$

(18) The direction vector of the straight line perpendicular to the y-axis could be ....

(a) (2 , 0) (b) (0 , 1) (c) (1 , 1) (d) (– 1 , – 1)

(19) Each of the following relations represents a straight line except ....

(a) $y = \sqrt{5}\,x$ (b) $x = 5$ (c) $\frac{x}{5} + \frac{y}{2} = 1$ (d) $y = \sqrt{x}$

(20) The equation of the straight line which passes through the two points $(4 , 0)$, $(0 , 3)$ is ....

(a) $3 X + 4 y = 12$ (b) $4 X + 3 y = 12$ (c) $3 X + 4 y = 0$ (d) $3 y + 4 X = 0$

(21) The equation of the straight line which passes through the point $(2 , -3)$ and is parallel to the X-axis is ....

(a) $X + 3 = 0$ (b) $y + 3 = 0$ (c) $X - 2 = 0$ (d) $y - 3 = 0$

(22) 🕮 The cartesian equation of the straight line which passes through the point $(-2 , 7)$ and is parallel to the y–axis is ....

(a) $y = 2$ (b) $y = -2$ (c) $X = 7$ (d) $X = -2$

(23) 🕮 The equation of the straight line which makes a positive angle of measure 45° with the positive direction of the X-axis and cuts 5 units from the positive part of the y-axis is ....

(a) $y = X - 5$ (b) $y = \frac{1}{2} X + 5$ (c) $y = \frac{1}{\sqrt{2}} X + 5$ (d) $y = X + 5$

(24) The equation of the straight line which passes through the point $(3 , -2)$ and is perpendicular to the straight line $y = 7$ is ....

(a) $X = 3$ (b) $X = 7$ (c) $y = -2$ (d) $y = 7$

(25) 🕮 The cartesian equation of the straight line which cuts the positive parts of the X-axis and the y-axis with magnitudes 2 , 3 respectively is ....

(a) $3 X + 2 y = 6$ (b) $3 X + 2 y = 1$

(c) $2 X + 3 y = 6$ (d) $2 X + 3 y = 1$

(26) The vector equation of the straight line which passes through the point $(-4 , 3)$ and its direction vector is $(2 , 5)$ is ....

(a) $\vec{r} = (2 , 5) + k (-4 , 3)$ (b) $\vec{r} = (-4 , 3) + k (2 , 5)$

(c) $\vec{r} = (-4 , 3) + k (5 , 2)$ (d) $\vec{r} = (2 , 5) + k (3 , -4)$

(27) 🕮 The vector equation of the straight line which passes through the origin point and the point $(1 , 2)$ is ....

(a) $\vec{r} = k (1 , 2)$ (b) $\vec{r} = k (2 , 1)$

(c) $\vec{r} = (1 , 2) + k (1 , 0)$ (d) $\vec{r} = (1 , 2) + k (0 , 1)$

(28) The vector equation of the X-axis is ....

(a) $\vec{r} = (1 , 1) + k (0 , 0)$ (b) $\vec{r} = (1 , 0) + k (1 , 1)$

(c) $\vec{r} = k (1 , 0)$ (d) $\vec{r} = k (0 , 1)$

(29) The vector equation of the straight line which passes through the point (3 , 5) and is parallel to the X-axis is ....

(a) $\vec{r} = k(3, 5)$ (b) $\vec{r} = (3, 5) + k(0, 1)$

(c) $\vec{r} = (3, 5) + k(1, 0)$ (d) $\vec{r} = k(1, 0)$

(30) All the following equations represent an equation for the straight line which passes through the two points (5 , 0) , (0 , 2) except the equation ....

(a) $\vec{r} = (5, 0) + k(5, -2)$ (b) $\vec{r} = (0, 2) + k(5, -2)$

(c) $\vec{r} = (5, 0) + k(2, 5)$ (d) $\vec{r} = (0, 2) + k(-10, 4)$

(31) The parametric equations of the straight line passes through (0 , 5) and its vector direction is (– 1 , 4) are ....

(a) $X = 1 - k$, $y = 5 + 4k$ (b) $X = k$, $y = 5 + 4k$

(c) $X = 5 + 4k$, $y = -k$ (d) $X = -k$, $y = 5 + 4k$

(32) The parametric equations of the straight line which makes with the positive direction of the X-axis a positive angle of measure 45° and passes through the point (3 , – 5) are ....

(a) $X = 3 + k$, $y = -5 + k$ (b) $X = 3 + k$, $y = 5 + k$

(c) $X = 1 + 3k$, $y = 1 - 5k$ (d) $X = 1 - 3k$, $y = 1 + 5k$

(33) The straight line L : $X = 1 - 2k$, $y = -1 + 4k$ passes through the point ....

(a) (1 , 1) (b) (1 , – 1) (c) (– 1 , – 1) (d) (– 1 , 1)

(34) The straight line whose vector equation is $\vec{r} = (2, -1) + k(3, -5)$, its perpendicular direction vector = ...

(a) (3 , – 5) (b) (2 , – 1) (c) (5 , 3) (d) (– 5 , 3)

(35) If the two straight lines $4X + by + 9 = 0$, $\vec{r} = (1, 5) + k(2, 6)$ are parallel , then b = ...

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$ (c) $\frac{-4}{3}$ (d) $\frac{-3}{4}$

(36) If a straight line passes through the point (2 , 1) and the vector $\vec{N} = (1, 3)$ is perpendicular to it , then the equation of the straight line is ....

(a) $X + 2y + 5 = 0$ (b) $X + 3y - 5 = 0$

(c) $X - 3y = 0$ (d) $3X - y - 5 = 0$

(37) The straight line which is perpendicular to the straight line $\vec{r} = (0, 5) + k\left(\sqrt{3}, 1\right)$ makes with the positive direction of the X-axis an angle of measure ....

(a) 30° (b) 60° (c) 120° (d) 150°

(38) The straight line : $\vec{r} = (1 , 4) + k (0 , 1)$ is parallel to ....

(a) the X-axis (b) the y-axis

(c) the straight line $y = X$ (d) the straight line $y = 2$

(39) The area of the triangle enclosed by the X-axis and the y-axis and the straight line $2X + 3y = 6$ equals .... square unit.

(a) 6 (b) 3 (c) 2 (d) 12

(40) Which of the following points lies on the line $\vec{r} = (-2 , 1) + k (1 , -3)$?

(a) $\left(\frac{-5}{3} , -2\right)$ (b) $\left(\frac{-3}{2} , \frac{1}{2}\right)$ (c) $\left(\frac{3}{2} , -\frac{1}{2}\right)$ (d) $\left(\frac{-7}{3} , 2\right)$

(41) The point which lies on the straight line $X = -1 + 2k$, $y = 3 - k$ and its X coordinate = 3 is ....

(a) $(3 , 1)$ (b) $(3 , -1)$ (c) $(3 , 0)$ (d) $(3 , 2)$

(42) If the straight line $aX + by = 12$ intercepts 6 length unit from the positive part of X-axis and 4 length unit from the negative part of y-axis , then $a + 2b =$ ....

(a) 8 (b) 4 (c) -4 (d) -2

(43) The equation of the line which lies at equidistant from the two straight lines $y = -2$ and $y = 10$ is ....

(a) $y = 8$ (b) $y = 4$ (c) $X = 4$ (d) $X = -12$

(44) The equation of the straight line which passes through the point C $(3 , 4)$ and intersects the two positive parts from the coordinate axes X-axis and y-axis at the two points A and B respectively where AC : CB = 2 : 3 is ....

(a) $2X + y = 10$ (b) $X + 2y = 10$ (c) $2X + y = 5$ (d) $X + y = 10$

(45) The area of the triangle bounded by the straight line passing through the point A $(2 , 3)$ and its slope $= \frac{1}{2}$ and the coordinate axes equals .... square unit.

(a) 2 (b) 3 (c) 4 (d) 5

(46) In the opposite figure :

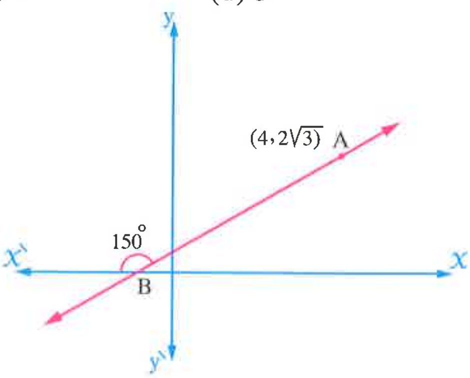
The equation of $\overleftrightarrow{AB}$ is ....

(a) $X - \sqrt{3}\,y - 1 = 0$

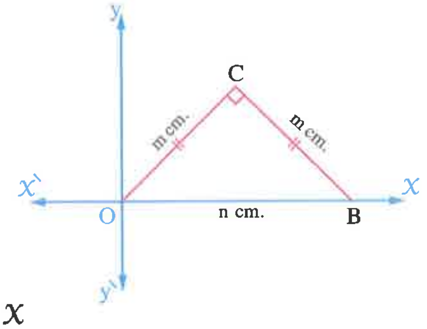
(b) $X - \sqrt{3}\,y + 2 = 0$

(c) $\sqrt{3}\,X + y - \sqrt{3} = 0$

(d) $3X - \sqrt{3}\,y - 6 = 0$



(47) In the opposite figure :



If OC = BC , m (∠ C) = 90°

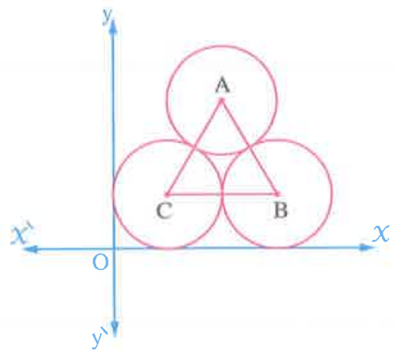
which of the following considered

an equation to the straight line $\overleftrightarrow{OC}$?

(a) $y = \frac{m}{n} X$ (b) $y = X$

(c) $y = \frac{n}{m} X$ (d) $y = m\,n\,X$

(48) In the opposite figure :



Three identical circles touching each other externally

If C = (4 , 4) , then the equation

of the straight line $\overleftrightarrow{AB}$ is ....

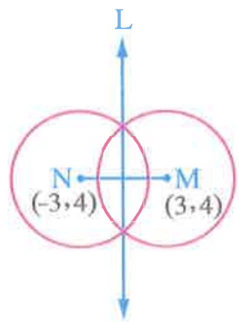
(a) $\vec{r} = (4 , 4) + k\left(1 , -\sqrt{3}\right)$

(b) $\vec{r} = (8 , 4) + k\left(1 , -\sqrt{3}\right)$

(c) $\vec{r} = (12 , 4) + k\left(1 , -\sqrt{3}\right)$

(d) $\vec{r} = (12 , 4) + k\left(-\sqrt{3} , 1\right)$

(49) In the opposite figure :



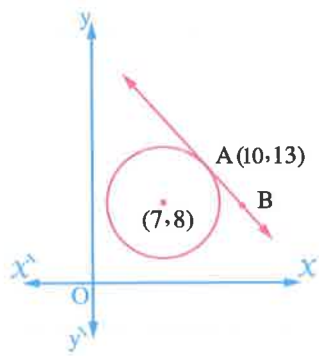
Two identical circles , then

the equation of the straight line L is ....

(a) $X = 0$ (b) $y = 0$

(c) $X + 4\,y = 0$ (d) $3\,X + 4\,y = 0$

(50) In the opposite figure :



The centre of the circle is (7 , 8) ,

the straight line $\overleftrightarrow{AB}$ is a tangent

to the circle at A , then the equation

of the straight line $\overleftrightarrow{AB}$ is ....

(a) $5\,X + 3\,y = 95$

(b) $3\,X + 5\,y = 35$

(c) $3\,X + 5\,y + 95 = 0$

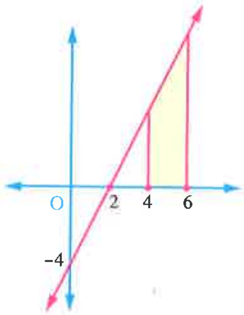
(d) $3\,X + 5\,y = 95$

(51) In the opposite figure :

The area of the shaded figure = ... square unit.

(a) 16 (b) 12

(c) 8 (d) 24



Second Essay questions

1 **Find the slope of the straight line passing through each pair of the following points and show which of these lines are parallel and which are perpendicular :**

(1) (3 , 1) , (– 2 , 5)

(2) (4 , 0) , (2 , – 1)

(3) (7 , – 1) , (3 , –3)

(4) (– 5 , –2) , (– 1 , 3)

2 If the equations of the straight lines L_1 and L_2 are $2x - 3y + a = 0$ and $3x + by - 6 = 0$ respectively

(1) Find the slope of the straight line L_1

(2) Find the value of b which makes the two lines L_1 and L_2 parallel.

(3) Find the value of b which makes the two lines L_1 and L_2 perpendicular.

(4) If the straight line L_1 passes through the point (1 , 3) , then find the value of a

« $\frac{2}{3}$, $\frac{-9}{2}$, 2 , 7 »

3 Which of the following straight lines is parallel to the y - axis , which of them is parallel to the x - axis and which of them passes through the origin point **, then find the coordinates of the points of intersection with the two axes (if found).**

(1) $2x + 3 = 0$

(2) $x + 3y = 0$

(3) $2x + 3y = 12$

(4) $y - 5 = 0$

4 **Find the different forms of the equation of the straight line which :**

(1) Passes through the point P = (3 , – 1) and the vector $\vec{u}$ = (– 3 , 5) is a direction vector to it.

(2) Passes through the point (5 , –1) and makes with the positive direction of the x-axis a positive angle of measure 135°

(3) Passes through the two points (2 , – 3) , (5 , 1)

(4) Passes through the point (2 , –1) and its slope = $\frac{1}{3}$

(5) Passes through the point A = (2 , – 3) and is perpendicular to the vector (– 1 , 2)

(6) Passes through the point $(1 , 3)$ and is perpendicular to the straight line $\vec{r} = (2 , 5) + k(-2 , 1)$

(7) Passes through the point $P = (3 , 5)$ and is perpendicular to the vector $\overrightarrow{AB}$ where $A = (2 , -3)$, $B = (5 , 4)$

(8) Carries a position vector $\vec{A} = (2 , -3)$

5 **Find the general equation of the straight line which :**

(1) Passes through the point $(3 , -4)$ and is parallel to the straight line $x + 2y - 7 = 0$

(2) Passes through the point $P(-1 , -3)$ and the vector $\overrightarrow{AB}$ is a direction vector to it where $A = (-4 , 3)$, $B = (-5 , -2)$

(3) Cuts a length = 4 units from the negative part of the y-axis and the vector $\vec{u} = (7 , -3)$ is a direction vector to it.

(4) Cuts a length = 3 units from the positive part of the x-axis and its slope $= -\frac{1}{2}$

(5) Cuts a length = 2 units from the negative part of the x-axis and cuts a length = 4 units from the positive part of the y-axis.

(6) Passes through the point $\left(-7 , 2\sqrt{3}\right)$ and makes with the positive direction of the x-axis a positive angle of measure $\left(\frac{2\pi}{3}\right)^{rad}$

(7) Passes through the point $(3 , -5)$ and is perpendicular to the straight line $x + 3y = 11$

(8) Passes through the point $(3 , 5)$ and is perpendicular to the straight line $\overleftrightarrow{AB}$ where $A = (2 , -3)$, $B = (5 , 4)$

(9) Is perpendicular to $\overline{AB}$ at the point A where $A = (-3 , 6)$, $B = (2 , 1)$

(10) Is perpendicular to $\overline{AB}$ at its midpoint where $A = (-4 , 1)$, $B = (-2 , 3)$

6 Find the equation of the straight line which passes through the point $(2 , -3)$ and its slope = 2 and if this straight line passes through the two points $(a , 7)$, $(5 , b)$ **, find the values of :** a and b

« 7 , 3 »

7 Find the equation of the straight line whose slope = m and passes through the point $(a , 0)$ What is the point of intersection of this straight line with the y-axis ?

8 **Prove that the straight line which passes through the two points :**

$A = (4 , -1)$, $B = (2 , 3)$ is parallel to the straight line which passes through the two points $C = (2 , 1)$, $D = (3 , -1)$, then find the equation of each of the two straight lines.

9 If $A = (0, 2)$, $B = (2, 1)$, $C = (-2, 3)$ are three points in the plane, find the vector equation of the straight line $\overleftrightarrow{AB}$, then prove that the points A , B , C are collinear.

10 Find the two intercepted parts from the two coordinate axes by the straight line : $2x - 3y + 12 = 0$

11 Find the equations of the two straight lines which pass through the point $(-3, 2)$ and parallel to the coordinate axes.

12 Find the equation of the straight line which passes through the point $(2, -2)$ and makes a positive angle whose cosine $= \frac{-\sqrt{2}}{2}$ with the positive direction of the x-axis.

13 If $A = (-4, 4)$, $B = (-1, -2)$, C divides $\overline{AB}$ by the ratio 1 : 2 internally, find the equation of the straight line which passes through C and the point $(2, 3)$

14 If $A = (1, 4)$, $B = (-4, 6)$, find the equation of the straight line which passes through the point of division of $\overline{AB}$ internally by the ratio 2 : 3 and is perpendicular to the straight line whose equation is $5x - 4y - 12 = 0$

15 **Connecting with geometry :** $\overline{AB}$ is a diameter in the circle M. If B $(-7, 11)$, $M = (-2, 3)$, find the equation of the tangent to the circle at the point A

16 **Connecting with geometry :** If the straight line whose equation is : $3x + 4y - 12 = 0$ intersects the x–axis and the y-axis at A and B respectively , **find :**

(a) The area of Δ OAB where O is the origin point.

(b) The equation of the straight line which is perpendicular to $\overline{AB}$ and passes through its midpoint.

17 Find the lengths of the two intercepted parts of the two coordinate axes by the straight line which passes through the two points $(-3, 1)$, $(4, 0)$

18 Find the lengths of the two intercepted parts from the two coordinate axes by the straight line : $\vec{r} = (3, -1) + k(2, 5)$

19 Find the equation of the straight line which passes through the point $(5, -2)$ perpendicular to the straight line which intercepts from the positive part of x-axis a part of length 4 units and from the negative part of y-axis a part of length 3 units.

20 Prove that the points $A = (2 , -3)$, $B = (7 , 2)$, $C = (1 , 1)$ are vertices of a triangle. If $D \in \overline{AB}$ such that $AD : AB = 2 : 5$, find the coordinates of D , then write the different forms of the equation of the straight line $\overleftrightarrow{CD}$

21 **Find the measure of the positive angle which the straight line L makes with the positive direction of x-axis if :**

(1) $L : 3y + x = 6$

(2) L passes through the two points $(0 , 0)$, $(2 , -2)$

(3) L intercepts from the two coordinate axes two positive parts of lengths 4 and 6 units respectively.

(4) $L : x = 2 + 3k$, $y = -1 + 2k$

(5) The vector $\vec{u} = \left(\sqrt{3} , 1\right)$ is a direction vector to it.

(6) The vector $\vec{N} = \left(\sqrt{3} , 1\right)$ is a direction vector perpendicular to it.

22 **Find the vector equation of the straight line L :** $2x - 3y - 6 = 0$

23 **Find the vector form and the general form of the equation of the straight line L :**

$x = 3 - 2k$, $y = -1 + 3k$

24 **Find the vector form of the equation of the straight line L :**

$\frac{x}{a} + \frac{y}{b} = 1$ where $a \neq 0$, $b \neq 0$

25 **Find the equation of the axis of symmetry of $\overline{AB}$ where :** $A = (2 , 3)$, $B = (-4 , 5)$

26 If $A (5 , -6)$, $B (3 , 7)$, $C (1 , -3)$, find the equation of the straight line which passes through the point A and bisects $\overline{BC}$

27 ABC is a triangle , its vertices are $A (-1 , 5)$, $B (4 , -2)$, $C (-3 , 0)$ Find the equation of the straight line which passes through the vertex A and perpendicular to $\overline{BC}$

28 **Prove that the two equations :**

$\vec{r} = (-1 , 3) + k (6 , -4)$, $\vec{r} = (5 , -1) + k (-3 , 2)$ are of the same straight line.

29 ABCD is a square in which $A = (3 , 2)$, $C = (-1 , 4)$

Find the equations of the straight lines carrying its two diagonals.

30 ABC is a triangle , its vertices are $A = (3, 1)$, $B = (-1, 1)$, $C = (2, 4)$ **Find :**

(1) The equations of the straight lines which carry the sides of the triangle.

(2) The equations of the straight lines which carry the medians of the triangle.

31 **Prove that :** The point $M = (5, -4)$ is the centre of the circumcircle of ΔABC where $A = (1, -1)$, $B = (1, -7)$, $C = (2, 0)$, then find the equation of the tangent to the circle at the point A

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) The straight line intersects X-axis at the point $(a, 0)$ and intersects y-axis at the point $(0, b)$ passes through the point ....

(a) (a, b) (b) $\left(\frac{1}{2} a, \frac{1}{2} b\right)$ (c) $\left(\frac{1}{3} a, \frac{1}{3} b\right)$ (d) $\left(\frac{1}{4} a, \frac{1}{4} b\right)$

(2) If the intercepted parts from the two positive parts of coordinate axes X-axis and y-axis by the straight line L_1 are a , b respectively and the intercepted parts from the two positive parts of coordinate axes X-axis and y-axis by the straight line L_2 are 2 a , 2 b respectively , then ....

(a) $L_1 \perp L_2$ (b) $L_1 // L_2$

(c) $L_1 \cap L_2 = \{(a, b)\}$ (d) something else.

(3) L_1 is a straight line passes through the point $A(2, 3)$ and makes an angle of measure 60° with the positive X-axis , if the straight line L_1 rotate around A with an angle of measure 15° clockwise , then the equation of the straight line in its new position is ....

(a) $y - X = 1$ (b) $y + X = 1$ (c) $2X + y = 1$ (d) $2X + 3y = 5$

(4) The equation of the straight line passes through the point $(3, 4)$ and intercepts two equal parts in length of the coordinate axes could be ....

(a) $X + y = 7$ (b) $X - y = -1$ (c) $y + 2X = 10$ (d) both (a) and (b)

(5) If $A(3, -5)$, $B(-4, 8)$, then the ratio in which the straight line $X + y = 0$ divides the line segment $\overline{AB}$ from A is ....

(a) 2 : 1 (b) 1 : 2 (c) 2 : 3 (d) 3 : 2

(6) The coordinates of the projection of the point (2 , 3) on the straight line L : $x + y = 11$ is ....

(a) (– 6 , 5) (b) (6 , 5) (c) (5 , 6) (d) (– 5 , 6)

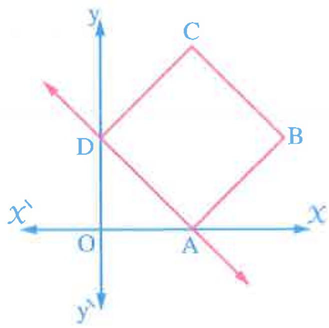
(7) The image of the point (3 , 8) by reflection in the straight line L : $x + 3y - 7 = 0$ is ....

(a) (– 1 , – 4) (b) (– 3 , – 8) (c) (1 , – 4) (d) (3 , 8)

(8) If the point A (0 , 0) is the image of the point B (4 , 2) by reflection in the straight line L , then the equation of the straight line L is ....

(a) $x = 2y$ (b) $2x + y = 5$ (c) $2x - y = 5$ (d) $x + y = 6$

(9) The opposite figure represents the square ABCD , the equation of the straight line $\overleftrightarrow{AD}$ is $x + y = 4$, then the equation of the straight line containes the diagonal $\overline{BD}$ is ....

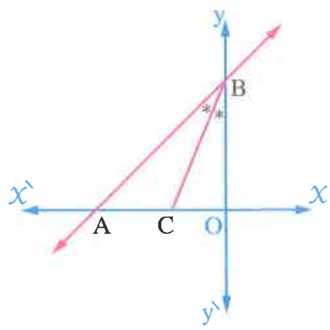


(a) $x = 4$ (b) $y = 4$

(c) $x + y = 2$ (d) $x + y = 4\sqrt{2}$

(10) In the opposite figure :

If the equation of the straight line $\overleftrightarrow{AB}$ is $\frac{y}{6} - \frac{x}{8} = 1$, then the equation of the straight line $\overleftrightarrow{BC}$ is ....

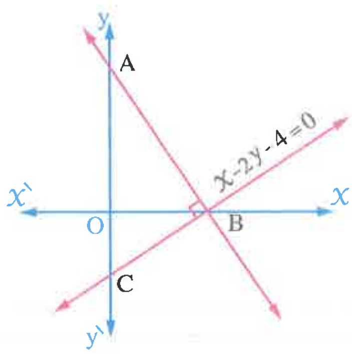


(a) $\frac{x}{3} + \frac{y}{6} = 1$ (b) $\frac{x}{-3} + \frac{y}{6} = 1$

(c) $\frac{x}{6} - \frac{y}{3} = 1$ (d) $x + y = 18$

(11) In the opposite figure :

The area of Δ ABC = ... square unit.



(a) 15 (b) 20

(c) 24 (d) 32

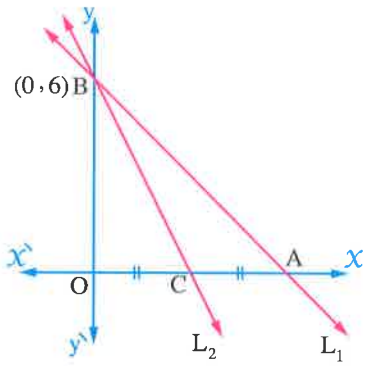
(12) In the opposite figure :

If the area of Δ ABC = 15 square unit

CA = CO , then the equation of L_1 is ....

(a) $\frac{x}{5} + \frac{y}{6} = 1$ (b) $\frac{x}{6} + \frac{y}{5} = 1$

(c) $\frac{x}{10} + \frac{y}{6} = 1$ (d) $\frac{x}{6} + \frac{y}{10} = 1$



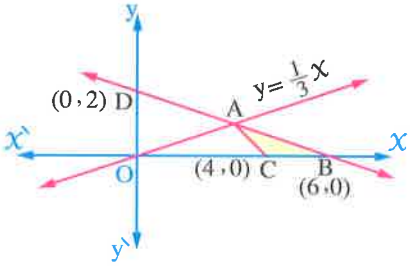
(13) In the opposite figure :

The area of Δ ABC

= .... square unit.

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) $\frac{3}{2}$



(14) In the opposite figure :

The vector equation of the

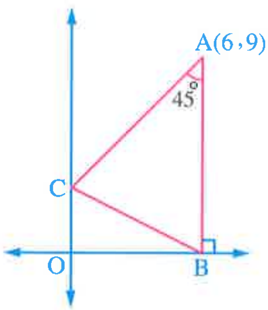
straight line $\overleftrightarrow{BC}$ is ....

(a) $\vec{r} = (0 , 3) + k (2 , -1)$

(b) $\vec{r} = (3 , 0) + k (2 , -1)$

(c) $\vec{r} = (0 , 3) + k (-1 , 2)$

(d) $\vec{r} = (3 , 0) + k (-1 , 2)$



(15) In the opposite figure :

If the equation of the straight line $\overleftrightarrow{AB}$

is $2x + 3y = 12$, then the vector equation

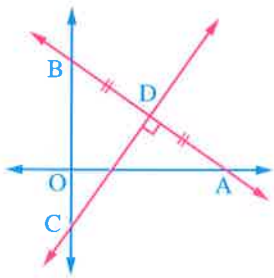
of the straight line $\overleftrightarrow{DC}$ is ....

(a) $\vec{r} = (2 , 3) + k (2 , 3)$

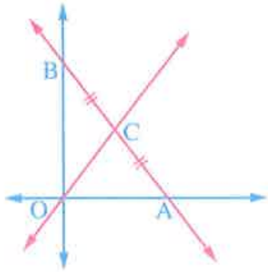
(b) $\vec{r} = (2 , 3) + k (3 , 2)$

(c) $\vec{r} = (3 , 2) + k (2 , 3)$

(d) $\vec{r} = (3 , 2) + k (3 , 2)$



(16) In the opposite figure :



If the equation of the straight line $\overleftrightarrow{AB}$ is $\frac{x}{6} + \frac{y}{8} = 1$

, then the parametric equation of the straight line $\overleftrightarrow{OC}$ is ....

(a) $x = 3 + 4k$, $y = 4 + 3k$

(b) $x = 4 + 3k$, $y = 3 + 4k$

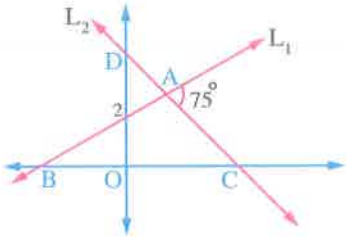
(c) $x = 3 + 3k$, $y = 4 + 4k$

(d) $x = 4 + 4k$, $y = 3 + 3k$

(17) In the opposite figure :

If $L_1 \cap L_2 = \{A\}$

, $OC = OD$, then the vector equation of the straight line L_1 is ....



(a) $\vec{r} = (0 , 2) + k\left(1 , \sqrt{3}\right)$

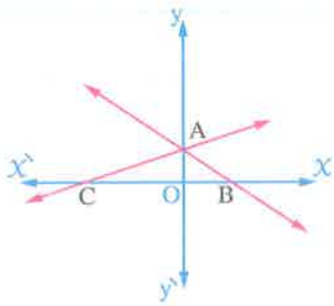
(b) $\vec{r} = (0 , 2) + k(-1 , -1)$

(c) $\vec{r} = (0 , 2) + k\left(\sqrt{3} , 1\right)$

(d) $\vec{r} = (0 , 2) + k\left(\sqrt{3} , -1\right)$

(18) In the opposite figure :

If $OC = 2\ OB$, the equation of the straight line $\overleftrightarrow{AB}$ is $2x + 3y = 6$, then the vector equation of the straight line $\overleftrightarrow{AC}$ is ....



(a) $\vec{r} = (-6 , 0) + k(1 , 3)$

(b) $\vec{r} = (-6 , 0) + k(3 , 1)$

(c) $\vec{r} = (6 , 0) + k(1 , 3)$

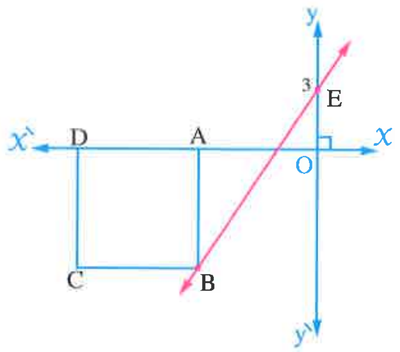
(d) $\vec{r} = (6 , 0) + k(3 , 1)$

(19) In the opposite figure :

If the area of the square ABCD = 36 square unit ,

D = (– 12 , 0) , then the vector equation of

the straight line $\overleftrightarrow{EB}$ is ....



(a) $\vec{r} = (0 , 3) + k (3 , 2)$

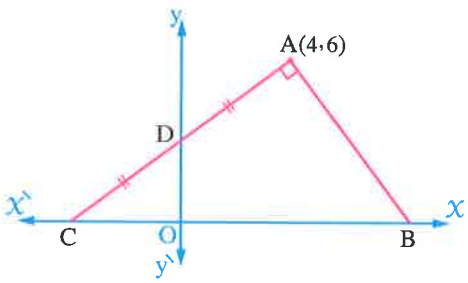
(b) $\vec{r} = (0 , 3) + k (-3 , 2)$

(c) $\vec{r} = (-6 , 6) + k (-2 , 3)$

(d) $\vec{r} = (-6 , -6) + k (2 , 3)$

(20) The parametric equations of

the straight line $\overleftrightarrow{AB}$ is ....



(a) $x = 4 + 3k$, $y = 6 + 4k$

(b) $x = 3 - 4k$, $y = 4 - 6k$

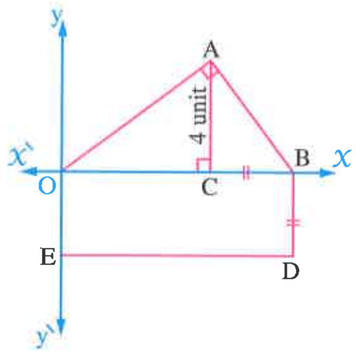
(c) $x = 4 + 3k$, $y = 6 - 4k$

(d) $x = 3 - 4k$, $y = 4 + 6k$

(21) In the opposite figure :

If the area of the rectangle OBDE = 20 square unit

, then the equation of $\overleftrightarrow{AB}$ is ....



(a) $2x + y + 20 = 0$

(b) $2x + y - 20 = 0$

(c) $2x - y = 20$

(d) $y = 2x + 20$

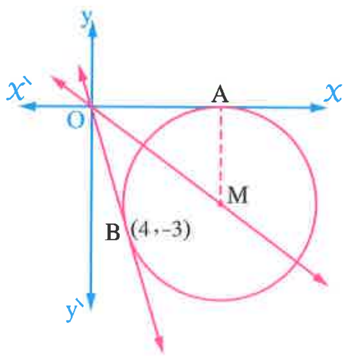
(22) In the opposite figure :

If $\overrightarrow{OA}$, $\overrightarrow{OB}$ are tangents to

the circle M at A and B

, then the vector equation

of the straight line $\overleftrightarrow{OM}$ is ....



(a) $\vec{r} = (0 , 0) + k (-1 , 3)$

(b) $\vec{r} = (-4 , 3) + k (3 , -1)$

(c) $\vec{r} = (0 , 0) + k (3 , -1)$

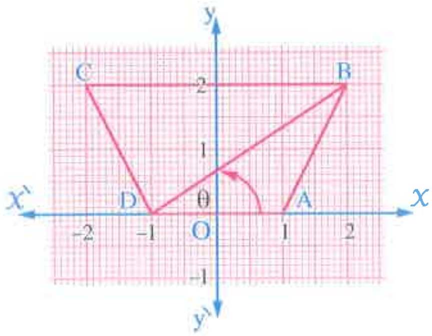
(d) $\vec{r} = (5 , 0) + k (3 , -1)$

2 **In the opposite figure :**

If ABCD is a quadrilateral , **find :**

(1) The slope of $\overleftrightarrow{BD}$, then deduce the value of θ

(2) The two equations of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$



3 Find the equation of the straight line which passes through the point (4 , 3) and intercepts from the coordinate axes two different positive parts , the sum of their lengths = 14

4 Find the equation of the straight line passing through the point (3 , 2) and its slope is negative and it makes with the two coordinate axes a triangle of area 12 square units.

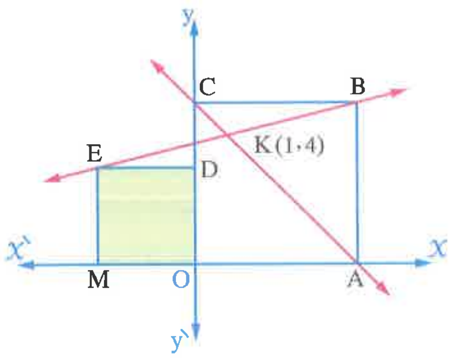
5 **In the opposite figure :**

Two squares OABC , OMED

, $\overleftrightarrow{BE} \cap \overleftrightarrow{CA} = \{K\}$, K = (1 , 4)

Find the area of the shaded square.

« 9 square unit »



Exercise **Seven**

Measure of the angle between two straight lines

Test yourself

From the school book

First Multiple choice questions

• Choose the correct answer from those given :

(1) The measure of the angle between the straight lines whose slopes are $2 , -\frac{1}{2}$ equals ....

(a) 30° (b) 60° (c) 90° (d) 45°

(2) The measure of the acute angle between the two straight lines whose slopes are $\frac{-3}{4} , -7$ equals ....

(a) 30° (b) 60° (c) 45° (d) 54°

(3) The measure of the acute angle between the two straight lines whose slopes are $\frac{1}{2} , \frac{2}{9}$ approximately equals ....

(a) 28° (b) 14° (c) 32° (d) 15°

(4) The measure of the angle between the straight lines whose equations are $x = 3$, $y = 4$ equals ....

(a) 90° (b) 45° (c) 60° (d) 30°

(5) The measure of the acute angle between the two straight lines $L_1 : \vec{r} = (0 , -2) + k (3 , -1)$ and $L_2 : \vec{r} = (0 , 5) + k (2 , 1)$ equals ....

(a) 30° (b) 45° (c) 60° (d) 90°

(6) The measure of the acute angle between the straight line :

$6x - 3y + 5 = 0$ and the straight line whose slope is $\frac{1}{3}$ equals ....

(a) 135° (b) 60° (c) 30° (d) 45°

(7) The measure of the acute angle between the two straight lines $L_1 : y - \sqrt{3}\, x - 5 = 0$ and $L_2 : x - \sqrt{3}\, y - 6 = 0$ equals ....

(a) 30° (b) 45° (c) 60° (d) 90°

(8) The measure of the acute angle between the two straight lines $L_1 : \vec{r} = (2 , 5) + k(-3 , 1)$, $L_2 : 2x = 3 - y$ equals ....

(a) 30° (b) 45° (c) 60° (d) 50°

(9) The measure of the angle between the two lines $L_1 : x + 2y + 5 = 0$, $L_2 : \vec{r} = (1 , 4) + k(1 , 2)$ equals ....

(a) zero (b) 45° (c) 90° (d) 135°

(10) The measure of the angle between the two straight lines $L_1 : \vec{r} = (1 , 2) + k(3 , -4)$, $L_2 : 4x + 3y - 5 = 0$ is ....

(a) 0° (b) 30° (c) 45° (d) 60°

(11) The measure of the acute angle between the two straight lines $L_1 : 2x + 3y = 15$, $L_2 : \vec{r} = (-2 , -1) + k(1 , -3)$ approximately equals ....

(a) 52° (b) 51° (c) 39° (d) 38°

(12) The measure of the acute angle between the two straight lines $L_1 : 2x - y - 3 = 0$, $L_2 : x = k , y = 1 + k$ approximately equals ....

(a) 19° (b) 71° (c) 18° (d) 72°

(13) The measure of the acute angle between the two straight lines $x = 3y$ and $x + 2y = 0$ is ....

(a) 15° (b) 30° (c) 45° (d) 60°

(14) The measure of the acute angle between the straight line : $x - 2y + 3 = 0$ and the straight line passes through the two points $(4 , -1)$, $(2 , 1)$ approximately equals ....

(a) 71° 34' (b) 19° 28' (c) 70° 32' (d) 18° 26'

(15) The measure of the acute angle between the two straight lines : $\sqrt{3}\, x - y = 5$, $y = 2$ equals ....

(a) 30° (b) 60° (c) 45° (d) 120°

(16) The measure of the angle between the straight line which passes through the two points $(0 , 3)$, $(-3 , 0)$ and the straight line $y = 0$ equals ....

(a) 30° (b) 60° (c) 45° (d) 90°

(17) The measure of the acute angle between the straight line whose equation is $\vec{r} = (2 , 2) + t (1 , 1)$ and the straight line whose equation is $X = 0$ equals ....

(a) 30° (b) 45° (c) 60° (d) 135°

(18) If the measure of the included angle between the two straight lines :

$X = 7$, $y = a X + 2$ equals 90° , then a = ...

(a) zero (b) 1 (c) 90 (d) – 1

(19) If A (– 2 , 1) , B (2 , 3) , C (– 2 , – 4)

, then the measure (1) of the acute angle between $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ is ....

(a) $\tan^{-1}(1)$ (b) $\tan^{-1}\left(\frac{2}{3}\right)$ (c) $\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\tan^{-1}\left(\frac{3}{2}\right)$

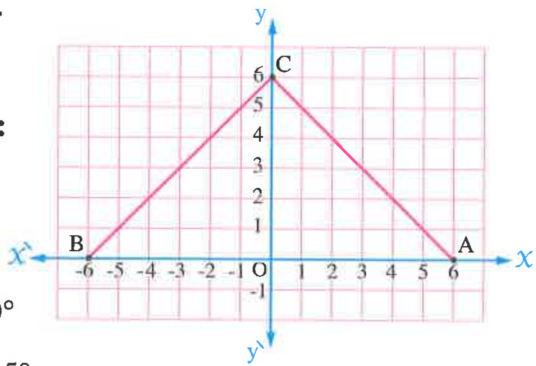
(20) The set of values of k which makes the measure of the acute angle between the two straight lines $X + k y - 8 = 0$, $2 X - y - 5 = 0$ equals $\frac{\pi}{4}$ is ....

(a) $\left\{3 , -\frac{1}{3}\right\}$ (b) $\left\{-3 , \frac{1}{3}\right\}$ (c) $\left\{3 , \frac{1}{3}\right\}$ (d) $\{3\}$

(21) The equation of the straight line L that has a positive slope and passes through the point (4 , 1) and makes with the straight line $3 X - y + 4 = 0$ an angle , the tangent of its measure = $\frac{1}{2}$, is ....

(a) $y - X - 3 = 0$ (b) $y - X + 3 = 0$ (c) $y + 7 X - 29 = 0$ (d) $y - 7 X + 29 = 0$

(22) **The opposite figure shows a triangular piece of land , its vertices are A (6 , 0) , B (– 6 , 0) , C (0 , 6), then :**



First : The measure of the acute angle between $\overleftrightarrow{AC}$ and the X-axis equals ....

(a) 30° (b) 60°

(c) 45° (d) 135°

Second : The measure of the angle between the straight lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ equals ....

(a) 30° (b) 60° (c) 45° (d) 90°

Third : The vector equation of the straight line $\overleftrightarrow{AC}$ is ....

(a) $\vec{r} = (6 , 0) + k (1 , 0)$ (b) $\vec{r} = (6 , 0) + k (0 , 1)$

(c) $\vec{r} = (6 , 0) + k (-1 , 1)$ (d) $\vec{r} = (6 , 0) + k (1 , 1)$

Fourth : The vector equation of the straight line $\overleftrightarrow{BC}$ is ....

(a) $\vec{r} = (0 , 6) + k(-1 , 1)$ (b) $\vec{r} = (0 , 6) + k(1 , 1)$

(c) $\vec{r} = (6 , 0) + k(-1 , 1)$ (d) $\vec{r} = (0 , 6) + k(1 , 0)$

Fifth : The cartesian equation of the straight line which passes through the point C and parallel to $\overleftrightarrow{AB}$ is ....

(a) $y = 6$ (b) $x = 6$ (c) $x + y = 6$ (d) $x - y = 6$

Sixth : The area of the triangle ABC equals .... square units.

(a) 24 (b) 12 (c) 36 (d) 48

Second Essay questions

1 Find the measure of the acute angle between each of the following pairs of straight lines :

(1) $L_1 : \vec{r} = k(1 , 0)$, $L_2 : \vec{r} = (3 , -2) + k(1 , -2)$

(2) 🕮 $L_1 : \vec{r} = (0 , 1) + k(1 , 1)$, $L_2 : 2x - y - 3 = 0$

(3) $L_1 : x + 2y + 5 = 0$, $L_2 : 4x - y - 3 = 0$

(4) 🕮 $L_1 : x + 2y + 3 = 0$, $L_2 : x - 3y + 1 = 0$

(5) $L_1 : 3y + 2x - 6 = 0$, $L_2 : \frac{x}{5} - y = 3$

(6) $L_1 : \frac{-x}{3} - \frac{y}{2} = 1$, $L_2 : y - 4x = 2$

(7) $L_1 : 3y = 5$, $L_2 : 2x + 5y = 1$

(8) $L_1 : \vec{r} = (-2 , -3) + k(1 , -2)$, $L_2 : x = 3 + 2k$, $y = 3k - 1$

---

2 If $L_1 : ax - 3y + 7 = 0$, $L_2 : 4x + 6y - 5 = 0$, $L_3 : \frac{x}{3} - \frac{y}{2} = 3$

, then find the value of a which makes :

(1) The measure of the angle between the two straight lines L_1 and L_3 equals 0° « 2 »

(2) The measure of the angle between the two straight lines L_1 and L_2 equals 90° « $\frac{9}{2}$ »

---

3 Find the equation of the straight line :

(1) Passing through the point $(-1 , 3)$ and makes with the straight line : $x + 2y + 6 = 0$ an angle θ where $\tan \theta = \frac{1}{7}$

(2) Passing through the point $(2 , -2)$ and makes with the straight line $\vec{r} = (2 , -1) + k(3 , -4)$ an angle of measure 45°

4 If θ is the measure of the angle between the two straight lines : $x - y + 6 = 0$ and $a\,x - 2\,y + 4 = 0$ where $\cos\theta = \frac{4}{5}$, **find the value of :** a « $\frac{2}{7}$ or 14 »

5 If the measure of the angle between the two straight lines : $k\,y + x = 6$, $2\,x + y = 3$ is θ where $\tan\theta = \frac{3}{4}$, **find the value of :** k « 2 or $\frac{-2}{11}$ »

6 If the measure of the angle between the two straight lines : $3\,x - 5\,y - 1 = 0$ and $k\,x - y = 3$ equals 45° , **find the value of :** k « $-\frac{1}{4}$ or 4 »

7 If the tangent of the measure of the angle between the two straight lines :
$\vec{r} = (0 , \frac{9}{2}) + k\,(2 , 3\,a)$, $\vec{r} = (4 , 1) + \grave{k}\,(1 , 2)$ equals $\frac{2}{3}$,
then find the value of : a « $\frac{8}{21}$ or $\frac{-16}{3}$ »

8 Find the equation of each of the two lines passing through (3 , – 1) and the measure of the angle between them is θ where $\tan\theta = \frac{5}{11}$,
knowing that their slopes are m and $\frac{3}{8}$m where m > 0

9 ABC is a triangle in which A = (0 , 2) , B = (3 , 1) , C = (– 2 , – 1)
Find the measure of angle A « 105° 15` 18``»

10 Find the measures of the angles of the triangle ABC whose vertices are A = (4 , 7) , B = (– 2 , – 1) and C = (2 , – 4) « 26° 34` , 90° , 63° 26` »

11 Find the measures of the angles of the triangle ABC whose vertices are A = (2 , 3) , B = (5 , 1) and C = (– 2 , 1) , then find the area of the triangle to the nearest unit.
« 7 square units »

12 ABC is a triangle in which A = (0 , 5) , B = (2 , – 1) and C = (6 , 3)
Prove that the triangle is isosceles , then find the measure of angle A and find its area to the nearest hundredth. « 53° 8` , 16 square units »

13 ABC is a right-angled triangle at A , the equation of $\overleftrightarrow{BC}$ is $\vec{r} = (1 , 1) + k\,(-1 , 3)$ and the equation of $\overleftrightarrow{AB}$ is $\vec{r} = (-1 , 5) + k\,(1 , 2)$ **Find :** m (∠ ACB) « 45° »

14 If the triangle ABC is right at B where A = (2 , 3) , B = (5 , 7) and C = (1 , y) , find the value of y , then find the measures of the other two angles. « 10 , 45° , 45° »

15 ABC is a triangle in which A = (– 3 , 2) , B = (– 3 , – 13) , C = (5 , 3) and D is the midpoint of $\overline{BC}$. **Find the measure of the acute angle between :** $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ « 56° 19`»

---

16 Δ ABC is a triangle in which A = (5 , 7) , B = (1 , 5) , C = (4 , 2)

(1) Find the coordinates of the point D which divides $\overline{BC}$ internally by the ratio 1 : 2

(2) **Prove that :** $\overline{AD} \perp \overline{BC}$

(3) **Prove that :** AD = BC

(4) **Find :** The measure of the acute angle B

(5) **Find :** The area of the triangle ABC

---

17 If A = (5 , 7) , B = (6 , – 1) , C = $\left(\frac{3}{2}, 5\right)$, **then prove that :** the straight line $\vec{r}$ = (3 , 1) + k (– 2 , 1) makes with the two straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ an isosceles triangle whose vertex is A

---

18 **Prove that :** The triangle whose equations of its sides are : $3x + 4y = 36$, $x - 7y + 13 = 0$ and $7x + y = 9$ is right-angled and isosceles.

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) The measure of the obtuse angle between the two straight lines $y = \left(2 - \sqrt{3}\right)(x + 5)$, $y = \left(2 + \sqrt{3}\right)(x - 7)$ is ....

(a) 150° (b) 60° (c) 135° (d) 120°

(2) The tangent of the angle between the two straight lines L_1 and L_2 is $\frac{1}{3}$, the slope of L_1 equals twice the slope of L_2 , then the slope of the straight line L_2 = ...

(a) $\pm \frac{1}{2}$ (b) ± 1 (c) $1 , \frac{1}{2}$ (d) All the previous.

(3) **In the opposite figure :**

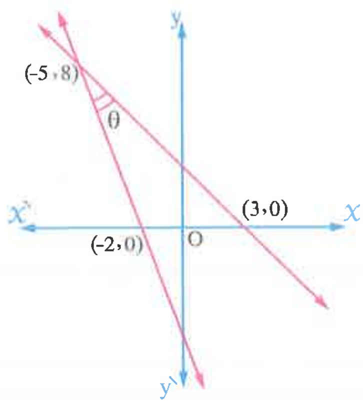
tan θ = ...

(a) $\frac{8}{3}$

(b) $\frac{3}{10}$

(c) $\frac{5}{11}$

(d) $\frac{7}{11}$



(4) In the opposite figure :

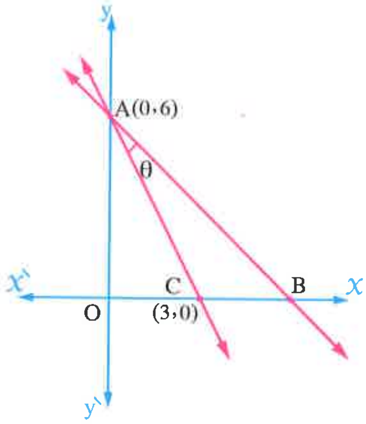
If $\cos\theta = \frac{2}{\sqrt{5}}$, then the coordinates of the point B = ...

(a) (8 , 0)

(b) (7 , 0)

(c) (6 , 0)

(d) (4 , 0)

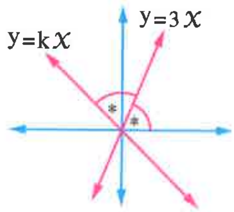


(5) In the opposite figure :

k = ...

(a) $\frac{-3}{4}$ (b) $\frac{-4}{3}$

(c) $\frac{3}{4}$ (d) $\frac{3}{2}$



(6) In the opposite figure :

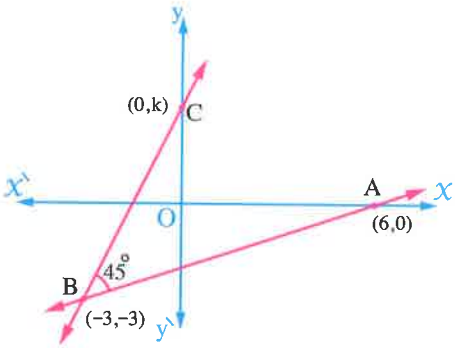
The value of k = ...

(a) 3

(b) – 3

(c) $\frac{9}{2}$

(d) $\frac{-9}{2}$



(7) In the opposite figure :

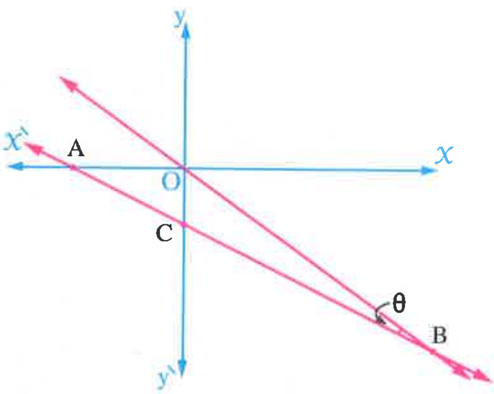
If the equation of $\overleftrightarrow{AB}$ is $X + 2y + 6 = 0$ and BC = 2 AC , then tan θ = ...

(a) $\frac{11}{10}$

(b) $\frac{2}{11}$

(c) $\frac{10}{11}$

(d) $\frac{11}{2}$



2 Find the equation of one of the two equal sides in right-angled triangle if the equation of the hypotenuse is $3X + 4y + 4 = 0$ and the coordinates of the right vertex is (2 , 2)

« $X - 7y + 12 = 0$ or $7X + y - 16 = 0$ »

3 If the line L makes an angle whose cosine is $\frac{3\sqrt{10}}{10}$ with the line $\grave{L}: 3x - y + 5 = 0$,

what is the slope of L ?

Find the equation of L if it passes through the point (1 , – 2) « undefined or $\frac{4}{3}$ »

---

4 **Prove that :** The angle between the two straight lines : $y = \frac{1+b}{1-b}x + 6$ and $y - bx + 1 = 0$

has a constant measure for all values of $b \neq 1$ and find the measure of this angle. « 45° »

---

5 **In the opposite figure :**

The equation of

L_1 is $x + 7y = 7$ and

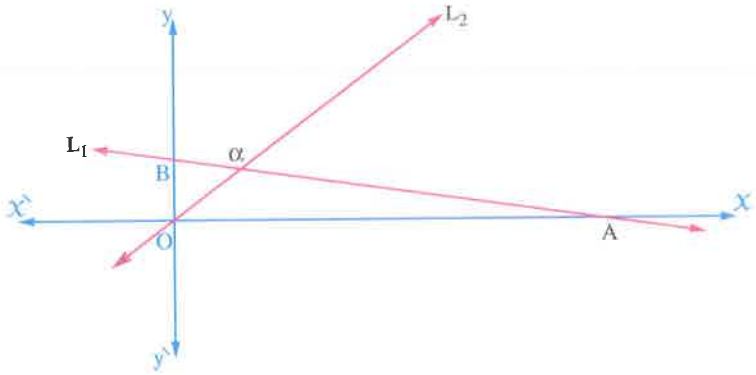
the equation of

L_2 is $3x - 4y = 0$

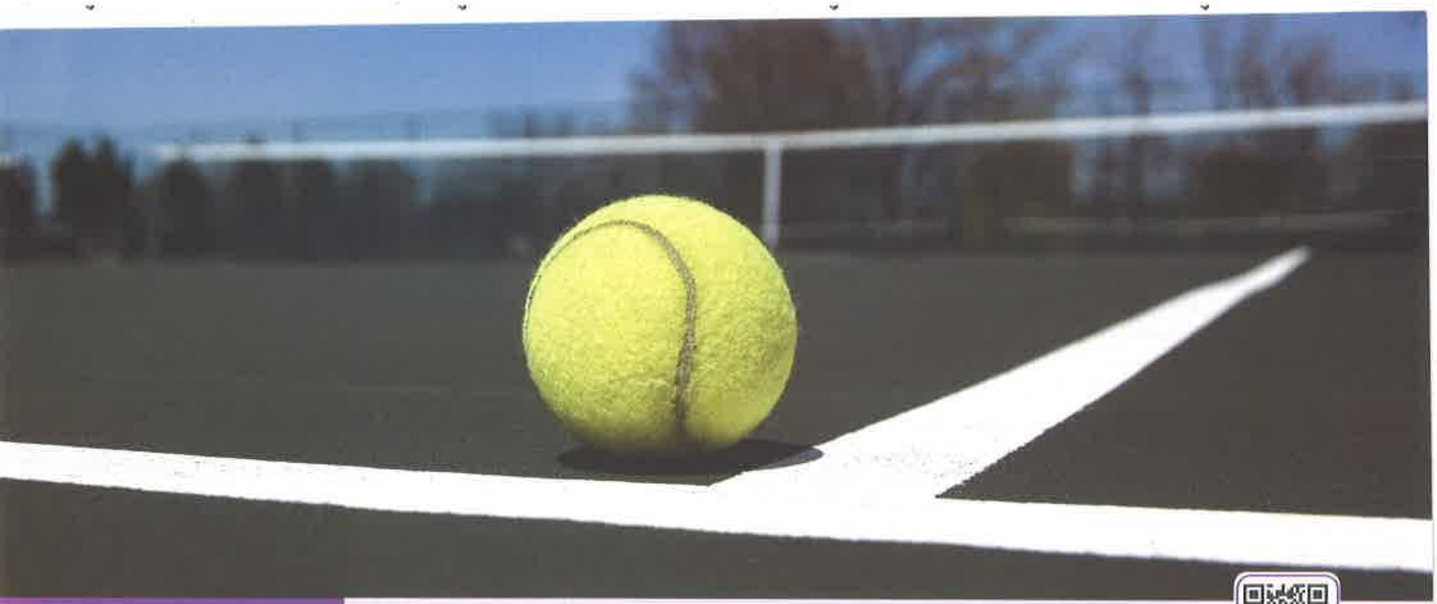
Find in degrees the measure of

the obtuse angle α , then find

the coordinates of the points A and B



« 135° , A (7 , 0) , B (0 , 1) »



Exercise Eight

The length of the perpendicular from a point to a straight line

Test yourself

From the school book

First Multiple choice questions

● **Choose the correct answer from those given :**

(1) The length of the perpendicular from the point $(-3 , 5)$ to y-axis equals .... length unit.

(a) 3 (b) 5 (c) 8 (d) -3

(2) The length of the perpendicular from the point $(-3 , 5)$ to X-axis equals .... length unit

(a) 3 (b) 5 (c) 8 (d) -3

(3) The length of the perpendicular drawn from the origin point to the straight line $3X - 4y - 15 = 0$ equals .... length unit.

(a) 15 (b) 5 (c) 3 (d) 4

(4) The length of the perpendicular drawn from the point $(3 , 1)$ to the straight line $4X + 3y - 5 = 0$ equals .... unit length.

(a) 2 (b) 3 (c) 4 (d) 5

(5) The length of the perpendicular drawn from the point $(0 , -4)$ to the straight line $X + 5 = 0$ equals .... length unit.

(a) 0 (b) 4 (c) 5 (d) -5

(6) 📖 The length of the perpendicular drawn from the point $(1 , 1)$ to the straight line whose equation is $X + y = 0$ equals .... length unit

(a) 2 (b) $\sqrt{2}$ (c) 1 (d) 0

(7) The length of the perpendicular drawn from the origin point to the straight line $\vec{r} = (1 , 2) + k (4 , 3)$ equals .... length unit.

(a) 3 (b) 4 (c) 5 (d) 1

(8) The length of the perpendicular drawn from the point (0 , 2) on the straight line $\vec{r} = (2 , 1) + k (3 , -4)$ equals .... length unit.

(a) 5 (b) 1 (c) 2 (d) 3

(9) The length of the perpendicular drawn from the point (– 2 , – 4) on the straight line $\vec{r} = (3 , 0) + k (6 , 8)$ equals .... length units.

(a) 1.6 (b) 2.6 (c) 0.6 (d) 3.6

(10) The length of the perpendicular drawn from the point (– 2 , – 5) on the straight line L : $x = -2 + 4k$, $y = -3k$ equals .... length units.

(a) 1 (b) 2 (c) 3 (d) 4

(11) The distance between the point (1 , 5) and the straight line passes through the two points (5 , – 3) , (1 , 0) equals .... length units.

(a) 2 (b) 3 (c) 4 (d) 5

(12) The distance between the point (1 , 5) and the straight line passes through the point (2 , – 3) and the vector (2 , – 1) is its direction vector equals .... length units.

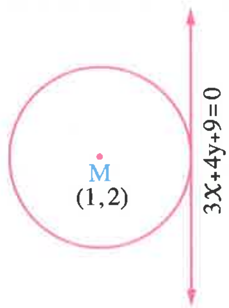
(a) $\sqrt{5}$ (b) $2\sqrt{5}$ (c) $3\sqrt{5}$ (d) $4\sqrt{5}$

(13) In Δ ABC : A (3 , 7) , B (7 , – 1) , C (3 , 2) , then the length of the perpendicular drawn from A to $\overleftrightarrow{BC}$ equals .... length units.

(a) 3 (b) 4 (c) 5 (d) 1

(14) **In the given figure :**

The straight line $3x + 4y + 9 = 0$ is a tangent to the circle M where M (1 , 2) , then the radius of the circle equals ....



(a) 5 (b) $\sqrt{5}$

(c) 4 (d) 3

(15) The area of the circle with centre (4 , – 1) and touches the straight line L : $\vec{r} = (1 , 1) + k (12 , 5)$ equals .... square units.

(a) 8π (b) 9π (c) 6π (d) 3π

(16) The distance between the straight lines whose equations are $y-3=0$, $y+2=0$ equals .... length unit.

(a) 3 (b) 2 (c) 1 (d) 5

(17) The distance between the two straight lines $\vec{r}=(-1,0)+k(4,-3)$, $6x+8y-9=0$ equals ....

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

(18) The distance between the two straight line $L_1: 3x-4y+20=0$, $L_2: 3x-4y+10=0$ equals .... length units.

(a) 2 (b) 3 (c) 4 (d) 5

(19) The distance between the two straight line : $\vec{r}=(2,0)+k(12,-5)$, $\vec{r}=(4.6,0)+k(12,-5)$ equals .... length units.

(a) 1 (b) 2 (c) 3 (d) 4

(20) If the length of the perpendicular drawn from the point $(2,k)$ on the straight line $2x+y+1=0$ equals $\sqrt{5}$ length unit then one of the values of $k=$ ....

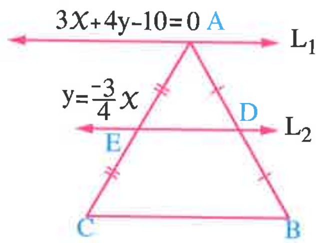
(a) -4 (b) -5 (c) -8 (d) -10

(21) If the distance between the two straight lines $L_1: 3x+4y-12=0$, $L_2: 6x+8y+c=0$ equals 3 length units , $c>0$, then $c=$ ....

(a) 54 (b) 6 (c) 30 (d) 3

(22) **In the given figure :**

The length of the perpendicular drawn from A to the straight line $\overleftrightarrow{BC}$ equals ....

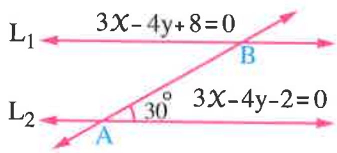


(a) 2 (b) 3

(c) 4 (d) 5

(23) **In the given figure :**

The length of $\overline{AB}=$ .... length unit.



(a) 1 (b) 2

(c) 3 (d) 4

(24) The equation of one of the two straight lines whose slope $=\frac{-5}{12}$ and the length of the perpendicular drawn on it from the point $(2,-1)$ equals 2 length units is ....

(a) $5x+12y-28=0$ (b) $5x+12y-24=0$

(c) $5x+12y+24=0$ (d) $5x-12y+28=0$

Second Essay questions

1 **Find the length of the perpendicular drawn from the point A to the straight line L if :**

(1) $A = (2 , 1)$, $L : 3x + 4y - 30 = 0$

(2) $A = (4 , 5)$, $L : \vec{r} = (1 , 2) + k (3 , -4)$

(3) 📖 $A = (0 , 0)$, $L : \vec{r} = (0 , 5) + k (3 , 4)$

(4) 📖 $A = (2 , -4)$, $L : 12x + 5y - 43 = 0$

(5) 📖 $A = (5 , 2)$, $L : 8x + 15y - 19 = 0$

(6) $A = (-2 , -7)$, $L : x + y + 9 = 0$

(7) $A = (2 , -6)$, $L : \frac{x}{3} + \frac{y}{2} = 2$

(8) $A = \left(2\sqrt{b} , -\sqrt{a}\right)$, $L : (a - b)x + 2\sqrt{ab}\, y - 2a\sqrt{b} = 0$

2 If ABC is a triangle in which $A = (3 , 5)$, $B = (-1 , 2)$ and $C = (7 , -4)$, find the length of the perpendicular drawn from A to $\overleftrightarrow{BC}$ « 4.8 length units »

3 **Calculate the radius length of the circle whose centre is $M = (3 , -1)$ and touches the straight line L :** $4x + 3y + 6 = 0$ « 3 length units »

4 Find the distance between the point $(1 , -2)$ and the straight line passing through the point $(2 , -3)$ and makes angles of same measure with each of the positive direction of the x-axis and the negative direction of the y-axis. « $\sqrt{2}$ length units »

5 If the length of the perpendicular drawn from the point $(1 , c)$ to the straight line $2x + 3y + 5 = 0$ equals $\sqrt{13}$ length units , **find the value of :** c « 2 or $\frac{-20}{3}$ »

6 📖 If the length of the perpendicular drawn from the point $(3 , 1)$ to the straight line $3x - 4y + c = 0$ equals 2 length units , **find the value of :** c « 5 or -15 »

7 If the length of the perpendicular drawn from the point $(7 , -1)$ to the straight line $ax + y = 0$ equals $2\sqrt{10}$ length units , **find the possible values of :** a « 3 or $\frac{-13}{9}$ »

8 **Prove that :** The two straight lines $L_1 : 2x + y - 3 = 0$, $L_2 : \vec{r} = (5 , 8) + k (-1 , 2)$ are parallel, then find the distance between them. « $3\sqrt{5}$ length units »

9 📖 **Prove that :** The straight lines whose equations are $3x - 4y - 12 = 0$ and $6x - 8y + 21 = 0$ are parallel , then find the distance between them. « 4.5 length units »

10 **Prove that :** The two straight lines $L_1 : \vec{r} = (0 , 2) + k (5 , 2)$, $L_2 : \vec{r} = (2 , -3) + k (10 , 4)$ are parallel, then find the distance between them.

« $\sqrt{29}$ length units »

---

11 **Roads :** Two adjacent roads, the path of the first road is represented by the equation $3x - 4y - 7 = 0$ and the path of the second road is represented by the equation $3x - 4y + 11 = 0$

Prove that : The two roads are parallel, then find the shortest distance between them.

« 3.6 length units »

---

12 If the straight line whose equation is $4x - 3y = 12$ intersects the two axes at the two points A , B , **find :**

(1) The area of the triangle OAB where O is the origin point.

(2) The shortest distance from the origin point to the straight line $\overleftrightarrow{AB}$

« 6 square units , 2.4 length units »

---

13 If the points $A = (-4 , 1)$, $B = (2 , 3)$ and $C = (-2 , 6)$ are the vertices of a triangle , **find :**

(1) The length of $\overline{BC}$

(2) The cartesian equation of the straight line $\overleftrightarrow{BC}$

(3) The length of the perpendicular drawn from A to $\overleftrightarrow{BC}$

(4) The area of Δ ABC

« 5 length units , $3x + 4y - 18 = 0$, 5.2 length units , 13 square units »

---

14 Find the area of the triangle whose vertices are $A = (3 , 2)$, $B = (-2 , 5)$ and $C = (1 , -3)$

« 15.5 square units »

---

15 ABCD is a parallelogram in which $A = (-1 , 4)$, $B = (3 , -2)$ and $C = (-1 , -5)$ **Find :**

(1) The coordinates of D

(2) The length of $\overline{BC}$

(3) The equation of $\overleftrightarrow{BC}$

(4) The length of the perpendicular drawn from A to $\overleftrightarrow{BC}$

(5) The area of the parallelogram ABCD

« $(-5 , 1)$, 5 length units , $3x - 4y - 17 = 0$, 7.2 length units , 36 square units »

16 **Prove that :** The points $A = (3 , -1)$, $B = (-5 , 2)$, $C = (-2 , 4)$ and $D = (6 , 1)$ are the vertices of a parallelogram and find its area. « 25 square units »

17 **Prove that :** The points $A = (2 , 3)$, $B = (6 , 2)$, $C = (-2 , -2)$ and $D = (-2 , 1)$ are the vertices of a trapezium and find its area. « 18 square units »

18 **Geometry :** ABCD is a trapezium in which $\overrightarrow{AD} \parallel \overrightarrow{BC}$, if $A (2 , 1)$, $B (5 , 3)$, $C (6 , 1)$, $D (4 , y)$, find the value of y , then find the area of the trapezium ABCD

« – 3 , 12 square units »

19 Find the equation of the straight line where the vector $(-1 , 7)$ is a direction vector to it and the length of the perpendicular drawn from the point $(3 , 1)$ to it equals $3\sqrt{2}$ length units. « $7x + y + 8 = 0$ or $7x + y - 52 = 0$ »

20 **Prove that :** The two points $(1 , 1)$ and $(-2 , 3)$ lie on two different sides of the straight line : $2x - y + 3 = 0$ and at equal distances from it.

21 Do the two points $(1 , 4)$ and $(-2 , 3)$ lie on the same or different sides of the line $2x - y + 3 = 0$?

22 **Prove that :** The straight line : $3x - 4y + 3 = 0$ touches each of the two circles in which their centres are the two points $N_1 = (5 , 2)$, $N_2 = (-2 , 3)$ and their radius lengths are 2 , 3 length units respectively, then show if the two circles lie on one side or on the two sides of the straight line.

23 **Geometry :** A circle of centre is the origin point in which two chords whose equations are $4x - 3y + 10 = 0$, $5x - 12y + 26 = 0$ **Prove that :** The two chords are equal in length.

24 **Prove that :** The point $(11 , 8)$ is the centre of the inscribed circle of the triangle in which the equations of its sides are : $\vec{r} = (-3 , 20) + k (1 , 0)$, $3x + 4y = 5$ and $5x + 12y + 5 = 0$

25 **Prove that :** The point $A (4 , 6)$ lies on one of the two bisectors of the angle between the two straight lines $9x - 13y - 8 = 0$ and $x - 3y + 4 = 0$

26 **Find the area of the quadrilateral ABCD whose vertices are :** $A = (2 , 0)$, $B = (4 , 1)$, $C = (3 , 5)$ and $D = (-4 , 8)$ « 23.5 square units »

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) ABCD is a square where A $(2 , -3)$, the equation of $\overleftrightarrow{BC}$ is $4x + 3y - 9 = 0$, then the area of the square = ... square unit.

(a) 2 (b) 4 (c) 6 (d) 8

(2) ABC is an equilateral triangle in which A $(2 , -1)$ and the equation of $\overleftrightarrow{BC}$ is $x + y = 2$, then the length of any side of Δ ABC = ... length unit.

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{6}}{2}$ (c) $\frac{\sqrt{6}}{3}$ (d) $\sqrt{2}$

(3) The equation of the straight line whose distance from the origin 4 units and makes an angle of measure 120° with positive direction of x-axis is ....

(a) $x + \sqrt{3}\, y \pm 8 = 0$ (b) $\sqrt{3}\, x + y \pm 4 = 0$

(c) $\sqrt{3}\, x + y \pm 2 = 0$ (d) $\sqrt{3}\, x + y \pm 8 = 0$

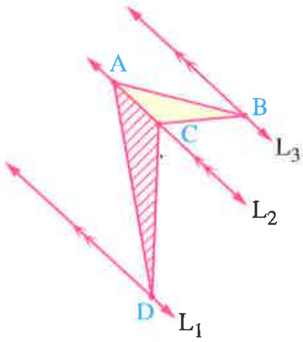
(4) The intersection point of altitudes of the triangle whose sides coincide with the straight lines : $x = 0$, $y = 0$, $x + y = 1$ is ....

(a) $(1 , 1)$ (b) $(0 , 0)$ (c) $(1 , 0)$ (d) $\left(\frac{1}{3} , \frac{1}{3}\right)$

(5) If c is the length of the perpendicular drawn from the origin to the straight line $x + by = 2c$, then b could be equals ....

(a) 1 (b) $\sqrt{3}$ (c) $\frac{1}{c}$ (d) c

(6) **In the opposite figure :**



If the equation of L_1 is $2x + 3y + k = 0$

, the equation of L_2 is $2x + 3y - 1 = 0$

, the equation of L_3 is $2x + 3y - 4 = 0$

and $\frac{\text{the area of } (\Delta \text{ ABC})}{\text{the area of } (\Delta \text{ ADC})} = \frac{1}{2}$, then k can be equal ....

(a) 3 (b) -5 (c) 5 (d) -3

2 Find point on the straight line $x + y + 9 = 0$ that far from the straight line $x + 2y + 2 = 0$ a distance $\sqrt{5}$ length unit. « $(-11, 2), (-21, 12)$ »

---

3 If A $(3, 4)$, B $(4, 6)$, C $(-1, 2)$ and D $(2, 0)$, find the length of $\overline{C`D`}$ where C` and D` are the two points of intersection of the two perpendiculars dropped from C and D to the line $\overleftrightarrow{AB}$ « $\frac{1}{\sqrt{5}}$ length unit »

---

4 Find the equation of the straight line which passes through the point $(2, -4)$ and the length of the perpendicular drawn to it from the point $(0, 0)$ equals 2 length units. Show that there are two straight lines satisfying these conditions.

« $x - 2 = 0$ or $3x + 4y + 10 = 0$ »

---

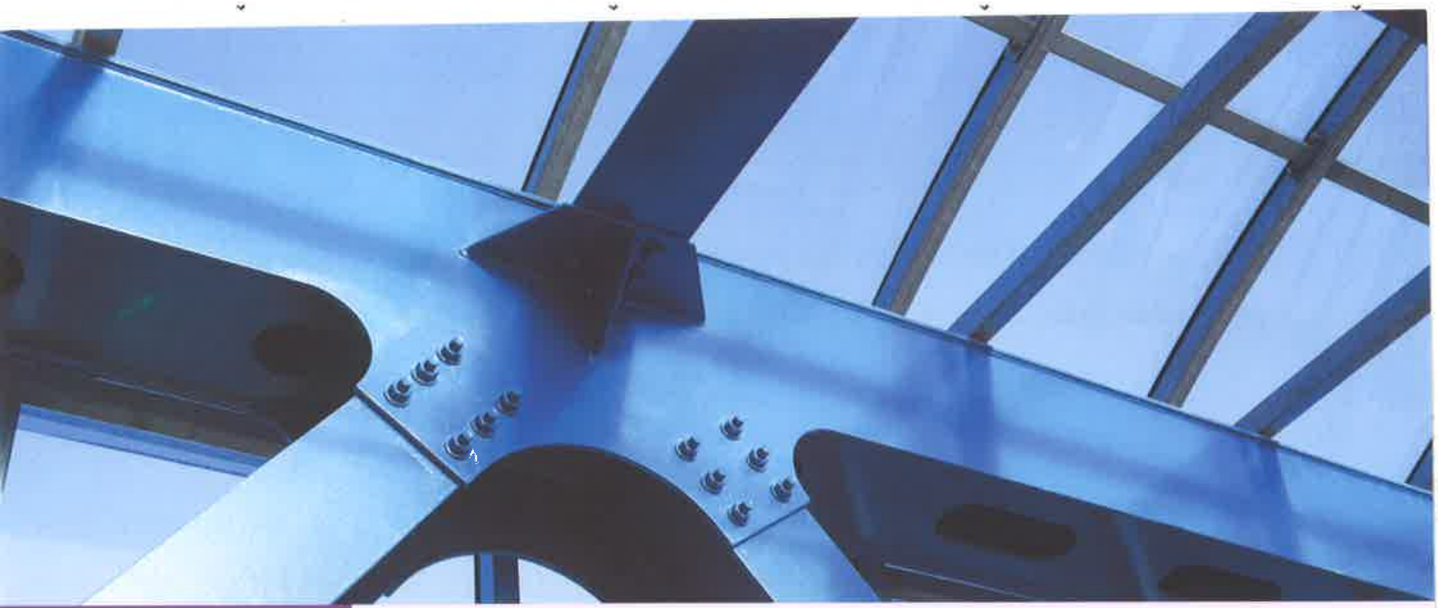
5 If $A = (3, 5)$ and $B = (11, 11)$ are two fixed points, find the point (or points) C which belongs to the x-axis where the area of Δ ABC = 30 square units.

« $\left(\frac{19}{3}, 0\right)$ or $\left(-\frac{41}{3}, 0\right)$ »

---

6 **Prove that :** The points $A = (-3, -3)$, $B = (4, 4)$, $C = (-1, 4)$ and $D = (-3, 2)$ are vertices of an isosceles trapezium, if $\overline{AC} \cap \overline{BD} = \{E\}$, then find the coordinates of the point E and the length of the perpendicular drawn from it to $\overline{BA}$

« $\left(-\frac{13}{9}, \frac{22}{9}\right)$, $\frac{35\sqrt{2}}{18}$ length units »



Exercise Nine

General equation of the straight line passing through the point of intersection of two lines

From the school book

First Multiple choice questions

- **Choose the correct answer from those given :**

(1) The point of intersection of the two straight lines : $X + 4 = 0$, $y - 3 = 0$ is ....

(a) $(4, 3)$ (b) $(3, 4)$ (c) $(-4, 3)$ (d) $(4, -3)$

(2) The equation of the straight line which passes through the origin point and the point of intersection of the two straight lines : $X = 2$, $y = 5$ is ....

(a) $5X - 2y = 0$ (b) $2X - 5y = 0$ (c) $2X + 5y = 0$ (d) $5X + 2y = 0$

(3) The equation of the straight line which is parallel to the y-axis and passes through the point of intersection of the two straight lines : $y = 4$, $X = \frac{3}{4}y$ is ....

(a) $y = 3$ (b) $X = 3$ (c) $y = 4$ (d) $X = 4$

(4) The equation of the straight line passes through the intersection point of the two straight lines $L_1 : X + 2y - 4 = 0$, $L_2 = X - 2y = 0$ and parallel to the X-axis is ....

(a) $X = 2$ (b) $y = 1$ (c) $y = 3$ (d) $y = 2$

(5) The equation of the straight line passes through the point of intersection of the two straight lines : $2X + y = 3$, $X + 4y + 2 = 0$ and parallel to the y-axis is ....

(a) $y - 2 = 0$ (b) $X - y = 2$ (c) $X - 2 = 0$ (d) $X + 2 = 0$

(6) The equation of the straight line passes through the point of intersection of the two straight lines : $X + y = 3$, $2X - y = 6$ and passes through the point $(2, -1)$ is ....

(a) $X - y + 3 = 0$ (b) $X - y - 3 = 0$

(c) $X + y + 3 = 0$ (d) $2X - y - 3 = 0$

(7) The vector equation of the straight line passes through the point $(3, 1)$ and the point of intersection of the two straight lines $3X + 2y - 7 = 0$, $X + 3y = 7$ is ....

(a) $\vec{r} = (2, -1) + k(3, 1)$ (b) $\vec{r} = (1, 3) + k(2, -1)$

(c) $\vec{r} = (3, 1) + k(2, -1)$ (d) $\vec{r} = (3, 1) + k(-1, 2)$

(8) The equation of the straight line which passes through the point $(3, 4)$ and the point of intersection between the two lines $L_1 : 3X + 2y - 7 = 0$, $L_2 : \vec{r} = (-2, 0) + k(3, 2)$ is ....

(a) $X - y + 1 = 0$ (b) $X - y + 2 = 0$

(c) $X + y - 1 = 0$ (d) $X + y + 2 = 0$

(9) The vector equation of the straight line passes through the intersection point of the two straight lines $L_1 : X - 5 = 0$, $L_2 : X + y = 13$ and its direction vector $(4, 1)$ is .... length units.

(a) $\vec{r} = (4, 1) + k(5, 8)$ (b) $\vec{r} = (5, 6) + k(4, 1)$

(c) $\vec{r} = (5, 8) + k(4, 1)$ (d) $\vec{r} = (5, -8) + k(1, 4)$

(10) The equation of the straight line passing through the point of intersection of the two straight lines : $3X - 5y - 13 = 0$, $2y - 3X + 7 = 0$ and parallel to the straight line : $\vec{r} = (-1, 1) + k(5, 4)$ is ....

(a) $4X + 5y - 14 = 0$ (b) $4X - 5y + 14 = 0$

(c) $5X - 4y - 14 = 0$ (d) $4X - 5y - 14 = 0$

(11) The equation of the straight line passing through the point of intersection of the two straight lines : $2X + 3y - 2 = 0$, $3X - y - 14 = 0$ and makes with the positive direction of the X-axis a positive angle of measure $135°$ is ....

(a) $X + y = 0$ (b) $X + y = 2$ (c) $X - y = 2$ (d) $y - X = 2$

Second Essay questions

1 Find the equation of the straight line which passes through the point of intersection of the two straight lines : $3x + 2y = 4$ and $2x + 3y = 1$ and passes through the point $(1, -1)$

« $y + 1 = 0$ »

2 Find the equation of the line passing through the point of intersection of the two lines : $2x + y = 2$, $4x + 3y - 3 = 0$ and the origin point. « $2x + 3y = 0$ »

3 Find the vector equation of the straight line which passes through the point of intersection of the two straight lines whose equations are $\vec{r} = k(-3, 2)$, $3x - 2y = 13$ and parallel to the y-axis. « $\vec{r} = (3, -2) + k(0, 1)$ »

4 Find the equation of the straight line passing through the point of intersection of the two straight lines $x - 3y + 5 = 0$ and $2x - y - 4 = 0$ and parallel to the straight line $x - 2y + 1 = 0$ « $5x - 10y + 11 = 0$ »

5 Find the equation of the line passing through the point of intersection of the two lines $5x - y = 5$ and $x + 2y = 1$ and perpendicular to the second line. « $2x - y - 2 = 0$ »

6 **Prove that :** The two straight lines : $2x - 3y + 4 = 0$, $\vec{r} = (1, 2) + k(-2, 3)$ are intersecting orthogonally, then find their point of intersection. « $(1, 2)$ »

7 **Prove that :** The two straight lines : $x - 4y + 14 = 0$ and $4x + y + 5 = 0$ are perpendicular, then find their point of intersection and the equation of the straight line passing through the point of intersection and the point $(2, 1)$ « $(-2, 3)$, $2y + x - 4 = 0$ »

8 Find the equation of the straight line passing through the point of intersection of the two straight lines : $2x + y - 1 = 0$ and $x - y + 3 = 0$ and cuts from the negative direction of the y-axis a part of length 3 length units. « $8x + y + 3 = 0$ »

9 Find the equation of the straight line passing through the point of intersection of the two straight lines $5x - y = 5$ and $x + 2y = 1$ and cuts from the positive directions of the coordinate axes two equal parts in length. « $x + y - 1 = 0$ »

10 If $L_1 : 3X + 2y - 7 = 0$, $L_2 : \vec{r} = (-2 , 0) + k(3 , 2)$

, then find :

(1) The Cartesian equation of the line L_2

(2) The measure of the angle between the two lines L_1 , L_2

(3) The point of intersection of the two lines L_1 , L_2

(4) The equation of the line passing through the point of intersection and the point $(3 , 4)$

(5) The length of the perpendicular drawn from the point of intersection of the two lines to the straight line whose equation is : $3X - 4y - 9 = 0$

(6) The area of the triangle determined by the two lines L_1 , L_2 and the X-axis.

---

11 **Life :** Two straight roads , the equation of the first is : $3X - 4y - 14 = 0$, the equation of the second is : $4X + 3y - 2 = 0$

Prove that : The two roads are perpendicular, **then find :**

(1) Their point of intersection.

(2) The equation of the line which passes through the point of intersection and the point $(3 , -2)$

(3) The shortest distance from the point of intersection of the two roads and other road whose equation is : $4X + 3 = 0$

(4) The area of the triangular region enclosed by the two roads and the y-axis.

Third Problems that measure high standard levels of thinking

1 **Choose the correct answer from those given :**

(1) The point of intersection of the two different straight lines : $\frac{X}{a} + \frac{y}{b} = 1$, $\frac{X}{b} + \frac{y}{a} = 1$ is ....

(a) (a , b)

(b) $\left(\frac{1}{2} a , \frac{1}{2} b\right)$

(c) $\left(\frac{a+b}{ab} , \frac{a+b}{ab}\right)$

(d) $\left(\frac{ab}{a+b} , \frac{ab}{a+b}\right)$

(2) For any value of k the equation $(2 + k) X + (1 + k) y = 5 + 7k$ represents ....

(a) parallel straight lines.

(b) straight lines intersect at $(-2 , 9)$

(c) straight lines intersect at $(2 , -9)$

(d) nothing of the previous.

2 Find the equation of the straight line passing through the point of intersection of the two straight lines : $x + y = 4$ and $x - y = 2$ and the length of the perpendicular drawn to it from the origin point = 1 length unit.

« $y - 1 = 0$ or $3x - 4y - 5 = 0$ »

---

3 Find the equation of the straight line passing through the point of intersection of the two straight lines : $\vec{r} = (5 , 2) + k (2 , 3)$, $\vec{r} = (11 , 4) + \acute{k} (3 , 1)$ and the length of the perpendicular drawn from the point $(-2 , 1)$ to it equals $5\sqrt{2}$ length units.

« $7x + y - 37 = 0$ »

---

4 If $A = (1 , 1)$, $B = (7 , 4)$ and $D = (2 , 7)$ are three vertices of the cyclic quadrilateral ABCD in which $m (\angle B) = 90°$, **find :**

(1) The equation of $\overleftrightarrow{BC}$

(2) The equation of $\overleftrightarrow{CD}$

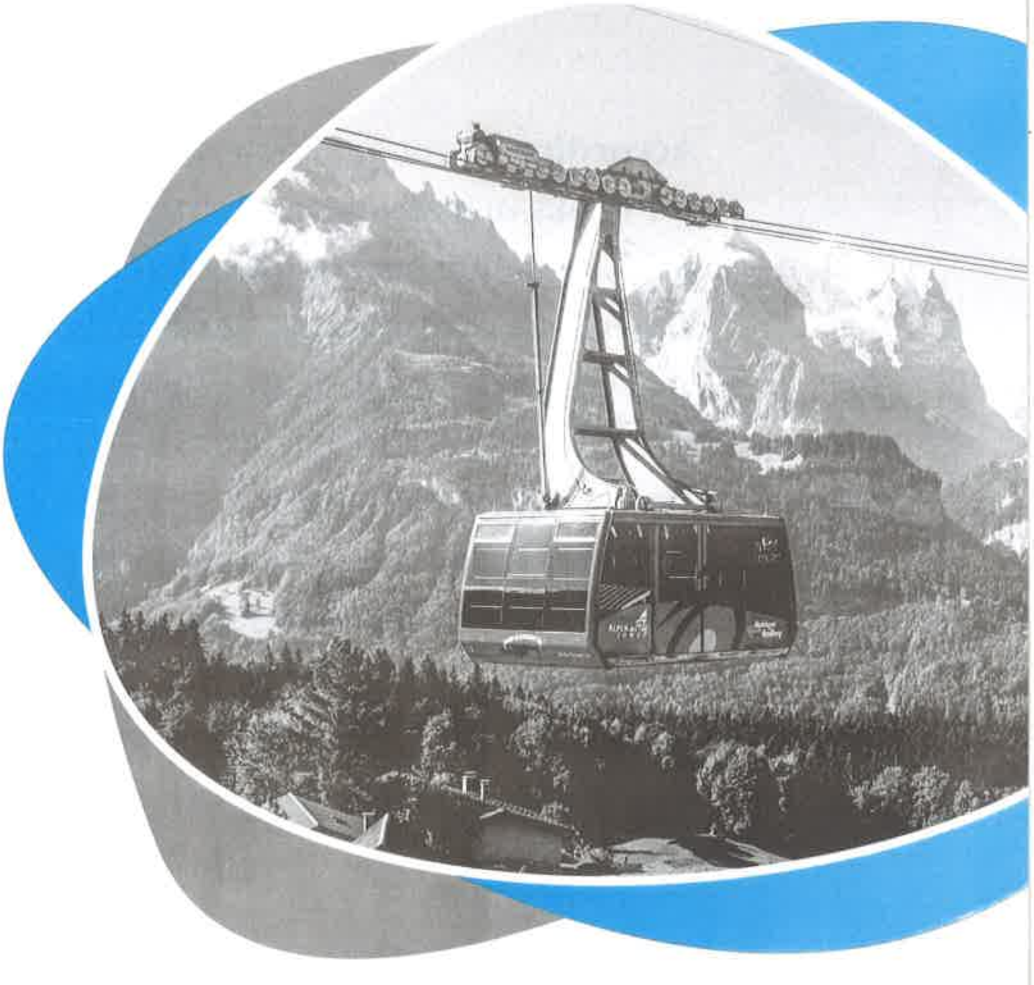
(3) The coordinates of the point C

« $y + 2x - 18 = 0$, $6y + x - 44 = 0$, $\left(\frac{64}{11} , \frac{70}{11}\right)$ »

Mathematics

By a group of supervisors

Final Revision and Examinations



SECOND TERM

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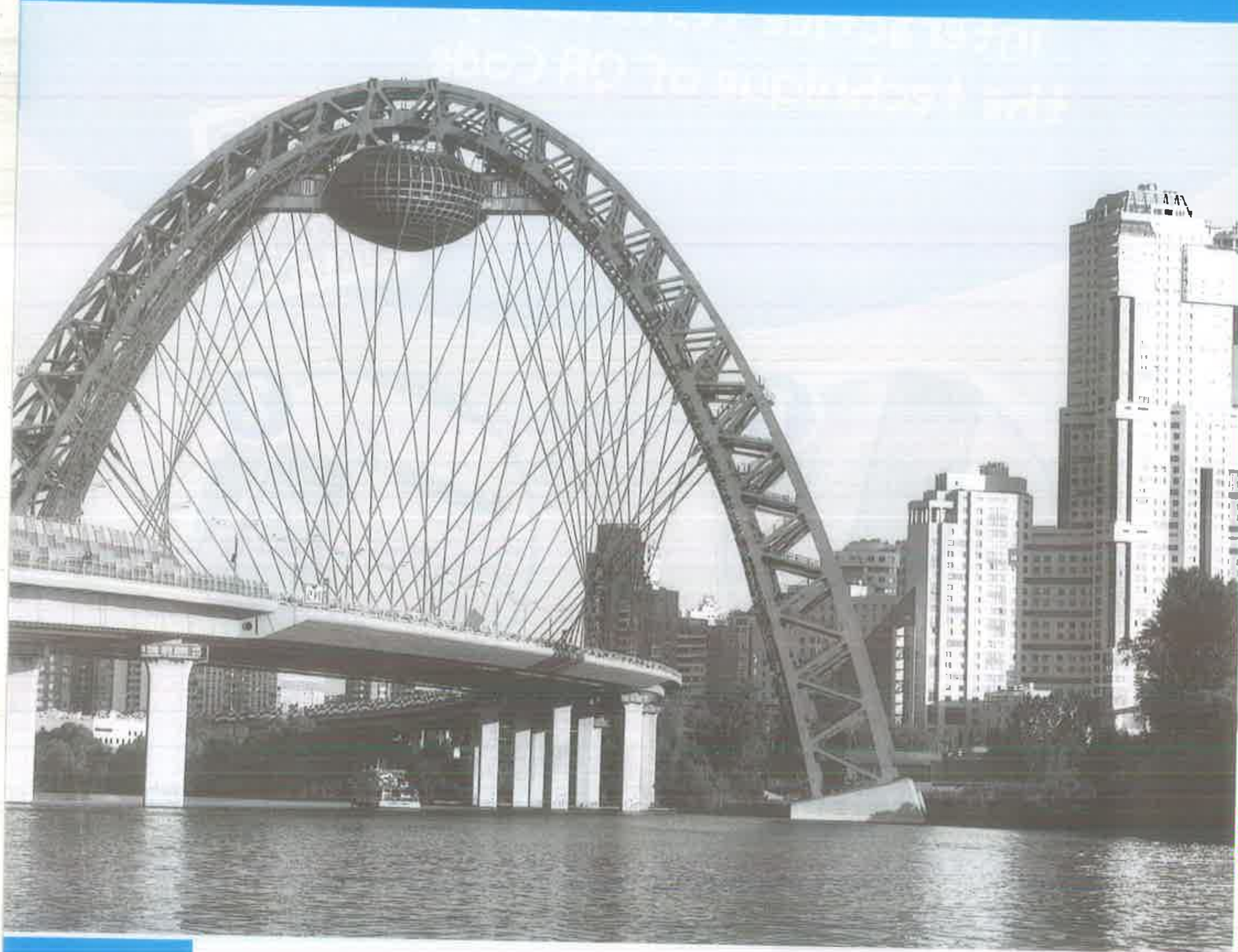
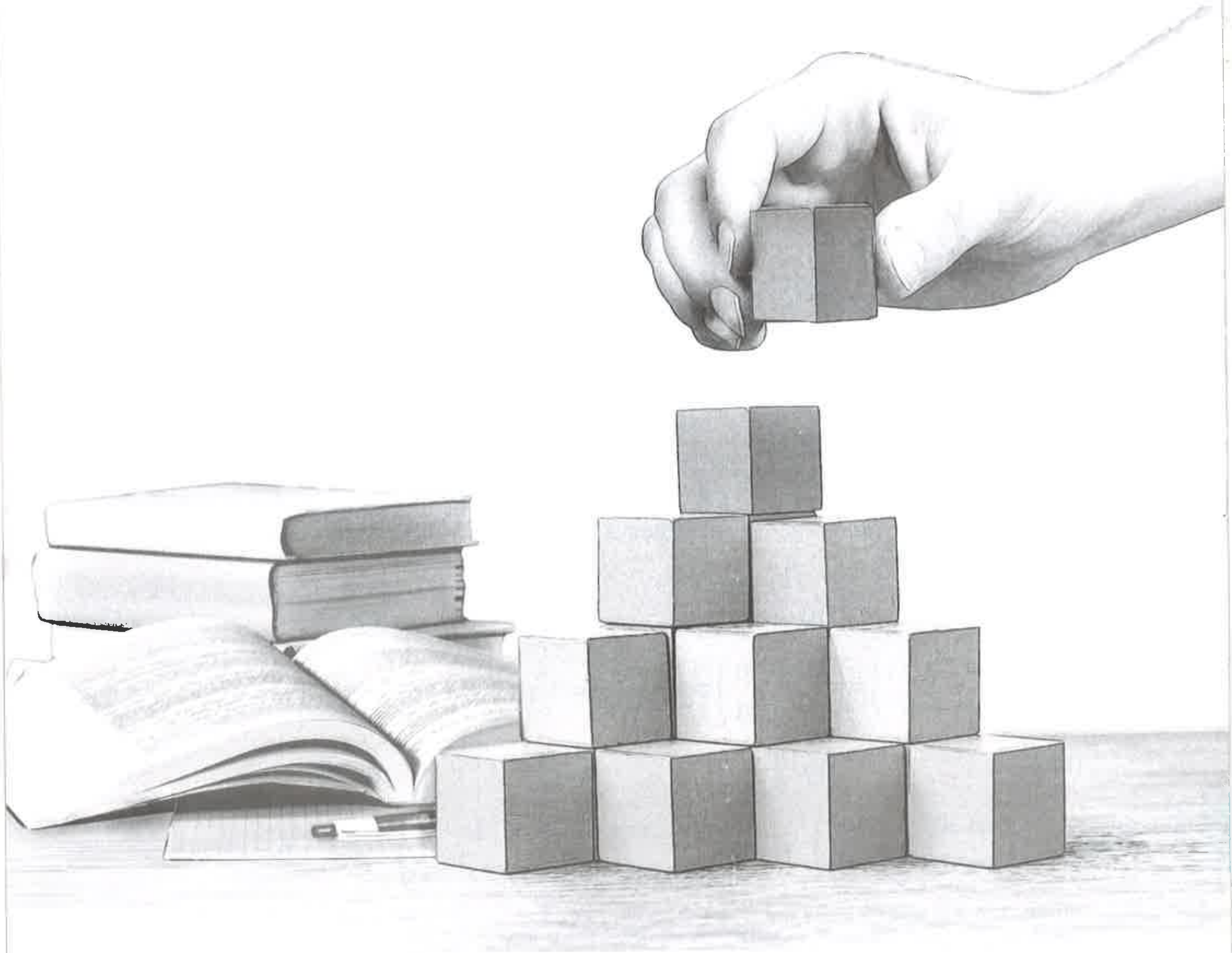


- Multiple choice accumulative quizzes.
- Final revision.
- School book examinations.
- Final examination models.
- Answers.

MULTIPLE CHOICE ACCUMULATIVE QUIZZES



- **First :** Accumulative quizzes on algebra
- **Second :** Accumulative quizzes on trigonometry
- **Third :** Accumulative quizzes on analytic geometry

First : Accumulative quizzes on algebra

Quiz 1 on lesson 1 – unit 1

Total mark 12

Choose the correct answer from those given :

(1) If $A = \begin{pmatrix} 1 & 1 & x-1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$ is a symmetric matrix , then $x =$ ....

(a) -1 (b) zero (c) 4 (d) 6

(2) If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ \frac{1}{2}k & 4 \end{pmatrix}$ where $A = B^t$, then $k =$ ....

(a) -2 (b) $-\frac{3}{2}$ (c) 8 (d) -6

(3) If $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \\ -1 & 7 \end{pmatrix}$, then $a_{12} + a_{32} =$ ....

(a) 8 (b) 12 (c) zero (d) 10

(4) If $\begin{pmatrix} 1 & x & 2 \\ -1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 6 \\ 2 & y \end{pmatrix}^t$, then $x\,y =$ ....

(a) -15 (b) -2 (c) 2 (d) 15

(5) If $\begin{pmatrix} 1+i & 1-i \\ \sin x & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a\,b\,c\,d = 1$, $0 < x < \frac{\pi}{2}$, then $x =$ ....

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

(6) If $A = \begin{pmatrix} \sin 30^\circ & 1 \\ \tan^2 60^\circ & \cot 45^\circ \end{pmatrix}$, $B = \begin{pmatrix} m - \cos 60^\circ & 1 \\ s + \sec^2 45^\circ & n + \cos^2 90^\circ \end{pmatrix}$ and $A = B$

, then $m + s - n =$ ....

(a) 2 (b) -1 (c) -2 (d) 1

(7) If $\begin{pmatrix} 1+i & 1-i \\ i & -i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the equation whose roots a^2 , $2\,d$ is ....

(a) $x^2 - 4 = 0$ (b) $x^2 + 4 = 0$

(c) $x^2 - 2 = 0$ (d) $x^2 + 2 = 0$

(8) If $X = \begin{pmatrix} a-b & 1 \\ 3 & b-2 \end{pmatrix}$, $Y = \begin{pmatrix} a-6 & 6 \\ 2 & 4-a \end{pmatrix}$ and $2X = Y^t$, then $a + 2b =$ ....

(a) 4 (b) 8 (c) 10 (d) 12

(9) The matrix $\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ of order ....

(a) 2×1 (b) 1×3 (c) 3×2 (d) 3×1

(10) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3I$ where I is the unit matrix , then $a + b + c + d =$ ....

(a) 5 (b) 6 (c) 7 (d) 8

(11) If the matrix A of order 2×2 and $a_{xy} = \frac{x}{y}$, then $a_{11} \times a_{12} \times a_{21} \times a_{22} =$ ....

(a) 4 (b) 2 (c) 1 (d) $\frac{1}{2}$

(12) If $A = \begin{pmatrix} x & 2 & 3 \\ z & y & n \\ -3 & 5 & m \end{pmatrix}$, which of the following sufficient to find the value of x ?

(a) $A = A^t$ (b) $A = -A^t$

(c) A is a diagonal matrix (d) nothing of the previous.

Total mark

....

12

Quiz 2 till lesson 2 – unit 1

Choose the correct answer from those given :

(1) If the matrix A is of order 2×3 , then the number of its elements equals ....

(a) 3 (b) 2 (c) 5 (d) 6

(2) If $A + A^t = O$, then A is a .... matrix.

(a) row (b) column (c) symmetric (d) skew

(3) If $\begin{pmatrix} x \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, then $x + y =$ ....

(a) 1 (b) 4 (c) 5 (d) 3

(4) $\begin{pmatrix} -5 & -2 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 6 & -4 \\ 2 & -6 \end{pmatrix}^t = \dots\dots\dots$

(a) O (b) $\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$ (d) I

(5) For any square matrix $B \neq O$, the matrix $A = B - B^t$ is ....

(a) symmetric. (b) skew symmetric.

(c) zero (d) unit matrix.

(6) If the matrix A is symmetric and skew symmetric in the same time , then ....

(a) $A = O$ (b) $A = I$

(c) A is a diagonal matrix. (d) A is a row matrix.

(7) If A is a matrix of order 2×2 where $a_{yz} = y - 2Z$, B is a matrix of order 2×2 where $b_{yz} = z - y$, then $A + B = \dots\dots\dots$

(a) $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

(8) If $\begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$ is a skew symmetric , then $\frac{a+b+c+f}{d+x+y+z} = \dots\dots\dots$

(a) 1 (b) zero (c) -1 (d) e

(9) If A is a symmetric , which of the following could be a rule to find the elements of the matrix A ?

(a) $a_{yz} = 2y - z$ (b) $a_{yz} = y + z$ (c) $a_{yz} = y^z$ (d) $a_{yz} = 3y + 2z$

(10) If $\begin{pmatrix} 4 & 2x-1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 5 & 3 \end{pmatrix}^t$, then $x = \dots\dots\dots$

(a) 1 (b) 2 (c) 3 (d) 5

(11) If $X + \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}^t = O$, then $X = \dots\dots\dots$

(a) $\begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$ (d) I

(12) $(X^t)^t - X = \dots\dots\dots$

(a) O (b) X (c) 2 X (d) zero

Total mark 12

Quiz 3 till lesson 3 – unit 1

Choose the correct answer from those given :

(1) If A , B are two matrices where $AB = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, then $B^t A^t =$ ....

(a) $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 3 \\ -1 & 2 \end{pmatrix}$

(2) If A is a matrix of order 2×3 , B^t is a matrix of order 1×3 , then the matrix AB is of order ....

(a) 3×1 (b) 2×1 (c) 1×2 (d) 3×3

(3) If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then $x =$ ....

(a) 3 (b) -3 (c) zero (d) 4

(4) If $\begin{pmatrix} 2 & x \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} y & -4 \\ 0 & -1 \end{pmatrix}$, then $x + y =$ ....

(a) 2 (b) -2 (c) 6 (d) -6

(5) If $A = \begin{pmatrix} 4 & 0 \\ 3 & -4 \end{pmatrix}$, then $A^{60} =$ ....

(a) $2^{30}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $2^{60}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $2^{90}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $2^{120}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(6) If $\begin{pmatrix} 3^x & 2x + y \\ x + y & 3^y \end{pmatrix} = \begin{pmatrix} 25 & b \\ a & 5 \end{pmatrix}$, then $\frac{b}{a} =$ ....

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) 15 (d) 5

(7) If A is a matrix of order 2×2 and $A + A^t = I$, then the sum of the elements of A equals ....

(a) 4 (b) 2 (c) 1 (d) zero

(8) $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \dots\dots$

(a) $\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(9) If A , B are symmetric matrices , then the matrix $(A\ B\ A)$ is ……………

(a) symmetric. (b) skew symmetric.

(c) diagonal. (d) zero matrix.

(10) If O is a zero matrix of order 2×2 , then the number of its elements is = ……………

(a) zero (b) $\varnothing$ (c) 2 (d) 4

(11) If $A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \end{pmatrix}$, then BA = ……………

(a) (-4) (b) (4) (c) $\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -4 \\ 3 & -6 \end{pmatrix}$

(12) If $A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$, then $A^2 = \dots\dots$

(a) $\begin{pmatrix} 4 & 1 \\ 9 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 9 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -3 \\ 9 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -3 \\ 9 & -2 \end{pmatrix}$

Total mark

12

Quiz 4 till lesson 4 – unit 1

Choose the correct answer from those given :

(1) The value of the determinant $\begin{vmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 4 & 2 & 5 \end{vmatrix}$ equals ……………

(a) 10 (b) 30 (c) 15 (d) 5

(2) If $A = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \end{pmatrix}$, then $(BA)^t = \dots\dots$

(a) $\begin{pmatrix} 4 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 14 \\ -14 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ -10 \end{pmatrix}$ (d) (-6)

(3) If A (3 , 5) , B (2 , 0) and C (– 3 , 3)

, then the area of Δ ABC equals .... square units.

(a) 28 (b) 14 (c) 7 (d) 2

(4) The solution set of the equation : $\begin{vmatrix} x^2 & -2 \\ 2 & 1 \end{vmatrix}$ = zero in $\mathbb{C}$ is ....

(a) $\varnothing$ (b) $\{-2 , 2\}$ (c) $\{-2i , 2i\}$ (d) $\{-i , i\}$

(5) If $\begin{vmatrix} x & y \\ z & \ell \end{vmatrix} = 4$, then $\begin{vmatrix} x-y & 4y \\ z-\ell & 4\ell \end{vmatrix} =$ ....

(a) 1 (b) 10 (c) 12 (d) 16

(6) If $A = \begin{pmatrix} x & f \\ e & y \end{pmatrix}$ is a skew symmetrix matrix , then 2 f + 3 e = ...

(a) f (b) $x + y$ (c) e + y (d) x

(7) If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A^{25} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then a + b + c + d = ...

(a) 25 (b) 26 (c) 27 (d) 28

(8) If $\begin{vmatrix} 3+\sin\theta & 1-\cos\theta \\ 4 & \sin\theta \end{vmatrix}$ = zero , then the value of θ = ...

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

(9) If $A^t B^t = \begin{pmatrix} 3 & -6 \\ 2 & 5 \end{pmatrix}$, then B × A = ...

(a) $\begin{pmatrix} -1 & -6 \\ 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 \\ -6 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -6 \\ 2 & 5 \end{pmatrix}$

(10) If $A = \begin{pmatrix} 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} x & 0 \\ 1 & y \end{pmatrix}$, $A \times B = \begin{pmatrix} 9 & 6 \end{pmatrix}$, then $x + y$ = ...

(a) 3 (b) 5 (c) 7 (d) 9

(11) If $\begin{vmatrix} x & -1 \\ 2 & x \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 2$, then x = ...

(a) 3 or – 2 (b) – 3 or 2 (c) 3 or 2 (d) – 3 or – 2

(12) If $\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \times \begin{pmatrix} x & 2y \\ 0 & z \end{pmatrix} + I = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}^t$, then $x \times y \times z$ = ...

(a) 1 (b) – 1 (c) 2 (d) 4

Total mark 12

Quiz 5 till lesson 5 – unit 1

Choose the correct answer from those given :

(1) If $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$, then $A^2 =$ ....

(a) $\begin{pmatrix} 1 & 4 \\ 9 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -4 \\ 6 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -6 \\ 9 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} -5 & -6 \\ 9 & -2 \end{pmatrix}$

(2) If $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, then $A^{-1} =$ ....

(a) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 1 \\ 5 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

(3) The values of x which make the matrix $\begin{pmatrix} x & 4 \\ 2 & x-2 \end{pmatrix}$ has no multiplicative inverse are ....

(a) 4 , – 2 (b) – 4 , 2 (c) – 4 , – 2 (d) 4 , 2

(4) The area of the triangle whose vertices are A (4 , 5) , B (6 , – 1) and C (1 , 6) equals .... square units.

(a) 16 (b) 8 (c) 32 (d) 24

(5) If $A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}$ and $A = A^{-1} \times B$, then B = ...

(a) $\begin{pmatrix} -7 & 4 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -7 & 8 \\ -4 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & -8 \\ 4 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & -8 \\ 4 & 4 \end{pmatrix}$

(6) If $\begin{vmatrix} 2k-1 & 2 \\ 3 & k+1 \end{vmatrix} = 2k^2 + 1$, then the value of k = ...

(a) 2 (b) 4 (c) 6 (d) 8

(7) If A is a square matrix of order 2 × 2 and | 2 A | = 8 , then | 3 A | = ...

(a) 9 (b) 12 (c) 18 (d) 24

(8) If A is a square matrix , then the matrix $(A + A^t)$ is ....

(a) symmetric. (b) skew symmetric.

(c) zero (d) diagonal.

(9) If $A = \begin{pmatrix} -1 & -2 \\ 1 & k \end{pmatrix}$ and $A^3 = -A$, then k = ... where $k \in \mathbb{Z}$

(a) 1 (b) – 1 (c) zero (d) 3

(10) If $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$, then a + b = ...

(a) 3 (b) 4 (c) 5 (d) 6

(11) If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 6$ and $\begin{vmatrix} ak & kc \\ b & d \end{vmatrix} = -24$, then k = ...

(a) 4 (b) 3 (c) – 3 (d) – 4

(12) If $\begin{vmatrix} a & 5 \\ 5 & b \end{vmatrix} = 0$, $\begin{vmatrix} b & 2 \\ 2 & c \end{vmatrix} = 0$, $\begin{vmatrix} a & 1 \\ 1 & c \end{vmatrix} = 0$, then $\begin{vmatrix} a & 0 & 0 \\ -1 & b & 0 \\ 2 & 5 & c \end{vmatrix} =$ ....

(a) ± 10 (b) ± 50 (c) ± 100 (d) ± 20

Quiz 6 till lesson 1 – unit 2

Total mark 12

Choose the correct answer from those given :

(1) The point that belongs to the S.S. of the inequality : $y < 2x + 3$ is ....

(a) (– 1 , 1) (b) (– 1 , – 1) (c) (0 , 3) (d) (– 3 , – 3)

(2) If $A = \begin{pmatrix} 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, then the only possible operation of the following is ....

(a) A + B (b) $A^t + B^t$ (c) A B (d) $A B^t$

(3) If $\begin{vmatrix} x & 1 & 3 \\ 0 & 2x & 5 \\ 0 & 0 & x \end{vmatrix} = 8x$, then the S.S. is ....

(a) $\{2 , -2\}$ (b) $\{0 , -2 , 2\}$ (c) $\left\{\frac{1}{2} , \frac{-1}{2}\right\}$ (d) $\left\{0 , \frac{-1}{2} , \frac{1}{2}\right\}$

(4) If $A = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 6 & -9 \\ \frac{-3}{2} & 3 \end{pmatrix}$, then AB = ...

(a) I (b) 2 I (c) $\frac{1}{2}$ I (d) 3 I

(5) When solving the two equations : $a X + b y = 4$, $c X + d y = 2$, the multiplicative inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ found to be equals $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$, then $X + y =$ ....

(a) 2 (b) 3 (c) – 4 (d) 2

(6) If $A = \begin{pmatrix} X & 2 \\ y & -2 \end{pmatrix}$, $A \times A^{-1} = A^2$, then $X \times y =$ ....

(a) – 3 (b) – 2 (c) 2 (d) 3

(7) If L , M are the roots of the equation : $X^2 - 4X - 10 = 0$, then the value of the determinant $\begin{vmatrix} 2L & -1 \\ 3 & M \end{vmatrix}$ equals ....

(a) – 17 (b) – 12 (c) – 8 (d) – 6

(8) If $X + \begin{pmatrix} 2 & 5 \\ 0 & 7 \end{pmatrix} = 0$, then X = ...

(a) $\begin{pmatrix} 2 & -5 \\ 0 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & -5 \\ 0 & -7 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 5 \\ 0 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 0 \\ 5 & 7 \end{pmatrix}$

(9) The point which belongs to the solution set of the inequalities $X \geq 2$, $y < 2$, $X + y > 3$ is ....

(a) (3 , 1) (b) (3 , 2) (c) (2 , 2) (d) (2 , 1)

(10) The area of the triangle with the vertices (1 , 6) , (0 , 10) , (0 , 0) equals .... square unit.

(a) 5 (b) 10 (c) 15 (d) 20

(11) If $\begin{vmatrix} \sin\theta & 0 & 0 \\ 2 & \csc\theta & 0 \\ 1 & 3 & X \end{vmatrix} + \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0$, then $X =$ ....

(a) 1 (b) – 1 (c) zero (d) 2 sin θ

(12) The solution set of the inequality : $X + 5 \leq 3X + 1 < 2X + 2$ in $\mathbb{R}$ is ....

(a) $\mathbb{R} - [1 , 2[$ (b) $]1 , 2]$ (c) $\varnothing$ (d) $\{1 , 2\}$

Total mark

12

Quiz 7 till lesson 2 – unit 2

Choose the correct answer from those given :

(1) The point that belongs to the S.S. of the inequalities : $x > 2$, $y > 1$, $x + y \geq 3$ is ....

(a) (3 , 1) (b) (1 , 2) (c) (3 , 2) (d) (1 , 3)

(2) If $\begin{vmatrix} 2x & 2 \\ 4 & 3 \end{vmatrix} = 10$, then $x =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(3) If A is a matrix of order 1×3 , B^t is a matrix of order 1×3 , then we can perform the operation ....

(a) A + B (b) $B^t + A^t$ (c) $A\,B^t$ (d) AB

(4) If the matrix $\begin{pmatrix} x & 1 \\ 6 & 3 \end{pmatrix}$ has no multiplicative inverse , then $x =$ ....

(a) – 2 (b) zero (c) 2 (d) 3

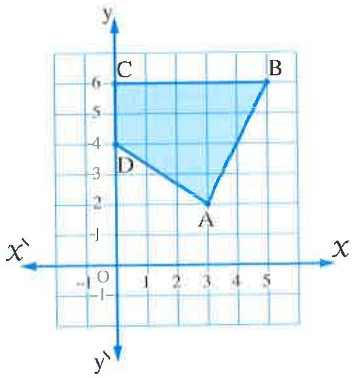
(5) If the area of the triangle with the vertices (k , 0) , (4 , 0) , (0 , 2) is 4 square units , then k = ...

(a) zero or – 8 (b) – 4 or 4 (c) zero or 8 (d) 8 or – 8

(6) If A is a square matrix and $A^2 = I$, then $A^{-1} =$ ....

(a) 0 (b) A (c) 2 A (d) A + I

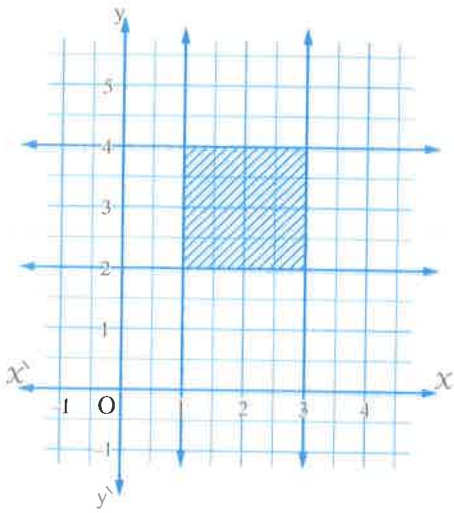
(7) The given figure represents the region of solution of system of inequalities , then the minimum value of the objective function $P = 3x + 2y$ is ....



(a) 6 (b) 8

(c) 12 (d) 13

(8) The shaded region in the given figure represents the solution set of the inequalities ....



(a) $X > 1$, $y > 2$

(b) $1 < X < 3$, $2 < y < 4$

(c) $1 \le X \le 3$, $2 \le y \le 4$

(d) $X + y \ge 3$, $X - y \le 7$

(9) If the matrix $A = \begin{pmatrix} 0 & X - 1 \\ X^2 - 1 & 0 \end{pmatrix}$ is skew symmetric , then $X \in$ ....

(a) $\{-1 , 2\}$ (b) $\{0 , -1\}$ (c) $\{1 , -2\}$ (d) $\{0 , 1\}$

(10) If $\begin{vmatrix} X & 12 \\ 3 & X \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 10 & 5 \end{vmatrix}$, then $X =$ ....

(a) 2 (b) 5 (c) 6 (d) ± 6

(11) If twice the number X is not less than 3 times the number y , then ....

(a) $2X < 3y$ (b) $2X \le 3y$ (c) $2X > 3y$ (d) $2X \ge 3y$

(12) The point that belongs to the region of solution of the inequalities : $X \ge 1$, $X + y \ge 5$, $y \ge 2$ and makes the objective function $p = 2X + y$ is as small as possible is ....

(a) (0 , 5) (b) (4 , 3) (c) (1 , 4) (d) (3 , 2)

Second : Accumulative quizzes on trigonometry

Total mark 12

Quiz 1 on lesson 1 – unit 3

Choose the correct answer from those given :

(1) $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta =$ .... (in its simplest form)

(a) $2\sin\theta\cos\theta$ (b) 1 (c) 2 (d) $\sin^2\theta - \cos^2\theta$

(2) $\frac{\tan\theta \cot\theta}{\sec\theta} =$ ....

(a) $\sin\theta$ (b) $\cos\theta$ (c) $\sec\theta$ (d) $\csc\theta$

(3) $\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos(2\pi - \theta)} =$ ....

(a) $\tan\theta$ (b) $\cot\theta$ (c) 1 (d) -1

(4) $\sin(90° - \theta)\csc(90° - \theta) =$ .... (in its simplest form)

(a) 1 (b) $\sin^2\theta$ (c) $\cos^2\theta$ (d) $\sin\theta\cos\theta$

(5) If $X + y = 30°$, then $\tan(X + 2y)\tan(2X + y) =$ ....

(a) 1 (b) zero (c) -1 (d) $2\sqrt{3}$

(6) If $\sin\theta - \cos\theta = \frac{4}{5}$ where $\theta \in]0, \frac{\pi}{2}[$, then $\sin\theta\cos\theta =$ ....

(a) $\frac{1}{5}$ (b) $\frac{9}{25}$ (c) $\frac{41}{50}$ (d) $\frac{9}{50}$

(7) If $\sin\theta + \csc\theta = 5$, then $\sin^2\theta + \csc^2\theta =$ ....

(a) 1 (b) 5 (c) 23 (d) 25

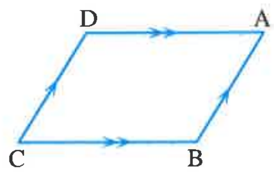
(8) If $\tan\theta = 3$, then $\sec^2\theta =$ ....

(a) 9 (b) 10 (c) -10 (d) 0.9

(9) $3\tan\theta\cot\theta + 2\sin\theta\csc\theta + \cos\theta\sec\theta =$ ....

(a) 1 (b) 3 (c) 5 (d) 6

(10) In the given figure :



ABCD is a parallelogram , then

$\cos A + \cos B + \cos C + \cos D =$ ....

(a) -1 (b) zero (c) 1 (d) 4

(11) If $3 \sin \theta + 4 \cos \theta = 5$, then $3 \cos \theta - 4 \sin \theta =$ ....

(a) 5 (b) 4 (c) 3 (d) zero

(12) The numerical value of the expression : $5 \cos \theta \times 3 \sec \theta =$ ....

(a) 1 (b) 21 (c) 8 (d) 15

Total mark 12

Quiz 2 till lesson 2 – unit 3

Choose the correct answer from those given :

(1) The general solution of the equation :

$\cos \theta = 1$ is .... where $n \in \mathbb{Z}$

(a) $n \pi$ (b) $2 n \pi$ (c) $\frac{\pi}{2} + n \pi$ (d) $\frac{\pi}{2} + 2 n \pi$

(2) $\frac{1 - \cos^2 \theta}{1 - \sin^2 \theta} + 1 =$ ....

(a) 2 (b) $2 \cos^2 \theta$ (c) $\sec^2 \theta$ (d) $\csc^2 \theta$

(3) $\sin \theta \cos \theta \cot \theta + \sin^2 \theta =$ .... (in its simplest form)

(a) -1 (b) zero (c) 1 (d) 2

(4) If $\cos \theta = \frac{1}{2}$, $\theta \in \left] \frac{3\pi}{2} , 2\pi \right[$, then $\theta =$ ....

(a) $\frac{5}{3} \pi$ (b) $\frac{1}{3} \pi$ (c) $\frac{2}{3} \pi$ (d) $\frac{11}{6} \pi$

(5) If $x , y \in [0 , 2\pi[$, $\theta = x + y$, then the set of values of θ satisfy $\sin x \sin y = 1$ equals ....

(a) $\{\pi , 2\pi\}$ (b) $\{\pi , 3\pi\}$ (c) $\left\{\frac{\pi}{2} , \frac{3\pi}{2}\right\}$ (d) $\left\{\frac{\pi}{2} , \frac{\pi}{3}\right\}$

(6) If $x \in [0 , \pi[$, then the solution set of the equation $\cos x = \frac{1}{2}$ is the same solution set of the equation ....

(a) $\tan x = 2 \sin x$ (b) $2 \cos^2 x = \cos x$

(c) $2 \cos^2 x + 3 \cos x = 2$ (d) $\cos \frac{1}{2} x = 1$

(7) $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta =$ ....

(a) 1 (b) $\sec^2 \theta$ (c) $\cot^2 \theta$ (d) $2 \tan^2 \theta$

(8) If $k = 4 \sin 3X - 5$, then $k \in$ ....

(a) $[-4 , 4]$ (b) $[8 , 11]$ (c) $[5 , 7]$ (d) $[-9 , -1]$

(9) If $\tan \theta = 1$, then one of the values of θ is ....

(a) 30° (b) 60° (c) 135° (d) 225°

(10) If $3^{\sin \theta} = 1$, where $\theta \in]0 , 2\pi[$, then $\theta =$ ....

(a) 45° (b) 90° (c) 180° (d) 270°

(11) If $\tan \theta = 4$, then $\dfrac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} =$ ....

(a) $\frac{17}{15}$ (b) 1 (c) $\frac{-7}{15}$ (d) -1

(12) The solution set of the equation : $\sin X + \cos X = 0$ where $180^\circ < X < 360^\circ$

equals ....

(a) $\{210^\circ\}$ (b) $\{225^\circ\}$ (c) $\{240^\circ\}$ (d) $\{315^\circ\}$

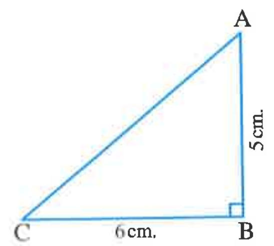
Total mark ....

Quiz 3 till lesson 3 – unit 3

12

Choose the correct answer from those given :

(1) In the opposite figure :

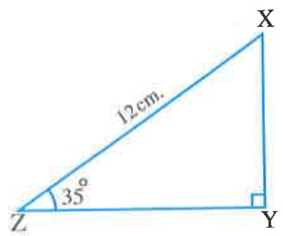


m (∠ C) = ...°

(a) $56^\circ\ 27'$ (b) $39^\circ\ 48'$

(c) $33^\circ\ 33'$ (d) $50^\circ\ 12'$

(2) In the opposite figure :



XY ≈ .... cm. (to the nearest cm.)

(a) 9.8 (b) 6.9

(c) 8.4 (d) 14.6

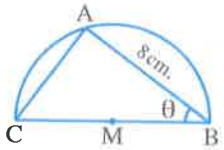
(3) If $\theta \in [0 , \pi]$, $\cos \theta + 1 = 0$, then $\theta =$ ....°

(a) 0 (b) 90 (c) 180 (d) 270

(4) If $\cot\theta = \frac{1}{2}$, then $\csc^2\theta =$ ....

(a) $\frac{5}{4}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

(5) **In the opposite figure :**



$\overline{BC}$ is a diameter in circle M , AB = 8 cm.

, m ($\angle$ ABC) = θ° , then the area of Δ ABC = ... cm^2.

(a) $8\cos\theta$ (b) $8\cot\theta$ (c) $32\tan\theta$ (d) $32\sin\theta$

(6) ABC is a right angled-triangle at B , AB = 3 cm. , its perimeter = 12 cm.

, then m ($\angle$ C) $\simeq$ ....

(a) 37° (b) 14° (c) 18° (d) 53°

(7) The general solution of the equation $\cos\theta = -1$ is .... where $n \in \mathbb{Z}$

(a) $\frac{\pi}{2} + \pi n$ (b) $\frac{\pi}{3} + \pi n$ (c) $\pi + 2\pi n$ (d) $\frac{\pi}{6} + 2\pi n$

(8) If $\csc\theta - \frac{1}{\tan\theta} = \frac{1}{5}$, then $\cot\theta + \frac{1}{\sin\theta} =$ ....

(a) $\frac{1}{10}$ (b) 5 (c) $\frac{1}{25}$ (d) 1

(9) If $2\sin x - 1 = 0$, where x is the greatest positive angle , $x \in [0, 360°]$

, then $x =$ ....

(a) 150° (b) 315° (c) 330° (d) 30°

(10) $\sin^2\left(\frac{\pi}{3} - \theta\right) + \cos^2\left(\theta - \frac{\pi}{3}\right) - 1 =$ ....

(a) zero (b) 1 (c) $\sin^2\theta$ (d) $\cos^2\theta$

(11) If $\sin\theta = \frac{a}{b}$, $\theta \in \left]0, \frac{\pi}{2}\right[$, then $\sqrt{1 + \tan^2\theta} =$ ....

(a) $\frac{a}{\sqrt{a^2 - b^2}}$ (b) $\frac{a}{\sqrt{a-b}}$ (c) $\frac{a}{\sqrt{1 + a^2}}$ (d) $\frac{b}{\sqrt{b^2 - a^2}}$

(12) The general solution of the equation :

$3\cot\left(\frac{\pi}{2} - \theta\right) = \sqrt{3}$ is .... (where $n \in \mathbb{Z}$)

(a) $\frac{\pi}{6} + 2\pi n$ (b) $\frac{\pi}{6} + \pi n$ (c) $\frac{7\pi}{6} + 2\pi n$ (d) $\frac{\pi}{3} + \pi n$

Total mark 12

Quiz 4 till lesson 4 – unit 3

Choose the correct answer from those given :

(1) If $0° \leq \theta < 180°$, $\sqrt{3} \tan \theta - 1 = 0$, then $\theta =$ ....

(a) 30° (b) 60° (c) 120° (d) 150°

(2) It is possible to solve the right-angled triangle except the case in which the givens are ....

(a) lengths of two sides of the triangle.

(b) lengths of two sides and measure of the included angle between them.

(c) measures of two angles of the triangle.

(d) length of one side and length of the hypotenuse.

(3) The general solution of the equation : $\tan \theta = \sqrt{3}$ is ....

(a) $\frac{\pi}{3} + n\pi$ (b) $\frac{\pi}{3} + 2n\pi$ (c) $\frac{4\pi}{3} + 2n\pi$ (d) $\frac{\pi}{6} + n\pi$

(4) $\sin (90° - \theta) \csc (180° - \theta) =$ .... (in the simplest form)

(a) -1 (b) 1 (c) $\tan \theta$ (d) $\cot \theta$

(5) If $\sin A + \sin B = 2$, then ....

(a) $\cos A + \cos B = 0$ (b) $\cos A - \cos B = 1$

(c) $\sin A - \sin B = 1$ (d) $\sin (A + B) = -1$

(6) If ΔABC is a right angled-triangle and the lengths of its sides are : a , $a + 1$, $a - 1$ where $a > 1$, then the measure of its greatest angle is approximately ....

(a) $36° 52'$ (b) $48° 18'$ (c) $53° 8'$ (d) $62° 42'$

(7) From a point on the ground surface 40 m. away from a tower base , the measure of elevation angle of the top of the tower is 72° , then the height of the tower to the nearest metre is .... m.

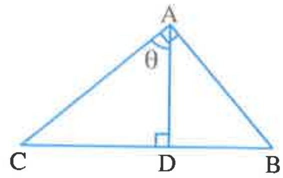
(a) 120 (b) 121 (c) 122 (d) 123

(8) $(\sin^2 40° + \cos^2 40°)^5 =$ ....

(a) 1 (b) -1 (c) 5 (d) -5

(9) In the opposite figure :

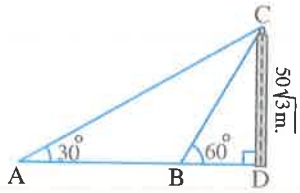
ABC is a right-angled triangle at A , $\overline{AD} \perp \overline{BC}$, AD = $\sin^2 \theta$, then BC = ...



(a) $\sin \theta$ (b) $\tan \theta$ (c) $\cos \theta$ (d) $\tan^2 \theta$

(10) In the opposite figure :

The angle of elevation of the top of a tower of length $50\sqrt{3}$ m. is measured from two points A and B on the same horizontal line as the tower base , their measures are 30° , 60° respectively , then the distance between the two points equals .... m.



(a) $100\sqrt{3}$ (b) $50\sqrt{3}$ (c) 100 (d) 50

(11) The expression : $\frac{1-\cos^2 \theta}{\sin^2 \theta - 1}$ in the simplest form is ....

(a) $-\tan^2 \theta$ (b) $-\cot^2 \theta$ (c) $\tan^2 \theta$ (d) $\cot^2 \theta$

(12) If $\csc \theta - \cot \theta = \frac{1}{5}$, then $\csc \theta + \cot \theta =$ ....

(a) $\frac{-1}{10}$ (b) 5 (c) $\frac{1}{25}$ (d) 1

Total mark 12

Quiz 5 till lesson 5 – unit 3

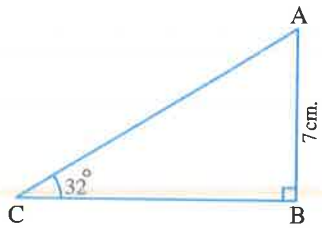
Choose the correct answer from those given :

(1) The perimeter of the circular sector whose area is 18 cm^2 and length of its arc is 6 cm. equals .... cm.

(a) 18 (b) 12 (c) 9 (d) 15

(2) **In the opposite figure :**

AC ≈ .... cm.



(a) 13.2 (b) 8.3

(c) 3.7 (d) 5.9

(3) The general solution of the equation : $\cos (90° - \theta) = 1$ is .... (where $n \in \mathbb{Z}$)

(a) πn (b) $\frac{\pi}{2} + 2\pi n$ (c) $2\pi n$ (d) $\pi + 2\pi n$

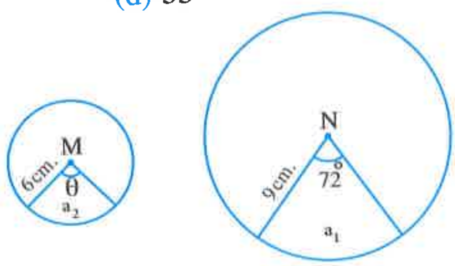
(4) $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta - \sec^2 \theta =$ .... (in its simplest form)

(a) 1 (b) -1 (c) zero (d) 2

(5) From a point on the ground surface 35 m. away from the base of a house , a man measures the elevation angle of the top of the house , its measure was 45° , then the height of the house is .... m.

(a) 45 (b) 35 (c) 25 (d) 55

(6) In the opposite figure :



M and N are two distant circles , a_1 and a_2 are the areas of the two sectors and $\frac{a_1}{a_2} = \frac{9}{5}$

, then $\theta =$ ....

(a) 72° (b) 80° (c) 90° (d) 100°

(7) The area of the circular sector , the measure of its central angle is 120° , in a circle of area 24 cm^2 equals .... cm^2.

(a) 24 (b) 16 (c) 8 (d) 36

(8) The perimeter of a circular sector is 12 cm. and its area is 9 cm^2 , then the measure of its central angle is ....

(a) $\frac{1}{2}^{rad}$ (b) 1^{rad} (c) $\frac{3}{2}^{rad}$ (d) 2^{rad}

(9) If $25 \sin \theta \cos \theta = 12$, then $\cos \theta - \sin \theta =$ ....

(a) $\frac{1}{5}$ (b) $\pm \frac{1}{5}$ (c) $\pm \frac{\sqrt{3}}{5}$ (d) $\frac{\sqrt{3}}{5}$

(10) $\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} =$ ....

(a) 1 (b) $\sin \theta + \cos \theta$ (c) $\csc \theta \sec \theta$ (d) $\tan \theta$

(11) If $\theta \in]0 , 2\pi[$, $2 \cos \theta + \sqrt{3} = 0$, then one of the values of θ is ....

(a) 30° (b) 60° (c) 210° (d) 240°

(12) If $0° \le X \le 360°$, then the number of solutions of the equation :

$3 \sin X = \tan X$ is ....

(a) 2 (b) 3 (c) 4 (d) 5

Total mark 12

Quiz 6 till lesson 6 – unit 3

Choose the correct answer from those given :

(1) $\sin^2 \theta + \cos^2 \theta + \cot^2 \theta =$ ....

(a) $\sec^2 \theta$ (b) $\csc^2 \theta$ (c) $\tan^2 \theta$ (d) $\cot^2 \theta$

(2) The area of the circular sector which the radius length of its circle is 4 cm. and the length of its arc is 6 cm. equals .... cm^2.

(a) 24 (b) 12 (c) 10 (d) 8

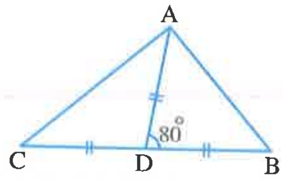
(3) If $\sin \theta + \cos \theta = 0$, where $0 < \theta < 180°$, then $\theta =$ ....

(a) 45° (b) 135° (c) 60° (d) 120°

(4) The general solution of the equation : $\cos \theta = 1$ is .... where $n \in \mathbb{Z}$

(a) $n\pi$ (b) $2n\pi$ (c) $\frac{\pi}{2} + n\pi$ (d) $\frac{\pi}{2} + 2n\pi$

(5) **In the opposite figure :**



$D \in \overline{BC}$, $AD = DB = DC = 5$ cm.

, $m(\angle ADB) = 80°$

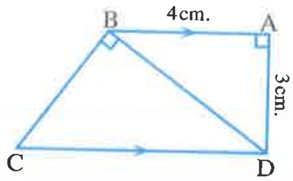
, $AC =$ .... cm.

(a) 10 sin 40° (b) 10 sin 50° (c) 5 sin 80° (d) 5 sin 40°

(6) If $\sec \theta - \tan \theta = 2$, then $\sec \theta =$ ....

(a) 5 (b) 1 (c) 4 (d) $\frac{5}{4}$

(7) **In the opposite figure :**



The length of $\overline{BC} =$ .... cm.

(a) 5 (b) $6\frac{2}{3}$

(c) $3\frac{3}{4}$ (d) 3

(8) The number of solutions of the equation :

$\cos^2 \theta - 4 \cos \theta + 4 = 0$ equals ....

(a) zero (b) 1 (c) 2 (d) 3

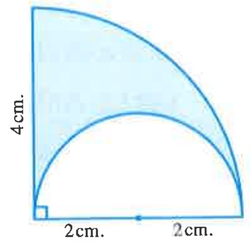
(9) The area of the circular segment whose central measure is 30° and the radius length of its circle is $2\sqrt{3}$ cm. equals .... cm^2.

(a) $\frac{\pi}{3}+2$ (b) $\pi-3$ (c) $\pi+3$ (d) $\frac{\pi}{3}-2$

(10) $\frac{3}{1+\tan^2\theta}+3\sin^2\theta=$ ....

(a) 3 (b) 1 (c) $3\sin^2\theta$ (d) $\sec^2\theta$

(11) In the opposite figure :

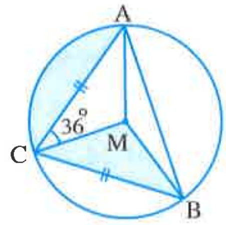


The area of the shaded region equals .... cm^2.

(a) 8π (b) 16π

(c) 4π (d) 2π

(12) In the opposite figure :



The radius of circle M is 10 cm.

, BC = AC , m (∠ ACM) = 36°

, then the area of the shaded region = ... cm^2.

(a) 20π (b) 30π

(c) 40π (d) 50π

Total mark

12

Quiz 7 till lesson 5 – unit 3

Choose the correct answer from those given :

(1) If X is the side length of the equilateral triangle whose area is $9\sqrt{3}$ cm^2, then $X=$ .... cm.

(a) 6 (b) $6\sqrt{3}$ (c) $3\sqrt{3}$ (d) 3

(2) The radius length of the circle of the circular sector whose area is 45 cm^2 and the length of its arc is 10 cm. equals .... cm.

(a) 4.5 (b) 3 (c) 9 (d) 6

(3) The area of the regular pentagon whose side length is 12 cm. equals .... cm^2 (to the nearest cm^2)

(a) 131 (b) 991 (c) 50 (d) 248

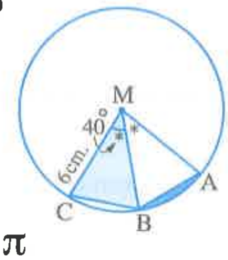
(4) The area of the convex quadrilateral whose diagonal lengths are 12 cm. , 8 cm. and the measure of the included angle between them is 30° equals .... cm^2.

(a) 48 (b) 108 (c) 24 (d) 96

(5) If $\theta \in [0 , \pi[$, then the value of θ which makes the roots of the equation : $x^2 + 2x + 2\cos\theta = 0$ are equal is ....

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2}{3}\pi$ (d) $\frac{5}{6}\pi$

(6) In the opposite figure :



M is a circle , MC = 6 cm.

, m ($\angle$ AMB) = m ($\angle$ CMB) = 40°

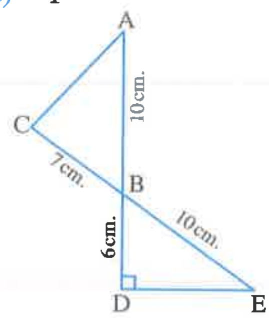
, then the area of the shaded region = ... cm^2.

(a) 4π (b) 5π (c) 6π (d) 7π

(7) If a + b = 30° , then the numerical value of the expression :
sin (3 a + 2 b) + sin (9 a + 8 b) = ...

(a) 1 (b) $\sqrt{3}$ (c) zero (d) – 1

(8) In the opposite figure :



The area of Δ ABC equals .... cm^2.

(a) 24

(b) 28

(c) 32

(d) 35

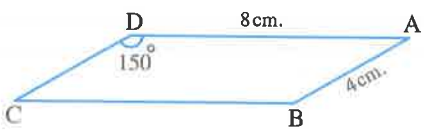
(9) The area of the circular segment whose chord length is 18 cm. , and the radius length of its circle 18 cm. approximately equals .... cm^2.

(a) 29 (b) 28 (c) 30 (d) 60

(10) The general solution of the equation : $\cos\theta = 0$ is .... (where $n \in \mathbb{Z}$)

(a) πn (b) $2\pi n$ (c) $\frac{\pi}{2} + \pi n$ (d) $\frac{\pi}{2} + 2\pi n$

(11) In the opposite figure :

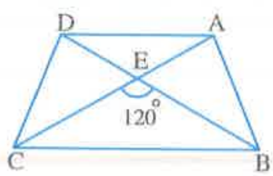


ABCD is a parallelogram

it's area = ... cm^2.

(a) 16 (b) 20 (c) 24 (d) 36

(12) In the opposite figure :



BD = 6 cm. , the area of the figure ABCD = $24\sqrt{3}$ cm^2.

, m ($\angle$ BEC) = 120° , then AC = ... cm.

(a) 12 (b) 14

(c) 15 (d) 16

Third : Accumulative quizzes on analytic geometry

Total mark 8

Quiz 1 on lesson 1 – unit 4

Choose the correct answer from those given :

(1) On the Cartesian plane , if A (4 , – 3) , B (4 , 4) , C (– 2 , – 1) and $\overrightarrow{CD}$ is equivalent to $\overrightarrow{AB}$, then the point D is ....

(a) (– 2 , 6) (b) (2 , – 6) (c) (0 , 7) (d) (0 , – 7)

(2) If ABCDEF is a regular hexagon whose centre (M) which of the following directed line segments are not equivalent ?

(a) $\overrightarrow{AB}$, $\overrightarrow{FM}$ (b) $\overrightarrow{AB}$, $\overrightarrow{ED}$ (c) $\overrightarrow{AB}$, $\overrightarrow{MC}$ (d) $\overrightarrow{AB}$, $\overrightarrow{MD}$

(3) A car covered 20 metres due north , then it covered the same distance due west , then the displacement of the car is ....

(a) 40 metres due west.
(b) 40 metres due western north.
(c) $20\sqrt{2}$ metres due western north.
(d) $20\sqrt{2}$ metres due western south.

(4) Which of the following is a vector quantity ?

(a) The time
(b) The temperature
(c) The displacement
(d) The mass

(5) If a cyclist travels 12 km. due to the north , then travels 4 km. due to the south from the point he reaches , then the total distance covered by the cyclist during the whole journey .... km.

(a) 12 (b) 4 (c) 8 (d) 16

(6) ABCD is a rectangle , if a particle moves from A to C then to D , then its displacement ....

(a) of magnitude AB in $\overrightarrow{BA}$ direction.
(b) of magnitude AD in $\overrightarrow{DA}$ direction.
(c) of magnitude AD in $\overrightarrow{AD}$ direction.
(d) of magnitude DC in $\overrightarrow{DC}$ direction.

(7) ABCD is a square , its diagonals intersect at M. If X , Y are the midpoints of $\overline{AB}$, $\overline{BC}$ respectively , then $\overrightarrow{XY}$ is equivalent to ....

(a) $\overrightarrow{CM}$ (b) $\overrightarrow{AC}$ (c) $\overrightarrow{CA}$ (d) $\overrightarrow{AM}$

(8) If the point B is the image of the point A (2 , 3) by reflection in the y-axis and C (1 , – 3) , $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$, then the coordinates of the point D is ....

(a) (– 2 , 3) (b) (2 , – 3) (c) (– 3 , 3) (d) (– 3 , – 3)

Total mark 12

Quiz 2 till lesson 2 – unit 4

Choose the correct answer from those given :

(1) If $\overrightarrow{A} = (6, -8)$, then $\|\overrightarrow{A}\| =$ ....

(a) 6 (b) 8 (c) 10 (d) 14

(2) If $\overrightarrow{A} = (3, -2)$, $\overrightarrow{C} = (-2, k)$ are parallel, then $k =$ ....

(a) -2 (b) 3 (c) $\frac{4}{3}$ (d) $\frac{-4}{3}$

(3) If $\overrightarrow{A} = (-4, 2)$, $\overrightarrow{B} = (3, -5)$, then $\overrightarrow{A} + 2\overrightarrow{B} =$ ....

(a) $(2, 8)$ (b) $(8, 2)$ (c) $(2, -8)$ (d) $(-2, 8)$

(4) If $\overrightarrow{A} = 6\sqrt{3}\,\overrightarrow{i} - 6\,\overrightarrow{j}$, then $\overrightarrow{A}$ in the polar form = ...

(a) $(12, 30°)$ (b) $(12, 150°)$ (c) $(12, 210°)$ (d) $(12, 330°)$

(5) If $\overrightarrow{A} = \left(3, \frac{\pi}{4}\right)$, then $2\overrightarrow{A} =$ ....

(a) $\left(6, \frac{\pi}{2}\right)$ (b) $\left(6, \frac{\pi}{4}\right)$ (c) $\left(3, \frac{\pi}{2}\right)$ (d) $\left(3, \frac{\pi}{4}\right)$

(6) If $\|-4\overrightarrow{A}\| = 5\|k\overrightarrow{A}\|$, then $k =$ ....

(a) $\pm\frac{4}{5}$ (b) $\pm\frac{5}{4}$ (c) ± 5 (d) ± 4

(7) If $\overrightarrow{C} = (5, 1)$, $\overrightarrow{D} = (-2, 4)$, then $\|\overrightarrow{C} + \frac{1}{2}\overrightarrow{D}\| =$ ....

(a) 25 (b) 7 (c) 5 (d) 1

(8) If $\overrightarrow{A} = (-1, 2)$, $\overrightarrow{B} = (3, 7)$, $\overrightarrow{C} = (7, 12)$, then $\overrightarrow{C} =$ ....

(a) $2\overrightarrow{A} - \overrightarrow{B}$ (b) $\overrightarrow{A} + 2\overrightarrow{B}$ (c) $2\overrightarrow{B} - \overrightarrow{A}$ (d) $3\overrightarrow{A} + 2\overrightarrow{B}$

(9) If $\overrightarrow{A} = (3, 6)$, $\overrightarrow{B} = (x - 5, 10)$, and $\overrightarrow{A} \perp \overrightarrow{B}$, then $x =$ ....

(a) -20 (b) -18 (c) -16 (d) -15

(10) If $\overrightarrow{A} = 3\overrightarrow{i} - 4\overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{j}$, $\overrightarrow{C} = \left(5, \frac{\pi}{18}\right)$, then $\|\overrightarrow{A}\| + \|\overrightarrow{B}\| + \|\overrightarrow{C}\| =$ ....

(a) 9 (b) 10 (c) 11 (d) 12

(11) If $\overrightarrow{A} = 2\overrightarrow{i} - \overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{C} = \overrightarrow{i} + 3\overrightarrow{j}$ and $\overrightarrow{A} \perp (k\overrightarrow{B} + \overrightarrow{C})$, then k = ...

(a) -1 (b) 1 (c) 3 (d) 4

(12) If $\overrightarrow{A} = -3\overrightarrow{B}$, then ....

(a) $-3\overrightarrow{A} = \overrightarrow{B}$ (b) $\overrightarrow{A}$, $\overrightarrow{B}$ act in the same direction

(c) $\overrightarrow{A} \perp \overrightarrow{B}$ (d) $\overrightarrow{A} \parallel \overrightarrow{B}$

Total mark 12

Quiz 3 till lesson 3 – unit 4

Choose the correct answer from those given :

(1) If $\overrightarrow{AB} = \overrightarrow{CD}$ where $\overrightarrow{AB} = (6 , 4)$, $\overrightarrow{C} = (-1 , 3)$, then $\overrightarrow{D}$ = ...

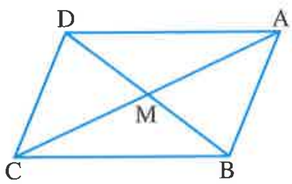
(a) $(5 , 7)$ (b) $(-5 , -7)$ (c) $(-5 , 7)$ (d) $(7 , 7)$

(2) The vector $\overrightarrow{M} = -12\overrightarrow{i} - 12\overrightarrow{j}$ is expressed in the polar form as ....

(a) $\overrightarrow{M} = \left(12 , \frac{\pi}{4}\right)$ (b) $\overrightarrow{M} = \left(12\sqrt{2} , \frac{\pi}{4}\right)$

(c) $\overrightarrow{M} = \left(12\sqrt{2} , \frac{3\pi}{4}\right)$ (d) $\overrightarrow{M} = \left(12\sqrt{2} , \frac{5\pi}{4}\right)$

(3) **In the opposite figure :**



ABCD is a parallelogram whose diagonals intersect at M , then each of the following expresses $\overrightarrow{AC}$ except ....

(a) $\overrightarrow{AB} + \overrightarrow{BD}$ (b) $2\overrightarrow{AM}$

(c) $\overrightarrow{AD} + \overrightarrow{DC}$ (d) $\overrightarrow{BC} + \overrightarrow{DC}$

(4) In Δ ABC , if D is the midpoint of $\overline{BC}$, then $\overrightarrow{BA} + \overrightarrow{CA} + \overrightarrow{AD}$ = ...

(a) $\overrightarrow{BC}$ (b) $\overrightarrow{DA}$ (c) $2\overrightarrow{DA}$ (d) $\overrightarrow{CB}$

(5) If $\overrightarrow{A}$, $\overrightarrow{B}$ are two non-zero vectors , and $\|\overrightarrow{A} + \overrightarrow{B}\| = \|\overrightarrow{A} - \overrightarrow{B}\|$, then ....

(a) $\overrightarrow{A} = -\overrightarrow{B}$ (b) $\overrightarrow{A} = \overrightarrow{B}$ (c) $\overrightarrow{A} \parallel \overrightarrow{B}$ (d) $\overrightarrow{A} \perp \overrightarrow{B}$

(6) The measure of the angle between the two vectors $\overrightarrow{A} = 3\overrightarrow{i} + 3\overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + \sqrt{3}\overrightarrow{j}$ is ....

(a) 45° (b) 30° (c) 60° (d) 15°

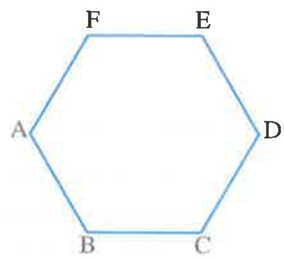
(7) $\overrightarrow{AB} - \overrightarrow{BA} = \dots\dots$

(a) zero (b) $2\,\overrightarrow{AB}$ (c) $2\,\overrightarrow{BA}$ (d) $\vec{O}$

(8) If $\vec{A} = 20\,\vec{i} - 15\,\vec{j}$, $\vec{B} = 7\,\vec{i} + 24\,\vec{j}$ and $\vec{M} = \vec{A} + \vec{B}$, $\vec{N} = \vec{A} - \vec{B}$, then $\dots\dots$

(a) $\vec{M} \,//\, \vec{N}$ (b) $\vec{M} \perp \vec{N}$ (c) $\vec{M} = \vec{N}$ (d) $\|\vec{M}\| = \|\vec{N}\|$

(9) **In the opposite figure :**



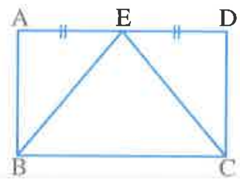
ABCDEF is a regular hexagon

, then $(\overrightarrow{AB} - \overrightarrow{CB}) + \overrightarrow{AF} + \overrightarrow{DE} = \dots\dots$

(a) $\overrightarrow{FE}$ (b) $\overrightarrow{AE}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{AC}$

(10) **In the opposite figure :**



ABCD is a rectangle , E is the midpoint of $\overline{AD}$

, then $\overrightarrow{EB} + \overrightarrow{BA} - \overrightarrow{DC} = \dots\dots$

(a) $\overrightarrow{EB}$ (b) $\overrightarrow{BE}$

(c) $\overrightarrow{EC}$ (d) $\overrightarrow{CE}$

(11) If $3\,\vec{A} + \vec{B} = (5 , -2)$, $\overrightarrow{AB} = (-3 , 10)$, then $\vec{A} = \dots\dots$

(a) $(2 , -1)$ (b) $(-1 , 2)$ (c) $(2 , -3)$ (d) $(-2 , 3)$

(12) If A (k , 2) , B (1 , −1) and $\|\overrightarrow{AB}\| = 5$, then k = $\dots\dots$

(a) 5 (b) −3 (c) 5 or −3 (d) 15

Quiz 4 till lesson 4 – unit 4

Total mark 12

Choose the correct answer from those given :

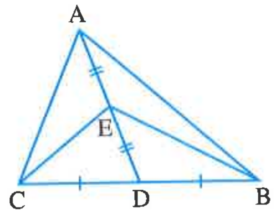
(1) If $\vec{F_1} = 2\,\vec{i} - 5\,\vec{j}$, $\vec{F_2} = \vec{i} + 2\,\vec{j}$ affect on one point , then the magnitude of the resultant force = $\dots\dots$ newton.

(a) $2\sqrt{3}$ (b) 9 (c) $3\sqrt{2}$ (d) $9\sqrt{2}$

(2) If $\vec{v_A} = 120\,\vec{e}$, $\vec{v_B} = -90\,\vec{e}$, then $\vec{v_{BA}} = \dots\dots$

(a) $210\,\vec{e}$ (b) $30\,\vec{e}$ (c) $-30\,\vec{e}$ (d) $-210\,\vec{e}$

(3) In the opposite figure :



ABC is a triangle , if D is the midpoint of $\overline{BC}$

, E is the midpoint of $\overline{AD}$, then

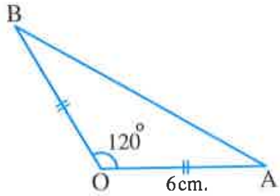
$\overrightarrow{AB} + \overrightarrow{AC} =$ .... $\overrightarrow{AE}$

(a) 1 (b) 2 (c) 4 (d) – 4

(4) If $\overrightarrow{A} = (6 , 8)$, $\overrightarrow{B} = (3 , 2k)$, $\overrightarrow{A} \perp \overrightarrow{B}$, then k = ...

(a) 2 (b) $\frac{-9}{4}$ (c) $\frac{-9}{8}$ (d) – 2

(5) In the opposite figure :



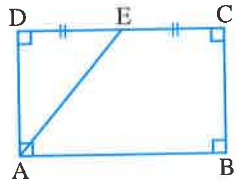
OA = OB = 6 cm. , m (∠ O) = 120°

, then $\|\overrightarrow{AB}\| =$ .... cm.

(a) 6 (b) 12

(c) $6\sqrt{3}$ (d) $6\sqrt{2}$

(6) In the opposite figure :



ABCD is a rectangle

E is the midpoint of $\overline{CD}$

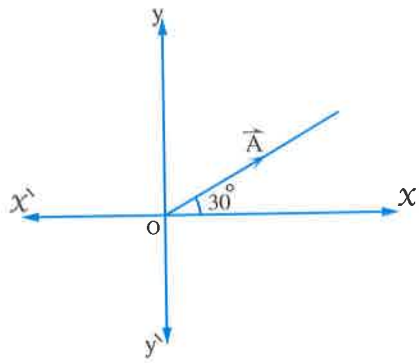
, then $\overrightarrow{AE} + \overrightarrow{DE} =$ ....

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$ (c) $2\,\overrightarrow{AD}$ (d) $\overrightarrow{CA}$

(7) If the vector $(\overrightarrow{A} - 2\,\overrightarrow{C})$ is equivalent to $\overrightarrow{BA}$, then $\overrightarrow{B}$ is equivalent to ....

(a) $2\,\overrightarrow{A}$ (b) $\overrightarrow{C}$ (c) $2\,\overrightarrow{BC}$ (d) $2\,\overrightarrow{C}$

(8) In the opposite figure :



$\|\overrightarrow{A}\| = 4$ length unit , then $\overrightarrow{A} =$ ....

(a) $(2 , 2\sqrt{3})$

(b) $(2\sqrt{3} , 2)$

(c) $(4 , \sqrt{3})$

(d) $(\sqrt{3} , 2)$

(9) If the forces $\overrightarrow{F_1} = \left(8\sqrt{2} , \frac{3}{4}\pi\right)$, $\overrightarrow{F_2} = a\,\overrightarrow{i} + 3\,\overrightarrow{j}$, $\overrightarrow{F_3} = -5\,\overrightarrow{i} + (b+2)\,\overrightarrow{j}$ act at one point and the system is in equilibrium , then $\frac{a}{b} =$ ....

(a) 13 (b) – 13 (c) 1 (d) – 1

(10) In the opposite figure :

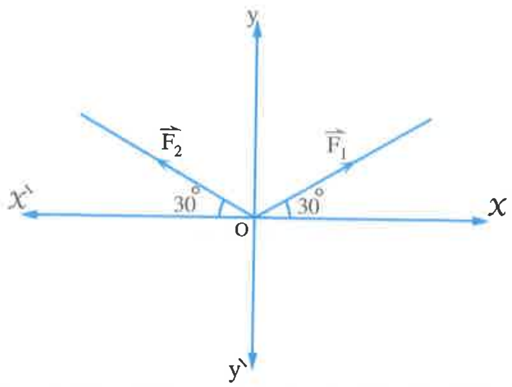
If $F_1 = F_2 = 3$ newton , then the resultant of the two forces F_1 and F_2 is $\overrightarrow{R} =$ ....

(a) (3 , 180°)

(b) (6 , 180°)

(c) (3 , 90°)

(d) (6 , 90°)



(11) In the opposite figure :

ABCD is a square , $\overrightarrow{AY} + \overrightarrow{XY} = k\,\overrightarrow{XC}$

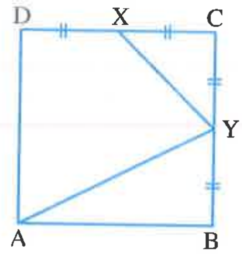
, then k = ...

(a) 1

(b) 2

(c) 3

(d) 4



(12) If $\overrightarrow{A}$, $\overrightarrow{B}$ are two unit vectors , then ....

(a) $\|\overrightarrow{A} + \overrightarrow{B}\| = 2$

(b) $\|\overrightarrow{A} - \overrightarrow{B}\| = 2$

(c) $\|\overrightarrow{A} + \overrightarrow{B}\| \geq 2$

(d) $\|\overrightarrow{A} + \overrightarrow{B}\| \leq 2$

Quiz 5 till lesson 1 – unit 5

Total mark 12

Choose the correct answer from those given :

(1) If $\overrightarrow{OC} = \left(12 , \frac{5\pi}{6}\right)$ is the position vector of the point C with respect to the origin point O , then the point C is ....

(a) (6 , 6)

(b) $\left(6 , 6\sqrt{3}\right)$

(c) $\left(-6\sqrt{3} , 6\right)$

(d) $\left(6\sqrt{3} , -6\right)$

(2) If A (7 , 5) , C (5 , – 2) , then $\overrightarrow{AB} + \overrightarrow{BC} =$ ....

(a) (12 , 3)

(b) (– 2 , – 7)

(c) (2 , 7)

(d) (7 , – 2)

(3) If C (4 , 6) is the midpoint of $\overline{AB}$ where B (2 , 8) , then A = ...

(a) (6 , 4)

(b) (6 , 14)

(c) (3 , 7)

(d) (2 , – 2)

(4) If $\overrightarrow{A} = (k , 5)$, $\overrightarrow{B} = (-20 , 16)$, $\overrightarrow{A} \perp \overrightarrow{B}$, then k = ...

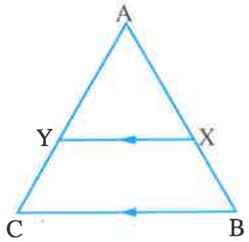
(a) – 4

(b) 4

(c) 5

(d) – 5

(5) In the opposite figure :



If A (1 , 2) , B (6 , 2) , $\overline{XY} // \overline{BC}$, $\frac{AY}{AC} = \frac{3}{5}$

, then the coordinates of the point X is ....

(a) (2 , 4) (b) (4 , 2)

(c) (– 2 , 4) (d) (– 4 , 2)

(6) The vector represents a uniform velocity 6 km./h. for a car moving due to the western north = ...

(a) $3\sqrt{2}\,\vec{i} + 3\sqrt{2}\,\vec{j}$ (b) $3\sqrt{2}\,\vec{i} - 3\sqrt{2}\,\vec{j}$

(c) $-3\sqrt{2}\,\vec{i} + 3\sqrt{2}\,\vec{j}$ (d) $-6\sqrt{2}\,\vec{i} + 6\sqrt{2}\,\vec{j}$

(7) All the following are unit vectors except ....

(a) (1 , 0) (b) (0 , – 1) (c) (0.6 , 0.8) (d) (1 , 1)

(8) If C $\in \overline{AB}$ and AB = 4 BC , A (– 1 , 4) , B (3 , 4) , the coordinates of C is ....

(a) (0 , 4) (b) (4 , 2) (c) (4 , 0) (d) (2 , 4)

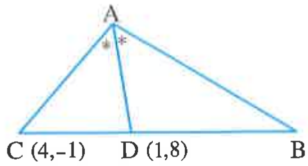
(9) The ratio of division that the X-axis divides the line segment $\overrightarrow{AB}$ where A (2 , 5) , B (7 , – 2) is ....

(a) 5 : 2 (b) 2 : 3 (c) 3 : 2 (d) 2 : 5

(10) In Δ ABC , A (3 , 5) , B (7 , 10) , C (2 , 3) , then the coordinates of the point of intersection of its medians is ....

(a) (4 , 6) (b) (6 , 4) (c) (6 , 9) (d) (9 , 6)

(11) In the given figure :



If 4 AC = 3 AB , then the coordinates of the point B is ....

(a) (– 5 , 14) (b) (– 4 , 16)

(c) (– 3 , 20) (d) (– 2 , 2)

(12) If the position vector $\vec{A} = (\sqrt{3} , 1)$ is rotated around the origin by an angle of measure 45° clockwise , then the polar form of the vector $\vec{A}$ after rotation is ....

(a) (2 , 30°) (b) (2 , 315°) (c) (2 , 345°) (d) (2 , 15°)

Quiz 6 till lesson 2 – unit 5

Total mark 12

Choose the correct answer from those given :

(1) The equation of the straight line passing through (2 , – 3) and is parallel to x-axis is ....

(a) $x + 3 = 0$ (b) $y + 3 = 0$ (c) $x - 3 = 0$ (d) $y - 3 = 0$

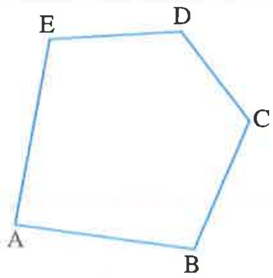
(2) The vector equation of the straight line : $4x + 3y = 12$ is ....

(a) $\vec{r} = (-4, 6) + k(4, 3)$ (b) $\vec{r} = (6, -4) + k(3, 4)$

(c) $\vec{r} = (6, -4) + k(-3, 4)$ (d) $\vec{r} = (6, -4) + k(-3, -4)$

(3) **In the opposite figure :**

All the following express $\overrightarrow{AE}$ except ....



(a) $\overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$

(b) $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{ED}$

(c) $\overrightarrow{AD} + \overrightarrow{DE}$

(d) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

(4) If $\vec{u} = (2, -3)$ is a direction vector of a straight line , then all the following vectors are direction vectors of the same straight line except the vector ....

(a) (– 2 , 3) (b) (– 2 , – 3) (c) (4 , – 6) (d) (– 4 , 6)

(5) If the point A (0 , 0) is the image of the point B (4 , 2) by reflection in the straight line L , then the equation of L is ....

(a) $x = 2y$ (b) $2x + y = 5$ (c) $2x - y = 5$ (d) $x + y = 6$

(6) The vector represents the displacement of a body covered distance 40 cm. due to eastern south is ....

(a) $20\sqrt{2}\,\vec{i} + 20\sqrt{2}\,\vec{j}$ (b) $-20\sqrt{2}\,\vec{i} + 20\sqrt{2}\,\vec{j}$

(c) $-20\sqrt{2}\,\vec{i} - 20\sqrt{2}\,\vec{j}$ (d) $20\sqrt{2}\,\vec{i} - 20\sqrt{2}\,\vec{j}$

(7) If A (3 , – 5) , B (– 1 , 5) , M (6 , k) and $\overrightarrow{AB} \parallel \overrightarrow{M}$, then k = ...

(a) – 15 (b) – 10 (c) – 5 (d) 5

(8) The ratio by which the y-axis divides $\overline{AB}$ where A (2 , 5) , B (6 , 7) equals ....

(a) 1 : 3 externally. (b) 3 : 1 internally.

(c) 1 : 2 externally. (d) 3 : 2 internally.

(9) In the opposite figure :

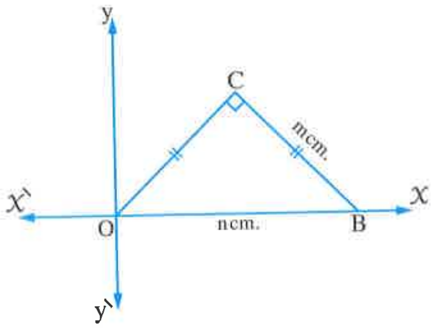
The equation of the straight line $\overleftrightarrow{OC}$ is ....

(a) $y = \frac{m}{n} x$

(b) $y = x$

(c) $y = \frac{n}{m} x$

(d) $y = mn\, x$



(10) The straight line : $6x - 8y = 48$ makes with the coordinate axes a triangle , its perimeter = ... length unit.

(a) 48 (b) 24 (c) 12 (d) 8

(11) The perpendicular direction vector on the straight line $x = 3 + 2k$, $y = 4 - k$ is ....

(a) (2 , – 1) (b) (1 , 2) (c) (2 , 1) (d) (4 , – 2)

(12) The equation of one of the two straight lines bisecting the angle between the coordinate axes is ....

(a) y = 2 (b) $x = 2y$ (c) $y = x$ (d) $y = 4x$

Total mark

12

Quiz 7 till lesson 3 – unit 5

Choose the correct answer from those given :

(1) The measure of the acute angle between the two straight lines $\vec{r} = (2 , 2) + k (1 , 1)$ and the straight line $x = 0$ equals ....

(a) 45° (b) 30° (c) 135° (d) 60°

(2) If the perpendicular direction vector on the straight line is $\vec{n} = (3 , 4)$, then the slope of this straight line is ....

(a) $\frac{-4}{3}$ (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) $\frac{-3}{4}$

(3) If ABCD is a parallelogram , then $\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{DB} =$ ....

(a) zero (b) $\overrightarrow{zero}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{DB}$

(4) The measure of the angle between the two straight lines $L_1 : x + 2y + 5 = 0$, $L_2 : \vec{r} = (1 , 4) + k (1 , 2)$ equals ....

(a) zero (b) 45° (c) 90° (d) 135°

(5) If B (0 , 3) , C (3 , 0) and A lies in the third distance between B and C , then the point A is ....

(a) (1 , 2) (b) (2 , 1) (c) (– 1 , – 2) (d) (– 2 , – 1)

(6) If $\vec{A} = (m - 1 , 2)$, $\vec{B} = (3 , - 4)$ and $\vec{A} \parallel \vec{B}$, then the value of m = ...

(a) $-\frac{1}{2}$ (b) zero (c) $\frac{1}{2}$ (d) 1

(7) The measure of the angle between the two straight lines $x = 1$, $y = 2$ equals ....

(a) 30° (b) 45° (c) 90° (d) 180°

(8) The measure of the obtuse angle between the two straight lines $y = \left(2 - \sqrt{3}\right)(x + 5)$, $y = \left(2 + \sqrt{3}\right)(x - 7)$ is ....

(a) 150° (b) 60° (c) 135° (d) 120°

(9) The perpendicular to the straight line $\vec{r} = (3 , 2) + k\left(1 , -\sqrt{3}\right)$ makes with the positive x-axis an angle of measure ....

(a) 120° (b) 30° (c) 60° (d) 150°

(10) Two cars A and B move in a straight line , if $\overrightarrow{v_A} = 30\,\vec{i}$, $\overrightarrow{v_B} = 50\,\vec{i}$, then $\overrightarrow{v_{AB}} =$ .... $\vec{i}$

(a) 80 (b) 20 (c) – 20 (d) – 80

(11) If the straight line : $\frac{x}{6} + \frac{y}{b} = 1$ makes with the coordinate axes a triangle , its area = 9 square units , then b = ...

(a) ± 3 (b) – 3 (c) 6 (d) ± 6

(12) The set of values of k which makes the measure of the acute angle between the two straight lines $x + k y - 8 = 0$, $2x - y - 5 = 0$ equals $\frac{\pi}{4}$ is ....

(a) $\left\{3 , -\frac{1}{3}\right\}$ (b) $\left\{-3 , \frac{1}{3}\right\}$ (c) $\left\{3 , \frac{1}{3}\right\}$ (d) $\{3\}$

Total mark

12

Quiz 8 till lesson 4 – unit 5

Choose the correct answer from those given :

(1) If (6 , 4) , (3 , m) are two direction vectors of two parallel straight lines , then m = ...

(a) 4 (b) 3 (c) 2 (d) $\frac{-9}{2}$

(2) The equation of the straight line passing through the two points (3 , 0) , (0 , – 2) is ....

(a) $\frac{x}{3}+\frac{y}{2}=0$ (b) $\frac{x}{3}+\frac{y}{2}=1$ (c) $\frac{x}{3}-\frac{y}{2}=0$ (d) $\frac{x}{3}-\frac{y}{2}=1$

(3) The measure of the angle between the two straight lines whose slopes are $\frac{1}{2}$, – 2 equals ....

(a) 45° (b) 30° (c) 90° (d) 60°

(4) The length of the perpendicular drawn from the point (1 , 1) on the straight line : $x + y = 0$ equals .... length unit.

(a) 2 (b) $\sqrt{2}$ (c) 1 (d) zero

(5) In triangle ABC , B (3 , 5) , C (– 3 , – 7) , D $\in \overline{BC}$ such that the area of Δ ABD = $\frac{1}{3}$ the area of Δ ABC , then D = ...

(a) $\left(3 , \frac{17}{3}\right)$ (b) $\left(\frac{3}{2} , 2\right)$ (c) (0 , – 1) (d) (1 , 1)

(6) If $\vec{u}$ = (3 , – 4) is a direction vector of the straight line , then all the following are direction vectors in the same direction as the line except ....

(a) (– 3 , 4) (b) (9 , – 12) (c) (3 , 4) (d) (1.5 , – 2)

(7) The distance between the two straight lines $3x - 4y + 20 = 0$, $3x - 4y + 10 = 0$ equals .... length unit.

(a) 2 (b) 3 (c) 4 (d) 5

(8) The length of the perpendicular drawn from the point (– 1 , 4) to the y-axis equals .... length unit.

(a) 7 (b) – 1 (c) 1 (d) 4

(9) ABCD is a square , A (2 , – 3) , the equation of $\overleftrightarrow{CD}$, $3x - 4y + 2 = 0$, then the area of the square ABCD = ... square units.

(a) 4 (b) 9 (c) 16 (d) 25

(10) The measure of the angle between the two vectors $\vec{A} = 3\vec{i} + \sqrt{3}\,\vec{j}$, $\vec{B} = -4\vec{i}$ equals ....

(a) 45° (b) 60° (c) 120° (d) 150°

(11) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} =$ ....

(a) $\overrightarrow{CD}$ (b) $\overrightarrow{BD}$ (c) $\vec{O}$ (d) $\overrightarrow{CB}$

(12) The length of the perpendicular drawn from the origion to the straight line $\vec{r} = (5 , 0) + k (4 , 3)$ equals .... length units.

(a) 15 (b) 5 (c) 3 (d) 4

Total mark 12

Quiz 9 till lesson 5 – unit 5

Choose the correct answer from those given :

(1) The measure of the acute angle between the two straight lines :

$x - 3y + 5 = 0$, $x + 2y - 7 = 0$ equals ....

(a) 60° (b) 30° (c) 45° (d) 75°

(2) The length of the perpendicular drawn from the origin on the straight line :

$\vec{r} = (10 , 0) + k (4 , 3)$ equals .... length units.

(a) 15 (b) 5 (c) 6 (d) 4

(3) The equation of the straight line passing through the origin and the intersection point of the two straight lines : $x = 1$, $y = 2$ is ....

(a) $x - y = 0$ (b) $x - 2y = 0$ (c) $2x - y = 0$ (d) $2x + y = 0$

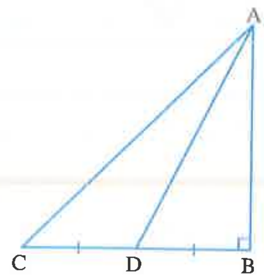
(4) **In the opposite figure :** $2\,\overrightarrow{AD} =$ ....

(a) $\overrightarrow{AB} + \overrightarrow{AC}$

(b) $\overrightarrow{AB} + \overrightarrow{BD}$

(c) $2\,\overrightarrow{AB} + 2\,\overrightarrow{CD}$

(d) $\overrightarrow{BA} + \overrightarrow{CA}$



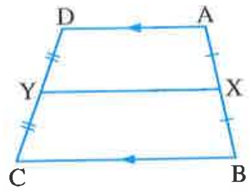
(5) Which of the following points lies on the straight line $\vec{r} = (-2, 1) + k(1, -3)$?

(a) $\left(\frac{-5}{3}, -2\right)$ (b) $\left(\frac{-3}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{3}{2}, -\frac{1}{2}\right)$ (d) $\left(\frac{-7}{3}, 2\right)$

(6) The straight line : $\frac{x}{4} + \frac{y}{7} = 1$ makes with the coordinate axes a triangle its area equals .... square unit.

(a) 4 (b) 7 (c) 14 (d) 28

(7) **In the opposite figure :**



ABCD is a trapezium $\overrightarrow{AD} + \overrightarrow{BC} = k\,\overrightarrow{YX}$

, then the value of k = ... where $k \in \mathbb{R}$

(a) – 2 (b) – 1 (c) 1 (d) 2

(8) If $\vec{u_1} = (2, 1)$, $\vec{u_2} = (-2, 4)$ are direction vectors of two straight lines , then the measure of the angle between these two straight lines equals ....

(a) 45° (b) 60° (c) 90° (d) 180°

(9) The vector equation of the straight line passes through the intersection point of the two straight lines $L_1 : x - 5 = 0$, $L_2 : x + y = 13$ and its direction vector (4 , 1) is .... length units.

(a) $\vec{r} = (4, 1) + k(5, 8)$ (b) $\vec{r} = (5, 6) + k(4, 1)$

(c) $\vec{r} = (5, 8) + k(4, 1)$ (d) $\vec{r} = (5, -8) + k(1, 4)$

(10) If the two straight lines : $y = 5x + c$, $y = ax + d$ are parallel , then a = ...

(a) 5 (b) – 5 (c) – 4 (d) 4

(11) The coordinates of the point lies at $\frac{2}{5}$ the distance between A and B where $\overline{AB}$ is a line segment and A (3 , – 2) , B (– 1 , 5) is ....

(a) (– 1 , 3) (b) $\left(\frac{7}{5}, \frac{4}{5}\right)$ (c) (3 , – 1) (d) $\left(\frac{4}{5}, \frac{7}{5}\right)$

(12) The equation of the straight line passes through the intersection point of the two straight lines $L_1 : x + 2y - 4 = 0$, $L_2 = x - 2y = 0$ and parallel to the x-axis is ....

(a) $x = 2$ (b) $y = 1$ (c) $y = 3$ (d) $y = 2$

FINAL REVISION

- **First :** Final revision on algebra.
- **Second :** Final revision on trigonometry.
- **Third :** Final revision on analytic geometry.

First : Final Revision on Algebra

Remember Matrices and determinants

Order of the matrix

If the number of rows of the matrix equals m and the number of its columns equals n , then the matrix is of order $m \times n$

For example : $A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -4 & 5 \end{pmatrix}$ is of order 2×3

and $a_{11} = 2$, $a_{12} = 3$, ... , $a_{23} = 5$

Some special matrices

The row matrix

Consists of one row and any number of columns , as :

$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \end{pmatrix}$

The column matrix

Consists of one column and any number of rows , as :

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

The square matrix

The number of its rows equals the number of its columns , as :

$\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

The zero matrix O

All of its elements are zeroes , as :

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The diagonal matrix

It is a square matrix and all of its elements are zeroes except the elements of its main diagonal , at least one of them is not equal to zero , as :

$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

The unit matrix I

It is a diagonal matrix and all elements of the main diagonal equal one , as :

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Matrix transpose

The transpose of the matrix A is denoted by A^t and it is the resulting matrix from replacing the rows by columns and the columns by rows.

For example : If $A = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 6 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 \\ -2 & 4 \\ 1 & 6 \end{pmatrix}$

Notice that : $(A^t)^t = A$

The symmetric and skew symmetric matrices

If A is a square matrix , then :

- A is called a symmetric matrix if and only if $A = A^t$
- A is called a skew symmetric matrix if and only if $A = -A^t$

For example : $A = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 3 & 6 \\ 5 & 6 & 4 \end{pmatrix}$ is a symmetric matrix

because : $A^t = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 3 & 6 \\ 5 & 6 & 4 \end{pmatrix} = A$

, $A = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & 4 \\ -2 & -4 & 0 \end{pmatrix}$ is a skew symmetric matrix

because : $A^t = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & -4 \\ 2 & 4 & 0 \end{pmatrix} = -A$

The equality of two matrices

The two matrices A and B are said to be equal if :

- They have the same order.
- Their corresponding elements are equal.

The concept of equality of two matrices is used in solving some equations.

For example :

If $\begin{pmatrix} x & 2 & -3 \\ y-x & 1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & -2 \end{pmatrix}$, then $x = 3$, $y - x = 5$, then $y = 8$

Multiplying a real number by a matrix

To multiply a real number by a matrix , we multiply the real number by each element of the elements of the matrix.

For example : If $A = \begin{pmatrix} -4 & 1 & 5 \\ 2 & -3 & -1 \end{pmatrix}$, then $2A = \begin{pmatrix} -8 & 2 & 10 \\ 4 & -6 & -2 \end{pmatrix}$

Adding and subtracting matrices

If A and B are two matrices of the same order $m \times n$, then :

- $A + B$ is a matrix of order $m \times n$ and each element in it is the sum of the two corresponding elements in A and B
- $A - B = A + (-B)$ is a matrix of order $m \times n$ and each element in it is the sum of the two corresponding elements in A and $-B$

For example : If $A = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -3 & 4 \\ -1 & 2 & -2 \end{pmatrix}$

, then $A + B = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 5 & -4 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 4 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -5 & 6 \\ -3 & 7 & -6 \end{pmatrix}$

, $A - B = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 5 & -4 \end{pmatrix} + \begin{pmatrix} 0 & 3 & -4 \\ 1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -2 \end{pmatrix}$

Notice that

A , B , A + B , A − B all of them are of the same order 2×3

Multiplying matrices

If A is a matrix of order $m \times \ell$ and B is a matrix of order $r \times n$, then :

AB is possible if $\ell = r$ (*i.e.* The number of columns of A = the number of rows of B) and the resulted matrix from the product is of order $m \times n$

For example : If $A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 5 \end{pmatrix}$

, then the number of columns of A = the number of rows of B = 3

$\therefore$ AB is possible and of order 2×2 , where $AB = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 5 \end{pmatrix}$

$$= \begin{pmatrix} (2)(3) + (-1)(-1) + (0)(4) & (2)(2) + (-1)(0) + (0)(5) \\ (3)(3) + (1)(-1) + (-2)(4) & (3)(2) + (1)(0) + (-2)(5) \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 0 & -4 \end{pmatrix}$$

Remarks

- $(A+B)^t = A^t + B^t$
- $(AB)^t = B^t A^t$

Determinants

- If A is a square matrix of order 2×2 where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then :

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12}$$

For example : $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix} = (4 \times 6) - (2 \times (-7)) = 24 + 14 = 38$

- If A is a square matrix of order 3×3 where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For example :

$$\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix} = 2\,(0 \times (-3) - 1 \times 4) - 3\,(-2 \times (-3) - 1 \times 4) - (-2 \times 1 - 1 \times 0)$$

$$= 2\,(0 - 4) - 3\,(6 - 4) - (-2 - 0)$$

$$= -8 - 6 + 2 = -12$$

Notice that

It is possible to expand the determinant using the elements of any row or column by using the rule of signs : $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

- The value of the determinant of the triangular matrix = the product of the elements of its main diagonal.

i.e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\, a_{22}\, a_{33}$

For example : $\begin{vmatrix} 2 & -1 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{vmatrix} = 2 \times 3 \times (-1) = -6$

Finding the area of a triangle using determinants

- If $X = (a , b)$, $Y = (c , d)$ and $Z = (e , f)$, then :

the area of ΔXYZ is $|A|$ where $A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$

Notice that : If $A = 0$, then the points X , Y and Z are collinear.

For example : If XYZ is a triangle where $X = (1 , 2)$, $Y = (3 , -4)$ and $Z = (-2 , 3)$

, then $A = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 1 \\ -2 & 3 & 1 \end{vmatrix}$ by using the elements of the 3rd column, you get :

$A = \frac{1}{2}[(9-8)-(3+4)+(-4-6)] = \frac{1}{2}(1-7-10) = -8$

$\therefore$ The area of $\Delta XYZ = |A| = |-8| = 8$ square units.

The multiplicative inverse of a 2 × 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplicative inverse of the matrix A which is denoted

by the symbol A^{-1} is defined (existed) when the determinant of $A \neq 0$

i.e. $|A| = \Delta \neq 0$ and : $\boxed{A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$ where $AA^{-1} = A^{-1}A = I$

For example : If $A = \begin{pmatrix} -2 & 2 \\ 3 & -4 \end{pmatrix}$, then A^{-1} is defined because $\Delta \neq 0$

where $\Delta = \begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} = -2 \times (-4) - 3 \times 2 = 2$

$\therefore A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & -2 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -\frac{3}{2} & -1 \end{pmatrix}$

Remember Solving equations

First Solving two simultaneous equations in two variables

It is possible to solve the two simultaneous equations :

$a_1 x + b_1 y = c_1$, $a_2 x + b_2 y = c_2$ by using :

1. The determinants (Cramer's rule).
2. The multiplicative inverse of a matrix.

1 By using determinants (Cramer's rule)

We find the values of the determinants :

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} , \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} , \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{, then } x = \frac{\Delta_x}{\Delta} , y = \frac{\Delta_y}{\Delta}$$

For example : If $2x - 3y = 4$, $3x + 4y = 23$, then : $\Delta = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 8 + 9 = 17$

$$, \Delta_x = \begin{vmatrix} 4 & -3 \\ 23 & 4 \end{vmatrix} = 16 + 69 = 85 , \Delta_y = \begin{vmatrix} 2 & 4 \\ 3 & 23 \end{vmatrix} = 46 - 12 = 34$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{85}{17} = 5 , y = \frac{\Delta_y}{\Delta} = \frac{34}{17} = 2$$

2 By using the multiplicative inverse of the matrix

We write the two equations in the form of the matrix equation : $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

i.e. In the form $AX = C$ where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

, then $X = A^{-1}C$ and from that we deduce the values of x and y

For example : If $2x - 3y = 4$, $3x + 4y = 23$

, then $A = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 4 \\ 23 \end{pmatrix}$

$\because X = A^{-1} C$ where $\Delta = |A| = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 8 + 9 = 17$

$\therefore A^{-1} = \frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}$ $\qquad \therefore X = \frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 23 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 85 \\ 34 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $\qquad \therefore x = 5$ and $y = 2$

Second Solving three equations in three variables (Cramer's rule)

To solve the equations : $a_1 x + b_1 y + c_1 z = d_1$, $a_2 x + b_2 y + c_2 z = d_2$

and $a_3 x + b_3 y + c_3 z = d_3$

We find the values of the determinants :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} , \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$, \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

, then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$

For example :

If $2x + 3y - z = 1$, $3x + 5y + 2z = 8$, $x - 2y - 3z = -1$

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = 2(-15+4) - 3(-9-2) + (-1)(-6-5)$$

$$= -22 + 33 + 11 = 22$$

$$\Delta_x = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = 1(-15+4) - 3(-24+2) + (-1)(-16+5)$$

$$= -11 + 66 + 11 = 66$$

$$\Delta_y = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix} = 2(-24+2) - 1(-9-2) + (-1)(-3-8)$$

$$= -44 + 11 + 11 = -22$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix} = 2(-5+16) - 3(-3-8) + 1(-6-5)$$

$$= 22 + 33 - 11 = 44$$

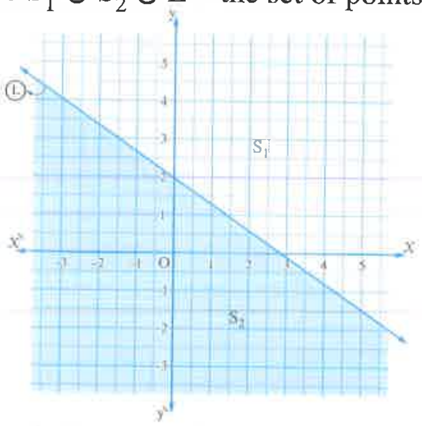
$\therefore x = \frac{\Delta_x}{\Delta} = \frac{66}{22} = 3$, $y = \frac{\Delta_y}{\Delta} = \frac{-22}{22} = -1$

, $z = \frac{\Delta_z}{\Delta} = \frac{44}{22} = 2$

Remember Solving linear inequalities

Solving first degree inequalities in two variables

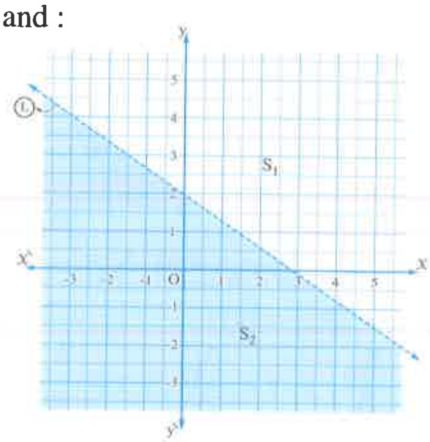
The boundary line $L : 2x + 3y = 6$ divides the plane into two halves S_1 and S_2 where $S_1 \cup S_2 \cup L$ = the set of points of the plane , and :



- The solution set of the inequality : $2x + 3y \geq 6$ is $L \cup S_1$
- The solution set of the inequality : $2x + 3y \leq 6$ is $L \cup S_2$

Notice that :

The boundary line L is represented by a solid line because the set of its points is a subset of the solution set.

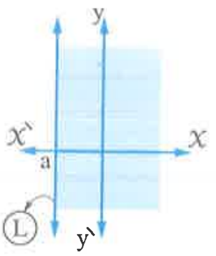


- The solution set of the inequality : $2x + 3y > 6$ is S_1
- The solution set of the inequality : $2x + 3y < 6$ is S_2

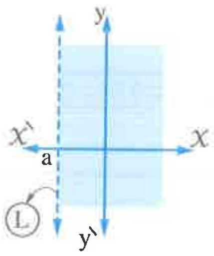
Notice that :

The boundary line L is represented by a dashed line because the set of its points is not a subset of the solution set.

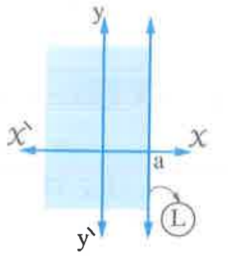
Remark **The following figures are the graphical representation of the solution sets of the given inequalities under each figure in $\mathbb{R} \times \mathbb{R}$**



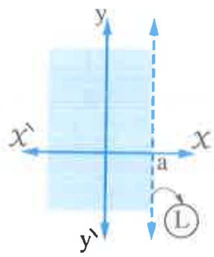
$x \geq a$



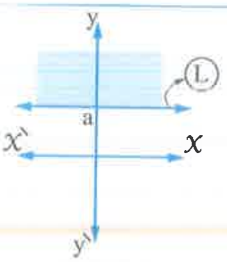
$x > a$



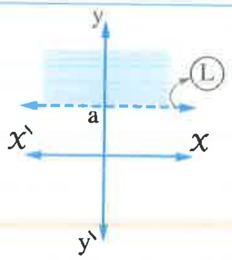
$x \leq a$



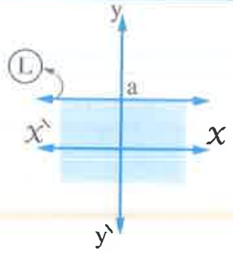
$x < a$



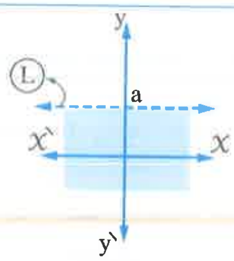
$y \geq a$



$y > a$



$y \leq a$



$y < a$

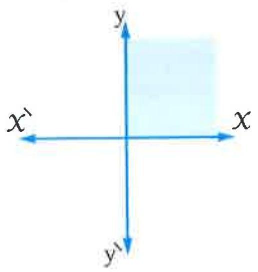
Solving systems of linear inequalities graphically

To find the graphical solution of two inequalities or more :

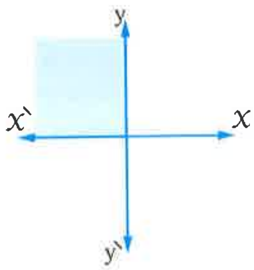
1. Determine the region of the solution of each inequality , as S_1 , S_2 , S_3 , ...
2. Determine the common region between S_1 , S_2 , S_3 , ...
   by finding $S_1 \cap S_2 \cap S_3$ ..... , then it is the solution set of the given inequalities.

Remark

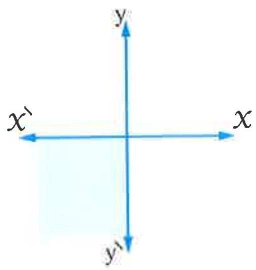
We can describe each quadrant of the four quadrants of the orthogonal cartesian plane by using a system of linear inequalities as the following :



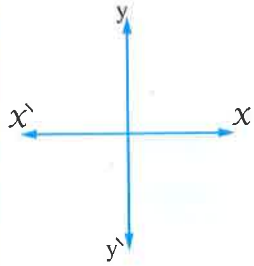
First quadrant :
$x > 0$, $y > 0$



Second quadrant :
$x < 0$, $y > 0$



Third quadrant :
$x < 0$, $y < 0$

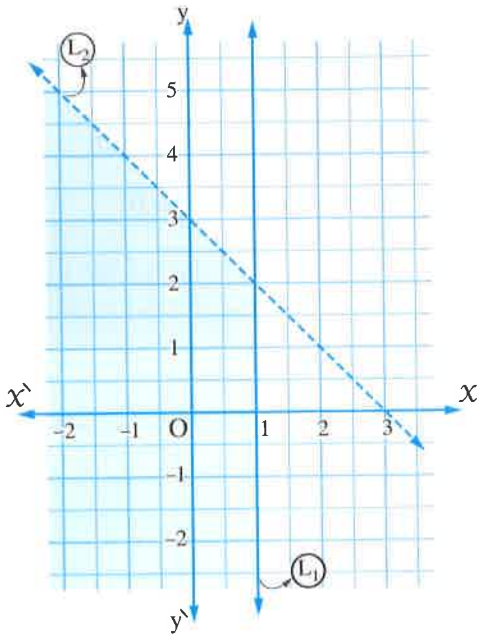


Fourth quadrant :
$x > 0$, $y < 0$

Example

Solve the following system of linear inequalities in $\mathbb{R} \times \mathbb{R}$ graphically : $x \leq 1$, $x + y < 3$

Solution



$L_1 : x = 1$ is parallel to the y-axis and intersects the x-axis at (1 , 0)

$L_2 : x + y = 3$ intersects the two axes at (3 , 0) and (0 , 3)

The solution set of the inequality :

$x \leq 1$ is S_1 where $S_1 = L_1 \cup$ the half plane in which the point (0 , 0) lies

, the solution set of the inequality : $x + y < 3$ is S_2

where S_2 = the half plane in which the point (0 , 0) lies

$\therefore$ The solution set = $S_1 \cap S_2$ and it is represented by the shaded region.

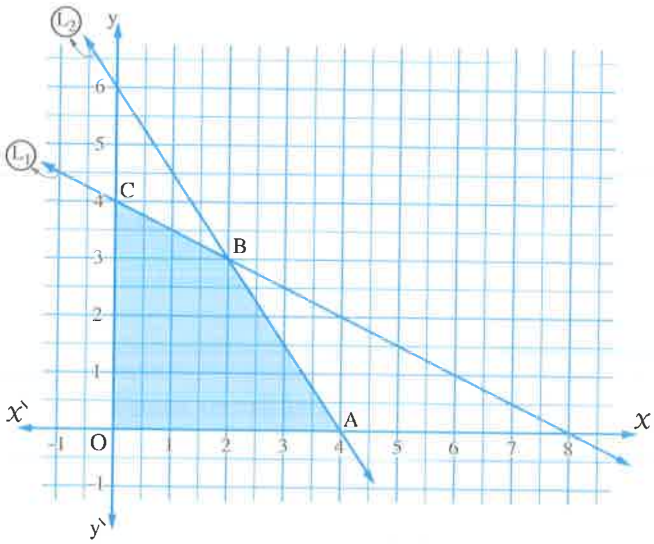
Linear programming and optimization

Linear programming is one of the scientific methods that is used to give the best decision of solving a problem that satisfies a certain object in view of some restrictions and available abilities and dependes on :

1. Representing the system of inequalities that expresses the stipulations.
2. Determining the objective function : $P = \ell X + m y$ where ℓ and m are constants.
3. Substituting by the resulting points from the intersections of the boundary lines that represent the solution set of systems of the inequalities , then chossing the point that satisfies the required.

For example :

If the opposite figure represents the solution set of the system of the inequalities : $X \geq 0$, $y \geq 0$, $X + 2y \leq 8$ and $3X + 2y \leq 12$, then to find the point (X , y) from the solution set that makes P maximum where $P = 50X + 75y$, we find :



$[P]_A = 50(4) + 75(0) = 200$

$[P]_B = 50(2) + 75(3) = 325$

$[P]_C = 50(0) + 75(4) = 300$

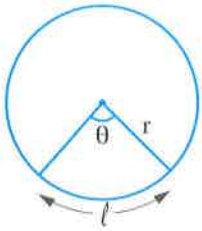
$[P]_O = 50(0) + 75(0) = 0$

$\therefore$ The maximum value of the objective function is 325 at the point B (2 , 3)

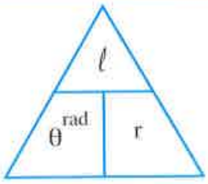
Second : Final Revision on Trigonometry

Remember **The relation between the radian measure and the degree measure of the angle**

If θ^{rad} and x° are the radian measure and the degree measure of a central angle in a circle of radius length r subtends an arc of length ℓ , then :



$$\theta^{rad} = \frac{\ell}{r} \text{ , then } \begin{cases} \ell = \theta^{rad}\, r \\ r = \dfrac{\ell}{\theta^{rad}} \end{cases}$$



Notice that

To convert from the radian measure (θ^{rad}) of the angle to the degree measure (x°) and vice versa , we use the relation :

$$\boxed{\frac{\theta^{rad}}{\pi} = \frac{x^\circ}{180^\circ}} \text{ , then } \begin{cases} x^\circ = \theta^{rad} \times \dfrac{180^\circ}{\pi} \\ \text{(to convert from the radian measure to the degree measure)} \\ \theta^{rad} = x^\circ \times \dfrac{\pi}{180^\circ} \\ \text{(to convert from the degree measure to the radian measure)} \end{cases}$$

Remember **The trigonometric identities**

If the terminal side of the directed angle of measure θ in the standard position intersects the unit circle at the point (x , y) , then :

$\boxed{\sin\theta = y}$, $\boxed{\cos\theta = x}$, $\boxed{\tan\theta = \frac{y}{x}}$

Reciprocals of the trigonometric functions

$\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$, $\cos\theta = \frac{1}{\sec\theta}$

$\sin\theta = \frac{1}{\csc\theta}$, $\tan\theta = \frac{1}{\cot\theta}$

The relation between θ and – θ

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$, $\csc(-\theta) = -\csc\theta$

$\sec(-\theta) = \sec\theta$, $\cot(-\theta) = -\cot\theta$

Trigonometry

Pythagorian identity

- $\sin^2\theta + \cos^2\theta = 1$

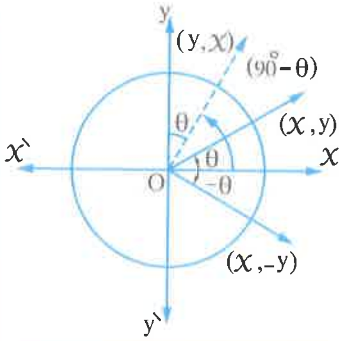
, then $\begin{cases} \sin^2\theta = 1 - \cos^2\theta \\ \cos^2\theta = 1 - \sin^2\theta \end{cases}$

- $1 + \tan^2\theta = \sec^2\theta$

, then $\begin{cases} \sec^2\theta - \tan^2\theta = 1 \\ \sec^2\theta - 1 = \tan^2\theta \end{cases}$

- $\cot^2\theta + 1 = \csc^2\theta$

, then $\begin{cases} \csc^2\theta - \cot^2\theta = 1 \\ \csc^2\theta - 1 = \cot^2\theta \end{cases}$



The relation between tan θ , sin θ and cos θ

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The relation between θ and (90° − θ)

- $\sin(90^\circ - \theta) = \cos\theta$
- $\cos(90^\circ - \theta) = \sin\theta$
- $\tan(90^\circ - \theta) = \cot\theta$
- $\csc(90^\circ - \theta) = \sec\theta$
- $\sec(90^\circ - \theta) = \csc\theta$
- $\cot(90^\circ - \theta) = \tan\theta$

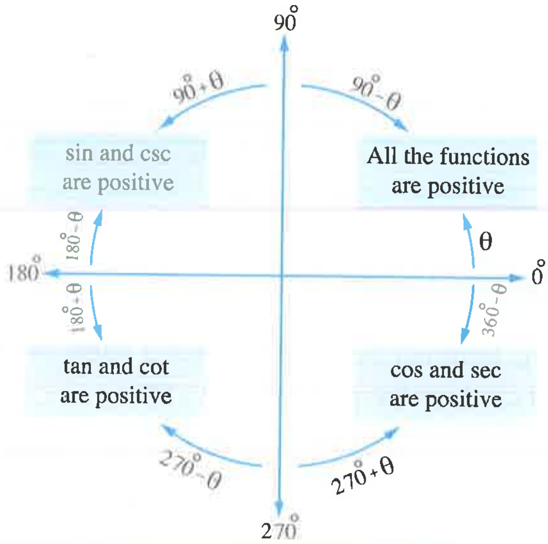
Remark

The relations between the trigonometric functions of the related angles are identities , and we can remember them from the opposite figure.

For example :

$\sin(90^\circ + \theta) = \cos\theta$

, $\tan(360^\circ - \theta) = -\tan\theta$, ... each of them is a trigonometric identity.



Remember **The general solution of the trigonometric equation**

If β is the smallest positive measure satisfying the equation , $n \in \mathbb{Z}$, then :

The general solution of the equation : $\cos\theta = a$ is $\theta = \pm\beta + 2\pi n$

For example :

If $\cos\theta = \frac{1}{2}$ (Positive)

$\therefore$ The smallest positive measure satisfying the equation is

$\beta = 60°$ (first quadrant)

$\therefore$ The general solution is $\theta = \pm\frac{\pi}{3} + 2\pi n$

The general solution of the equation : $\sin\theta = a$
is $\theta = \beta + 2\pi n$, $\theta = (\pi - \beta) + 2\pi n$

For example :

If $\sin\theta = \frac{-1}{2}$ (negative)

$\therefore$ The smallest positive measure satisfying the equation is $\beta = 180° + 30° = 210°$ (third quadrant)

$\therefore$ The general solution is $\theta = \frac{7}{6}\pi + 2\pi n$

, $\theta = \left(\pi - \frac{7}{6}\pi\right) + 2\pi n = -\frac{1}{6}\pi + 2\pi n$

The general solution of the equation : $\tan\theta = a$ is $\theta = \beta + \pi n$

For example :

If $\tan\theta = -1$ (negative)

$\therefore$ The smallest positive measure satisfying the equation is

$\beta = 180° - 45° = 135°$ (Second quadrant)

$\therefore$ The general solution is $\theta = \frac{3}{4}\pi + \pi n$

The general solution of the trigonometric equations of the quadrantal angles

$\sin\theta = 0$	its general solution is : $\theta = \pi n$
$\sin\theta = 1$	its general solution is : $\theta = \frac{\pi}{2} + 2\pi n$
$\sin\theta = -1$	its general solution is : $\theta = \frac{3\pi}{2} + 2\pi n$
$\cos\theta = 0$	its general solution is : $\theta = \frac{\pi}{2} + \pi n$
$\cos\theta = 1$	its general solution is : $\theta = 2\pi n$
$\cos\theta = -1$	its general solution is : $\theta = \pi + 2\pi n$

Remarks

1. $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all real values of θ

So , we find that the two equations : $\sin\theta = a$, $\cos\theta = a$ don't have solution in $\mathbb{R}$, if $a \notin [-1 , 1]$

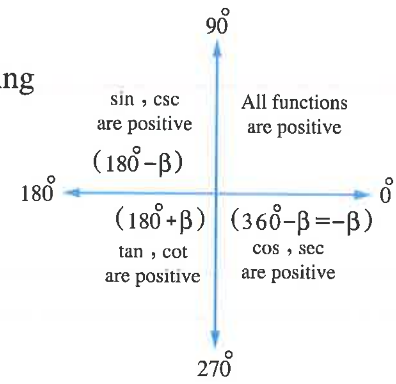
2. **To find the general solution of the trigonometric equation in the form :**

$\cos\theta = a$, $\sin\theta = a$ **or** $\tan\theta = a$ **, on an interval follow the following steps :**

(1) Let β be the measure of the acute angle which satisfies the equation :

$\cos\theta = |a|$, $\sin\theta = |a|$ or $\tan\theta = |a|$

(2) Determine the quadrant in which the angle lies according to the sign of a "Look to the opposite figure" :



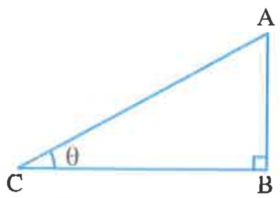
(3) Find the values of the angle θ where :

- If θ lies in the first quadrant , then $\theta = \beta$
- If θ lies in the second quadrant , then $\theta = 180° - \beta$
- If θ lies in the third quadrant , then $\theta = 180° + \beta$
- If θ lies in the fourth quadrant , then $\theta = 360° - \beta$

Remember Solving the right-angled triangle

Solving the right-angled triangle is evaluating the unknown measures of its angles and the unknown lengths of its sides.

To solve the right-angled triangle , we use



- **Pythagoras' theorem :**

$(AC)^2 = (AB)^2 + (BC)^2$

- **Trigonometric ratios :**

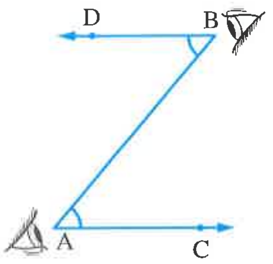
$\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{AB}{AC}$, $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{BC}{AC}$, $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{AB}{BC}$

Remember Angles of elevation and angles of depression

Angle of elevation

If a person looked from the point A to an object at the point B above his horizontal sight , then the included angle between the horizontal ray $\overrightarrow{AC}$ and the seeing ray to above $\overrightarrow{AB}$ is called the elevation angle of B with respect to A

i.e. ∠ CAB is the elevation angle of B with respect to A



Angle of depression

If a person looked from the point B to an object at the point A down his horizontal sight , then the included angle between the horizontal ray $\overrightarrow{BD}$ and the seeing ray to down $\overrightarrow{BA}$ is called the depression angle of A with respect to B

i.e. ∠ DBA is the depression angle of A with respect to B

Notice that

The measure of the elevation angle of B with respect to A = the measure of the depression angle of A with respect to B

i.e. m (∠ CAB) = m (∠ ABD)

Remember Areas of some geometric figures

Triangle

- The area of the triangle = $\frac{1}{2}$ length of the base × height

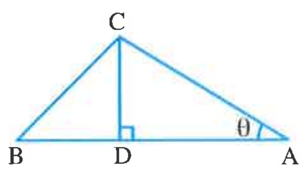
  $= \frac{1}{2}$ AB × CD

- The area of the triangle = $\frac{1}{2}$ the product of the lengths of two sides × sine of the included angle between them

  $= \frac{1}{2}$ AB × AC × sin θ

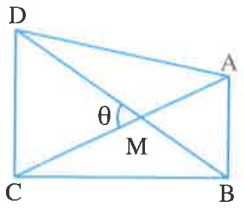
- The area of the triangle = $\sqrt{S(S-AB)(S-BC)(S-AC)}$

  where S equals half of the perimeter of the triangle ABC



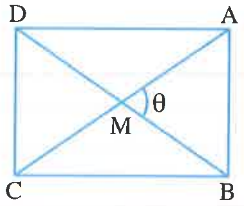
Notice that : The area of the equilateral triangle $= \frac{\sqrt{3}}{4}\ell^2$ where ℓ is its side length.

Quadrilateral



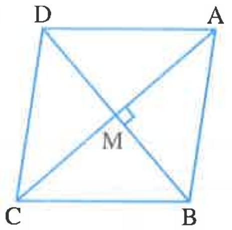
- The area of the quadrilateral $= \frac{1}{2}$ product of the lengths of its diagonals × sine of the included angle between them $= \frac{1}{2} AC \times BD \times \sin\theta$

Rectangle



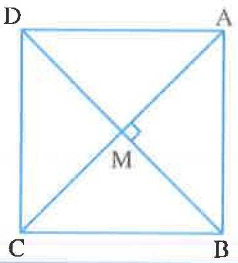
- The area of the rectangle = length × width = BC × AB
- The area of the rectangle $= \frac{1}{2}$ the square of its diagonal length × sine of the included angle between the two diagonals $= \frac{1}{2} (AC)^2 \times \sin\theta$

Rhombus



- The area of the rhombus $= \frac{1}{2}$ the product of its diagonal lengths $= \frac{1}{2} AC \times BD$

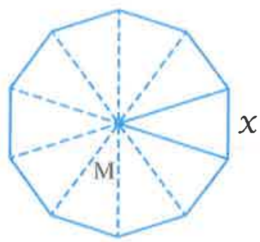
Square



- The area of the square = the square of its side length $= (AB)^2$
- The area of the square $= \frac{1}{2}$ the square of its diagonal length $= \frac{1}{2} (AC)^2$

Regular polygon

- It is a polygon in which all interior angles are equal in measure and all sides are equal in length.
- The area of the regular polygon in which the number of its sides is n sides and the length of its side is $x = \frac{1}{4} n x^2 \cot \frac{\pi}{n}$

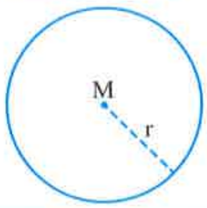


Notice that

1. The measure of vertex angle of a regular polygon in which the number of its sides is n sides $= \frac{(n-2) \times 180°}{n}$
2. We can divide the regular polygon in which the number of its sides is n sides into a number n of the congruent isosceles triangles and the measure of the vertex angle of each of them $= \frac{2\pi}{n}$

The circle and its parts (circular sector and circular segment)

The circle



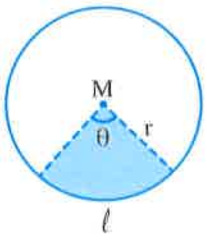
The area of the circle = πr^2

where r is its radius length

Notice that :

The circumference of the circle = $2 \pi r$

The circular sector



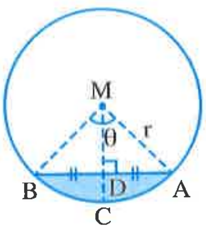
The area of the circular sector = $\frac{1}{2} \ell r$

$= \frac{1}{2} \theta^{rad} r^2 = \frac{x^\circ}{360^\circ} \times \pi r^2$

Notice that :

- x° and θ^{rad} are the degree measure and the radian measure of the angle of the sector.
- The perimeter of the sector = $2 r + \ell$

The circular segment



The area of the circular segment = $\frac{1}{2} r^2 (\theta^{rad} - \sin \theta)$

Notice that :

- If $\overline{MC} \perp \overline{AB}$, then the length of $\overline{CD}$ is the height of the circular segment.
- The perimeter of the circular segment
  = the length of its arc + the length of its chord.

Third : Final Revision on Analytic Geometry

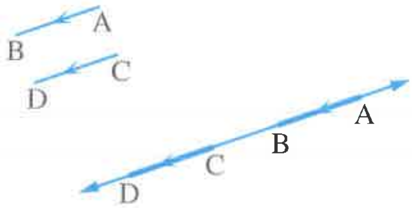
Remember Vectors

The directed line segment

It is a line segment which has a starting point , an ending point and a direction from the starting point to the ending point.

- $\overrightarrow{AB} \neq \overrightarrow{BA}$ (because they are different in the direction) but $\overrightarrow{AB} = -\overrightarrow{BA}$
- The norm of $\overrightarrow{AB}$ is denoted by $\|\overrightarrow{AB}\|$ = the length of $\overline{AB}$
- $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\|$

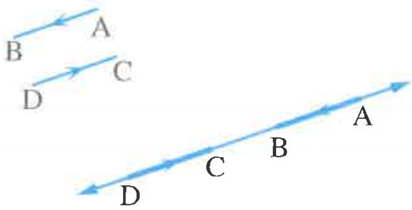
Equivalent two directed line segments



$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ $\left(\overrightarrow{AB} = \overrightarrow{CD}\right)$

If the following two conditions verified :

1. $\|\overrightarrow{AB}\| = \|\overrightarrow{CD}\|$
2. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction.



$\overrightarrow{AB} = -\overrightarrow{DC}$

If the following two conditions verified :

1. $\|\overrightarrow{AB}\| = \|\overrightarrow{DC}\|$
2. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are in opposite directions.

Notice that

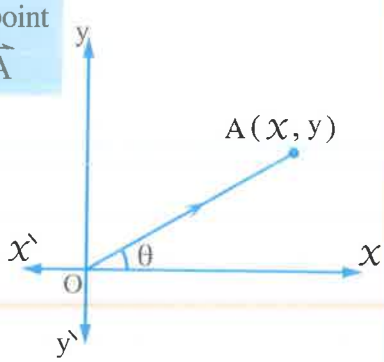
$\overrightarrow{AB}$ and $\overrightarrow{CD}$ can not be in the same direction or in opposite directions unless one straight line carries them or two parallel straight lines.

Remember the position vector

The position vector of a given point A with respect to the origin point is the directed line segment $\overrightarrow{OA}$ and it is denoted by the symbol $\vec{A}$

★ If $\overrightarrow{OA}$ is the position vector of the point A (x , y) , then :

- $\|\vec{A}\|$ = the length of $\overline{OA} = \sqrt{x^2 + y^2}$
- If $\|\vec{A}\| = 1$ (the unit) , then $\vec{A}$ is called the unit vector.
- $\vec{i} = (1 , 0)$ and $\vec{j} = (0 , 1)$ are the fundamental unit vectors in the two coordinates axes.



- $\overrightarrow{O} = (0 , 0)$ is the zero vector and it has no direction and sometimes denoted by the symbol $\overrightarrow{O}$
- If $\overrightarrow{OA}$ is the position vector of the point $A(x , y)$ and θ is the measure of the positive angle which $\overrightarrow{OA}$ makes with the positive direction of x-axis , then there are three forms for $\overrightarrow{A}$:

1 Cartesian form	2 Polar form	3 In terms of $\overrightarrow{i}$ and $\overrightarrow{j}$
$\overrightarrow{A} = (x , y)$	$\overrightarrow{A} = (\Vert\overrightarrow{A}\Vert , \theta)$	$\overrightarrow{A} = x\overrightarrow{i} + y\overrightarrow{j}$

Notice that

- $x = \Vert\overrightarrow{A}\Vert \cos\theta$, then $\cos\theta = \dfrac{x}{\Vert\overrightarrow{A}\Vert}$
- $y = \Vert\overrightarrow{A}\Vert \sin\theta$, then $\sin\theta = \dfrac{y}{\Vert\overrightarrow{A}\Vert}$

Example 1

Write the vector $\overrightarrow{A} = (3 , -\sqrt{3})$ in the polar form.

Solution

$\because \Vert\overrightarrow{A}\Vert = \sqrt{9+3} = 2\sqrt{3}$

$\therefore \cos\theta = \dfrac{x}{\Vert\overrightarrow{A}\Vert} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} > 0$

$, \sin\theta = \dfrac{y}{\Vert\overrightarrow{A}\Vert} = \dfrac{-\sqrt{3}}{2\sqrt{3}} = \dfrac{-1}{2} < 0$

$\therefore$ θ lies in the fourth quadrant.

$\therefore \theta = 360° - 30° = 330°$

$\therefore \overrightarrow{A} = (2\sqrt{3} , 330°)$

Example 2

Write in terms of the fundamental unit vectors $\overrightarrow{A} = (10 , 135°)$

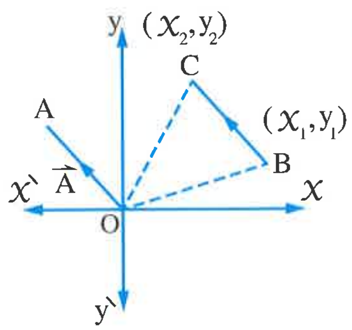
Solution

$\because x = \Vert\overrightarrow{A}\Vert \cos\theta = 10\cos 135° = -5\sqrt{2}$, $y = \Vert\overrightarrow{A}\Vert \sin\theta = 10 \sin 135° = 5\sqrt{2}$

$\therefore \overrightarrow{A} = (-5\sqrt{2} , 5\sqrt{2})$

$\therefore \overrightarrow{A} = -5\sqrt{2}\,\overrightarrow{i} + 5\sqrt{2}\,\overrightarrow{j}$

- If $B(x_1 , y_1)$ and $C(x_2 , y_2)$, then the position vector $\overrightarrow{OA}$ which is equivalent to the vector $\overrightarrow{BC}$ is gotten from the relation :



$\overrightarrow{OA} = \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

i.e. $\overrightarrow{A} = (x_2 , y_2) - (x_1 , y_1) = (x_2 - x_1 , y_2 - y_1)$

For example :

If $B = (3 , 1)$ and $C = (2 , 5)$, then :

$\overrightarrow{BC} = \overrightarrow{C} - \overrightarrow{B} = (2 , 5) - (3 , 1) = (-1 , 4)$

i.e. The equivalent position vector to $\overrightarrow{BC} = (-1 , 4)$

Analytic Geometry

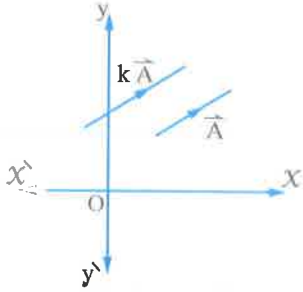
Remember — Operations on vectors

Multiplying a vector by a real number

If $\overrightarrow{A} = (x , y) \in \mathbb{R}^2$, $k \in \mathbb{R}$

, then :

$k\overrightarrow{A} = k(x , y) = (kx , ky)$

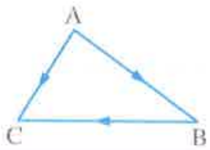


Notice that

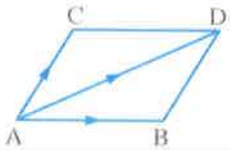
1. If $\overrightarrow{A} \,//\, k\overrightarrow{A}$ and
   - $k > 0$ then → $\overrightarrow{A}$ and $k\overrightarrow{A}$ have the same direction
   - $k < 0$ then → $\overrightarrow{A}$ and $k\overrightarrow{A}$ are in opposite directions
2. $\| k\overrightarrow{A} \| = |k| \, \| \overrightarrow{A} \|$

Adding and subtracting vectors

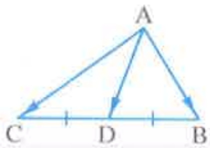
Geometrically



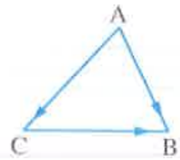
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$



$\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$



$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$

Algebraically

For any two vectors $\overrightarrow{A} = (x_1 , y_1)$ and $\overrightarrow{B} = (x_2 , y_2)$

$\overrightarrow{A} + \overrightarrow{B} = (x_1 + x_2 , y_1 + y_2)$

Notice that

- $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$
- $\left(\overrightarrow{A} + \overrightarrow{B}\right) + \overrightarrow{C} = \overrightarrow{A} + \left(\overrightarrow{B} + \overrightarrow{C}\right) = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$
- $\overrightarrow{A} + \left(-\overrightarrow{A}\right) = \overrightarrow{O}$
- $\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{A}$
- If $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{A} + \overrightarrow{C}$, then $\overrightarrow{B} = \overrightarrow{C}$
- $\overrightarrow{A} - \overrightarrow{B} = \overrightarrow{A} + \left(-\overrightarrow{B}\right)$

Parallel and perpendicular vectors

For any two non zero vectors $\overrightarrow{A} = (x_1 , y_1)$ and $\overrightarrow{B} = (x_2 , y_2)$

$\overrightarrow{A} // \overrightarrow{B}$

if :

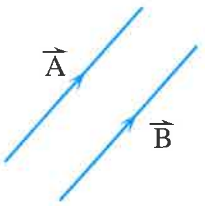
The slope of $\overrightarrow{A}$

= the slope of $\overrightarrow{B}$

i.e. $\frac{y_1}{x_1} = \frac{y_2}{x_2}$

i.e. $\boxed{x_1 y_2 - x_2 y_1 = 0}$

and vice versa.



$\overrightarrow{A} \perp \overrightarrow{B}$

if :

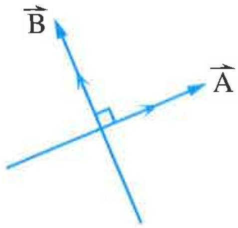
The slope of $\overrightarrow{A}$ ×

the slope of $\overrightarrow{B} = -1$

i.e. $\frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$

i.e. $\boxed{x_1 x_2 + y_1 y_2 = 0}$

and vice versa.



Remember Applications on vectors

Geometric applications

1. To prove that the points A , B and C are collinear using vectors , we prove that : $\overrightarrow{AB} = k\ \overrightarrow{AC}$ where k is constant.

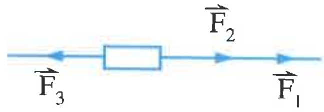
2. To prove that the quadrilateral ABCD in which $\overline{AC} \cap \overline{BD} = \{M\}$ is a parallelogram using vectors , we prove that :
   The two diagonals bisect each other
   i.e. We prove that : $\overrightarrow{AM} = \overrightarrow{MC}$, $\overrightarrow{BM} = \overrightarrow{MD}$
   or two opposite sides are parallel and equal in length
   i.e. We prove that : $\overrightarrow{AB} = \overrightarrow{DC}$ or $\overrightarrow{AD} = \overrightarrow{BC}$

3. To prove that the quadrilateral is a rectangle , rhombus or square , then we should prove first that the quadrilateral is a parallelogram as previous , then :
   - **To prove that the parallelogram is a rectangle , we prove one of the following properties :**
     (1) Two adjacent sides are perpendicular. *For example :* $\overrightarrow{AB} \perp \overrightarrow{BC}$
     (2) The two diagonals are equal in length. *For example :* $\| \overrightarrow{AC} \| = \| \overrightarrow{BD} \|$
   - **To prove that the parallelogram is a rhombus , we prove one of the following properties :**
     (1) Two adjacent sides are equal in length. *For example :* $\| \overrightarrow{AB} \| = \| \overrightarrow{BC} \|$
     (2) The two diagonals are perpendicular. *For example :* $\overrightarrow{AC} \perp \overrightarrow{BD}$
   - **To prove that the parallelogram is a square , we prove one of the properties of the rectangle and one of the properties of the rhombus together.**

Physical applications

The resultant force $\vec{F}$

The resultant force of some forces acting on an object are subjected to the operation of adding vectors.

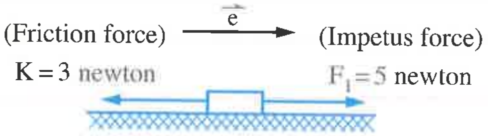


i.e. The resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$

For example :

If we considered the unit vector $\vec{e}$ in the direction of body motion , then in the case of :

Body motion on a rough plane



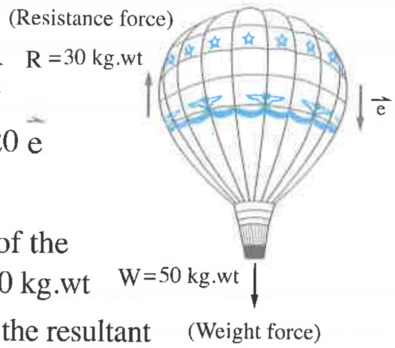
The resultant force $\vec{F} = 5\,\vec{e} + (-3)\,\vec{e} = 2\,\vec{e}$

i.e.

- Magnitude of the resultant = 2 newtons.
- Direction of the resultant is in the direction of body motion.

The vertical motion

The resultant force $\vec{F} = 50\,\vec{e} + (-30\,\vec{e}) = 20\,\vec{e}$



i.e.

- Magnitude of the resultant = 20 kg.wt
- Direction of the resultant is in the direction of weight body.

The relative velocity

If the actual velocity of the body A = $\vec{V}_A$ and the actual velocity of the body B = $\vec{V}_B$, then :

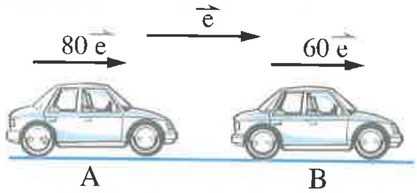
The relative velocity of the body B with respect to the body A = $\vec{V}_{BA}$ and :

$\boxed{\vec{V}_{BA} = \vec{V}_B - \vec{V}_A}$, $\boxed{\vec{V}_{AB} = \vec{V}_A - \vec{V}_B}$

For example :

If we considered the unit vector $\vec{e}$ in the direction of the motion of the car A , the velocity of the car A = 80 km./h. and the velocity of the car B = 60 km./h.

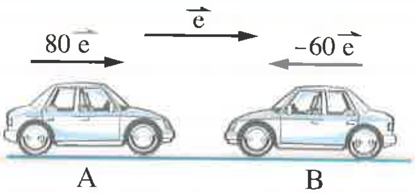
- If the two cars A and B move in the same direction , then $\vec{V}_{BA} = 60\,\vec{e} - 80\,\vec{e} = -20\,\vec{e}$



i.e.

The driver of the car A feels that the car B retreats with velocity 20 km./h.

- If the two cars A and B move in opposite directions , then $\vec{V}_{BA} = -60\,\vec{e} - 80\,\vec{e} = -140\,\vec{e}$

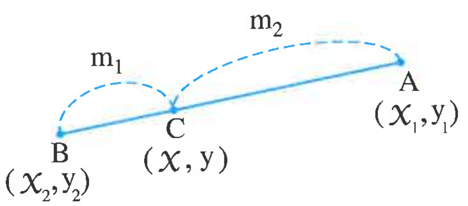


i.e.

The driver of the car A feels that the car B moves in the opposite of its direction with velocity 140 km./h.

Remember Division of a line segment

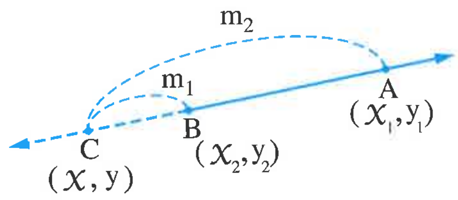
Internally division



If $C \in \overline{AB}$

, then C divides $\overrightarrow{AB}$ internally.

Externally division



If $C \in \overleftrightarrow{AB}$, $C \notin \overline{AB}$

, then C divides $\overrightarrow{AB}$ externally.

And we can find the coordinates of the division point internally or externally by using :

- **The vector form :** $\boxed{\vec{r} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}}$

where $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ are the position vectors $\overrightarrow{OC}$, $\overrightarrow{OA}$ and $\overrightarrow{OB}$

for the points C , A and B respectively.

- **The cartesian form :** $\boxed{(x , y) = \left(\dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} , \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2}\right)}$

Remarks

1. In the case of internally division , the two values m_1 and m_2 are positive.

   i.e. $\dfrac{m_2}{m_1} > 0$

2. In the case of externally division , one of the two values m_1 and m_2 is positive and the other is negative.

   i.e. $\dfrac{m_2}{m_1} < 0$

3. If C is the midpoint of $\overline{AB}$, then $\vec{r} = \dfrac{\vec{r}_1 + \vec{r}_2}{2}$ (the vector form)

   , $(x , y) = \left(\dfrac{x_1 + x_2}{2} , \dfrac{y_1 + y_2}{2}\right)$ (the cartesian form)

Remember The slope of the straight line

If the straight line L passes through the two points (x_1 , y_1) and (x_2 , y_2)

, then its slope (m) $= \dfrac{y_2 - y_1}{x_2 - x_1}$

For example :

The straight line which passes through the two points (1 , 3) and (4 , 2)

, its slope (m) $= \frac{2-3}{4-1} = \frac{-1}{3}$

If θ is the measure of the positive angle which the straight line L makes with the positive direction of X-axis , then its slope (m) = tan θ

For example :

If the straight line L makes a positive angle of measure 135° with the positive direction of X-axis , then its slope (m) = tan 135° = – 1

If $\vec{u} = (a , b)$ is a direction vector of the straight line L , then its slope (m) $= \frac{b}{a}$

For example :

If $\vec{u} = (2 , -3)$ is a direction vector of the straight line L , then its slope (m) $= \frac{-3}{2}$

If the equation of the straight line L is in the form :

$a X + b y + c = 0$, then its slope (m) $= \frac{-a}{b}$

For example :

The straight line whose equation is $2X - 3y + 1 = 0$, its slope (m) $= \frac{-2}{-3} = \frac{2}{3}$

If the equation of the straight line L is in the form : $y = bX + c$, then its slope (m) = b

For example :

The straight line whose equation is $y = \frac{-1}{2}X + 5$, its slope (m) $= \frac{-1}{2}$

Remarks

1. The slope of X-axis and the slope of any horizontal straight line (parallel to X-axis) are equal to zero.
2. The slope of y-axis and the slope of any vertical straight line (parallel to y-axis) are undefined.
3. If L_1 and L_2 are two straight lines of slopes m_1 and m_2 respectively , then :
   - $L_1 // L_2 \Leftrightarrow m_1 = m_2$

   i.e. The two parallel straight lines have equal slopes and vice versa.

   - $L_1 \perp L_2 \Leftrightarrow m_1 \times m_2 = -1$

   (unless one of them is parallel to one of the two coordinate axes)
4. If the slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{BC}$, then the points A , B and C are collinear.

Remember The direction vector of a straight line

1 The straight line whose slope $m = \frac{a}{b}$, then its direction vector $\overrightarrow{u} = (b , a)$

For example :

The straight line whose equation is $2X - 3y + 5 = 0$, its slope $(m) = \frac{-a}{b} = \frac{-2}{-3} = \frac{2}{3}$

, then its direction vector $\overrightarrow{u} = (3 , 2)$

2 The straight line which passes through the two points

$C(X_1 , y_1)$ and $D(X_2 , y_2)$

, then its direction vector $\overrightarrow{u} = \overrightarrow{CD} = \overrightarrow{D} - \overrightarrow{C} = (X_2 - X_1 , y_2 - y_1)$

Remarks

1 If the direction vector of the straight line L is $\overrightarrow{u} = (a , b)$, then the direction vector of the perpendicular straight line to the straight line L is $\overrightarrow{N} = (-b , a)$ or $(b , -a)$

2 The direction vector of any straight line parallel to X-axis $\overrightarrow{i} = (1 , 0)$

3 The direction vector of any straight line parallel to y-axis $\overrightarrow{j} = (0 , 1)$

Remember The different forms of the equation of the straight line

1 The different forms of the equation of the straight line which passes through the point $A(X_1 , y_1)$ and its direction vector $\overrightarrow{u} = (a , b)$

The vector equation

$\overrightarrow{r} = \overrightarrow{A} + k\overrightarrow{u}$

i.e.

$(X , y) = (X_1 , y_1) + k(a , b)$

The parametric equations

$X = X_1 + k\,a$

$, y = y_1 + k\,b$

The cartesian equation

$\frac{y - y_1}{X - X_1} = m$

2 The different forms of the equation of the straight line which passes through the two points $P(X_1 , y_1)$ and $N(X_2 , y_2)$

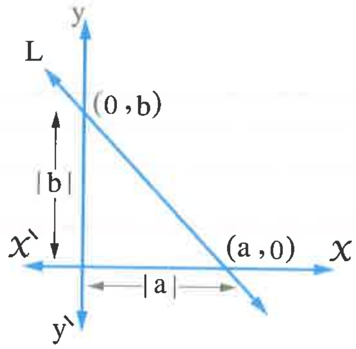
i.e.

its direction vector $\overrightarrow{u} = \overrightarrow{N} - \overrightarrow{P} = (X_2 - X_1 , y_2 - y_1)$

Analytic Geometry

The vector equation	The parametric equations	The cartesian equation
$\vec{r} = \vec{P} + k(\vec{N} - \vec{P})$ *i.e.* $(x, y) = (x_1, y_1) + k(x_2 - x_1, y_2 - y_1)$	$x = x_1 + k(x_2 - x_1)$ $y = y_1 + k(y_2 - y_1)$	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

3. The cartesian equation of the straight line whose slope is m and intercepts from y-axis a part of length $|b|$ (*i.e.* intersects y-axis at the point $(0, b)$) is $y = mx + b$



4. The cartesian equation of the straight line which intersects x-axis at $(a, 0)$ and y-axis at $(0, b)$ (*i.e.* intersects from the two axes two parts of lengths $|a|$ and $|b|$) is $\frac{x}{a} + \frac{y}{b} = 1$

5. If the two straight lines $L_1 : a_1 x + b_1 y + c_1 = 0$ and $L_2 : a_2 x + b_2 y + c_2 = 0$ intersect at a point , then the general equation of any straight line passing through the point of intersection of L_1 and L_2 other than L_1 and L_2 is $a_1 x + b_1 y + c_1 + k(a_2 x + b_2 y + c_2) = 0$

Where $k \neq 0$

Remarks

1. The equation of the straight line which passes through the origin point O $(0, 0)$ is :
   - The vector equation is : $\vec{r} = k\,\vec{u}$, where $\vec{u}$ is the direction vector of the straight line.
   - The cartesian equation is : $y = mx$, where m is the slope of the straight line.
2. The straight line which is parallel to x-axis and passes through the point (x_1, y_1)
   - The vector equation is : $\vec{r} = (x_1, y_1) + k(1, 0)$
   - The cartesian equation is : $y = y_1$
3. The straight line which is parallel to y-axis and passes through the point (x_1, y_1)
   - The vector equation is : $\vec{r} = (x_1, y_1) + k(0, 1)$
   - The cartesian equation is : $x = x_1$

Remember The measure of the angle between two straight lines

If θ is the measure of the included angle between the two straight lines L_1 and L_2 whose slopes are m_1 and m_2 , then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ where $\theta \in \left[0 , \frac{\pi}{2}\right]$

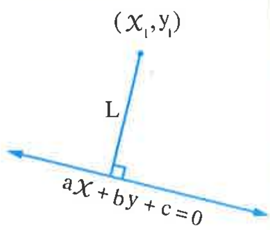
With noticing the following :

1. If the tangent is positive , then we obtain an acute angle.
2. If the tangent is zero , then the measure of the included angle = zero , then $m_1 = m_2$ and the two straight lines are parallel or coincident.
3. If the tangent is undefined , then the measure of the included angle is 90° , then $m_1 m_2 = -1$ and the two straight lines are orthogonal (perpendicular).
4. The measure of the obtuse angle = the measure of the supplementary angle of the acute angle.

Remember The length of the perpendicular from a point to a straight line

- The length of the perpendicular (L) drawn from the point (x_1 , y_1) to the straight line whose equation is : $ax + by + c = 0$ is determined by the relation :

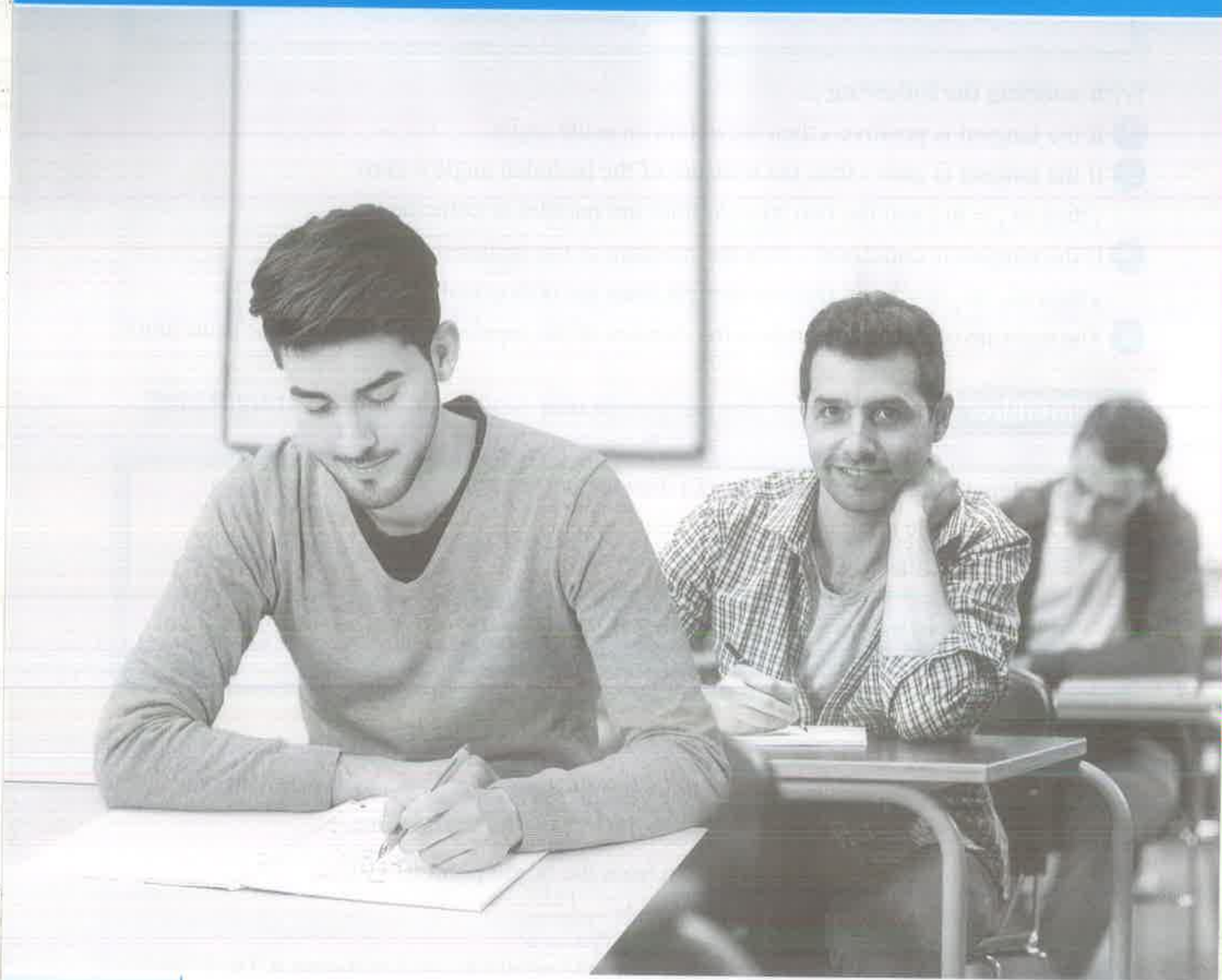
The length of the perpendicular (L) $= \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$



Remarks

1. If the length of the perpendicular drawn from the point (x_1 , y_1) to the straight line $ax + by + c = 0$ equals zero , then the point lies on the straight line.
2. The length of the perpendicular drawn from the origin point (0 , 0) to the straight line : $ax + by + c = 0$ equals $\frac{|c|}{\sqrt{a^2 + b^2}}$
3. The length of the perpendicular drawn from the point (x_1 , y_1) to x-axis = $|y_1|$
4. The length of the perpendicular drawn from the point (x_1 , y_1) to y-axis = $|x_1|$
5. If (x_1 , y_1) and (x_2 , y_2) are two points in the Cartesian plane which contains the straight line : $ax + by + c = 0$ and the two expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign , then the two points (x_1 , y_1) and (x_2 , y_2) are on the same side of the straight line , and if they have different signs , then the two points are in two different sides of the straight line.

SCHOOL BOOK EXAMINATIONS



First : School book examinations in algebra and trigonometry

Second : School book examinations in analytic geometry

First : School book examinations in algebra and trigonometry

Model 1

Answer the following questions :

1 Choose the correct answer from the given answers :

(1) The point which belongs to the solution set of the inequalities :

$x > 2$, $y > 1$, $x + y \geq 3$ is ....

(a) (2 , 1) (b) (1 , 2) (c) (3 , 2) (d) (1 , 3)

(2) If A is a matrix of order 1×3 , B^t is a matrix of order 1×3 , then it is possible to carry out the operation ....

(a) A + B (b) $B^t + A^t$ (c) AB^t (d) AB

(3) The solution set of the equations : $2x - 3y = 1$, $3x + 2y = 8$ is ....

(a) {(1 , 2)} (b) {(2 , 1)} (c) {(2 , 3)} (d) {(3 , 2)}

(4) The perimeter of a circular sector is 10 cm. , and the length of its arc equals 2 cm. , then its area in square centimetres equals ....

(a) 4 (b) 8 (c) 10 (d) 20

(5) The solution set of the equation : $\sin x + \cos x = 0$ where $180° < x < 360°$ equals ....

(a) {210°} (b) {225°} (c) {240°} (d) {315°}

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2 [a] Solve the system of the following linear equations using matrices :

$2x - 3y = 4$, $3x + 4y = 23$

[b] Prove the identity : $\sin\theta \sin(90° - \theta) \tan\theta = 1 - \cos^2\theta$

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3 [a] Find the area of the triangle whose vertices are (– 4 , 2) , (3 , 1) , (– 2 , 5) using matrices.

[b] Find the solution set of the equation : $2 \sin x + 1 = 0$ where $x \in]0 , 2\pi[$

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4 [a] Find the values of x which satisfy the equation : $\begin{vmatrix} x & 0 & 0 \\ 1 & x & x \\ 5 & 2 & x \end{vmatrix} = 3x$

[b] A boat was observed from the top of a lighthouse of height 50 metres , it was found that its depression angle is of measure 35°. Find the distance between the boat and the top of the lighthouse.

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5 [a] $\overline{AB}$ is a chord of length 8 cm. , is opposite to a central angle of measure 60°

Find to the nearest tenth the area of the minor circular segment whose chord is $\overline{AB}$

[b] **Determine the solution set of the following inequalities graphically in $\mathbb{R} \times \mathbb{R}$:**

$x \geq 0$, $y \geq 0$, $x + 3y \leq 7$, $3x + 4y \leq 14$

, then find from the solution set the values of x , y which make the value of the function : $P = 30x + 50y$ is greatest as possible.

Model (2)

Answer the following questions :

1 Choose the correct answer from the given answers :

(1) If A is a matrix of order 2×3 , B^t is a matrix of order 1×3 , then AB is a matrix of order ....

(a) 3×3 (b) 3×1 (c) 2×1 (d) 1×2

(2) The point which belongs to the solution set of the inequalities :

$x \geq 0$, $y \geq 0$, $2x + y < 4$, $x + 3y < 6$ is ....

(a) (1 , – 3) (b) (3 , 0) (c) (2 , 3) (d) (1 , 1)

(3) If $\begin{vmatrix} 2x & 2 \\ 4 & 3 \end{vmatrix} = 10$, then $x =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(4) $1 + \cot^2 \theta =$ .... in the simplest form.

(a) $\sin^2 \theta$ (b) $\cos^2 \theta$ (c) $\sec^2 \theta$ (d) $\csc^2 \theta$

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2 [a] **Solve the system of the following linear equations using Cramar's rule :**

$2x - 3y = 3$, $x + 2y = 5$

[b] **Prove the identity :** $\frac{\cos x \times \tan x}{\csc x} = 1 - \cos^2 x$

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3 [a] **Find the matrix A which satisfies the relation :**

$$A \times \begin{pmatrix} -2 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 23 \\ 8 & 13 \end{pmatrix}$$

[b] **Find the general solution of the equation :** $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$

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4 [a] If $A^t = \begin{pmatrix} 2 & -4 \\ 4 & 3 \end{pmatrix}$, **prove that :** $A^2 - 5A + 22I = O$

[b] The measure of the central angle of a circular segment is 90° and the area of its surface is 56 cm². Find the length of its radius.

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5 [a] From a point on the ground 50 metres distant from the base of a vertical pole , it is found that the measure of the elevation angle of the top of the pole is 19° 24`

Find to the nearest metre , the height of the pole from the ground.

[b] **Find the maximum value of the objective function : $P = 2x + y$ given that :**

$x \geq 0$, $y \geq 0$, $2x + 3y \leq 18$, $-4x + y \geq -8$

Second : School book examinations in analytic geometry

Model 1

Answer the following questions :

1 **Complete the following sentences :**

(1) If $\overrightarrow{A} = 2\,\overrightarrow{i} + 3\,\overrightarrow{j}$, $\overrightarrow{B} = 3\,\overrightarrow{i} - \overrightarrow{j}$, then $2\,\overrightarrow{A} - \overrightarrow{B} =$ ....

(2) If $\overrightarrow{A} = (-2 , 1)$, $\overrightarrow{C} = (-3 , k)$ are parallel , then $k =$ ....

(3) If $A = (-4 , 4)$, $B = (5 , -8)$, $C \in \overline{AB}$ where $AC : CB = 2 : 1$
, then $C = ($ .... , ... $)$

(4) If the two lines $L_1 : 3\,x - 2\,y + 7 = 0$, $L_2 : a\,x + 3\,y + 5 = 0$ are perpendicular
, then $a =$ ....

(5) The vector equation of the line which passes through the point $(2 , -3)$ and its direction vector is $(3 , 4)$, is ....

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2 [a] If $\| -8\,\overrightarrow{A} \| = 5\,\| k\,\overrightarrow{A} \|$, find the value of k

[b] Find the length of the perpendicular drawn from the point $(1 , 2)$ to the line whose equation is : $5\,x - 12\,y - 7 = 0$

---

3 [a] ABCD is a quadrilateral , E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{CD}$

Prove that : $\overrightarrow{BC} + \overrightarrow{AD} = 2\,\overrightarrow{EF}$

[b] Find the equation of the line which passes through the point of intersection of the two lines whose equations are : $2\,x + y = 5$ and $\overrightarrow{r} = (1 , 0) + t\,(1 , 1)$
and passes through the point $(5 , 3)$

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4 [a] If the point $C\,(2 , 5)$ divides $\overrightarrow{AB}$ by the ratio $4 : 1$, where $A\,(8 , 3)$
, find the coordinates of the point B

[b] Prove that the triangle whose vertices are the points $Y\,(4 , 2)$, $X\,(3 , 5)$, $Z\,(-5 , -1)$ is a right-angled triangle at Y , then calculate the area of the circle which passes through its vertices.

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5 If $L_1 : 3\,x + 2\,y - 7 = 0$, $L_2 : 2\,x - 3\,y + 4 = 0$, **find :**

(1) The measure of the angle between L_1 , L_2

(2) The vector equation of the line which passes through the point of intersection of the two lines L_1 , L_2 and the point $(3 , 4)$

Model 2

Answer the following questions :

1 **Complete the following sentences :**

(1) If $\overrightarrow{A} = (2 , 3)$, $\overrightarrow{B} = (-1 , 2)$, then $\overrightarrow{AB} =$ ....

(2) If $\overrightarrow{A} = (4 , 2)$, $\overrightarrow{B} = (1 , -2)$, then $\| \overrightarrow{A} - \overrightarrow{B} \| =$ ....

(3) If $A = (-3 , 4)$, $B = (6 , -8)$, then the X-axis divides $\overline{AB}$ by the ratio ....

(4) The measure of the angle between the two lines whose slopes are $\frac{1}{2} , -2$ equals ....

(5) The length of the perpendicular from $(1 , 1)$ to the line whose equation is : $X + y = 0$ equals ....

2 [a] If $k \| 4 \overrightarrow{A} \| = \| -3 \overrightarrow{A} \|$, find the value of k

[b] Find the equation of the line which passes through the point $(-1 , 0)$, and the point of intersection of the two lines whose equations are : $2X - y + 4 = 0$, $X + y + 5 = 0$

3 [a] If $A = (3 , 4)$, $B = (5 , -1)$, $C = (2 , -2)$ are three vertices of the parallelogram ABCD , find the coordinates of the fourth vertex D

[b] Prove that the two lines : $\overrightarrow{r} = (0 , 4) + t (1 , -2)$, $2X + y + 2 = 0$ are parallel , then find the shortest distance between them.

4 [a] If $A = (-1 , 4)$, $B = (5 , -1)$, find the coordinates of the point C which divides $\overline{AB}$ internally by the ratio 2 : 1

[b] A circle whose centre is the origin point, prove that the two chords drawn in the circle whose equations are :

$3X + 4y + 10 = 0$ and $5X - 12y + 26 = 0$ are equal in length.

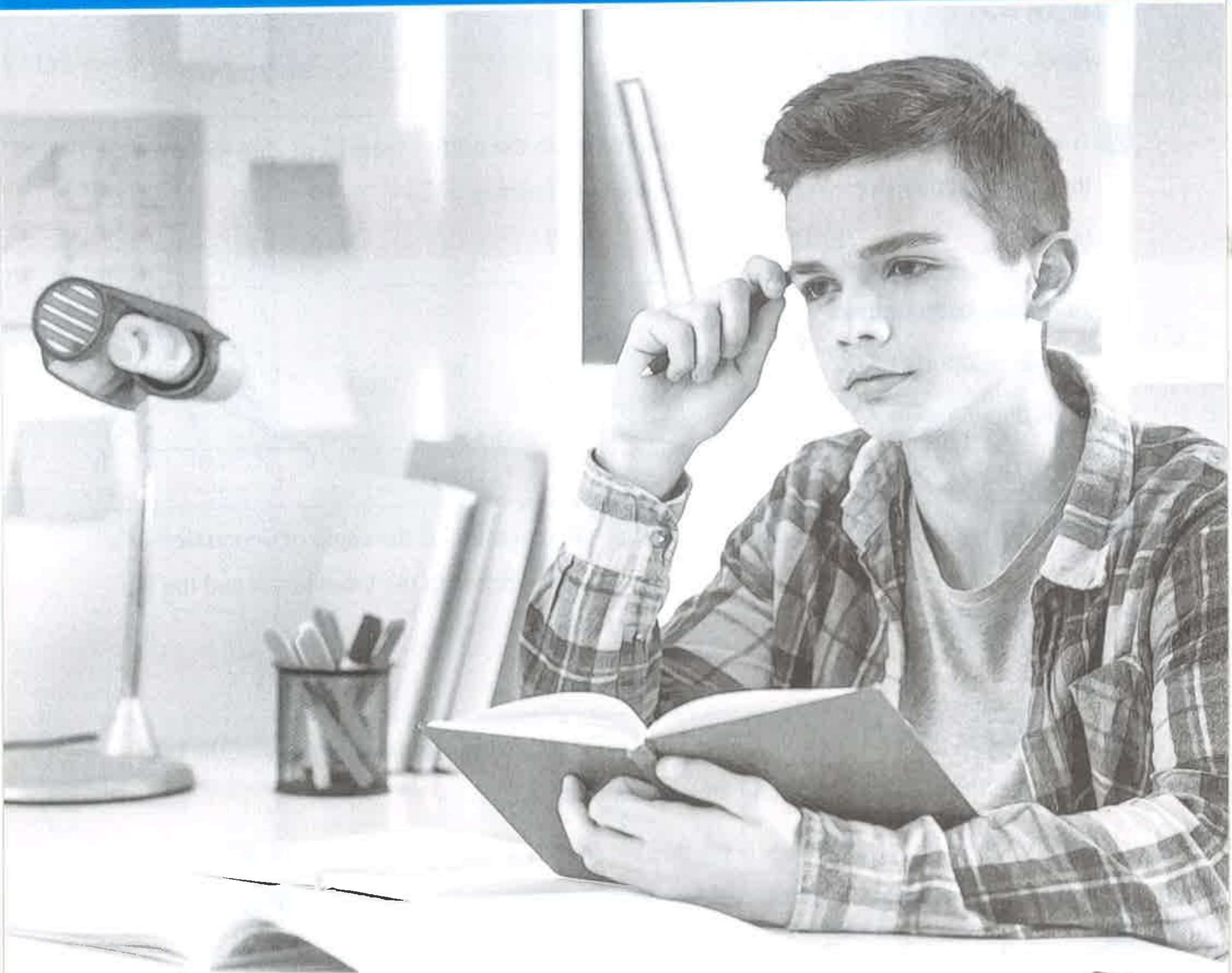
5 ABCD is a trapezium in which $\overline{AD} // \overline{BC}$

If $A (7 , -1)$, $B (3 , -1)$, $C (2 , 1)$, $D (5 , y)$:

(1) Find the value of y

(2) Find the area of the trapezium ABCD

FINAL EXAMINATION MODELS



Scan the QR codes to solve interactive tests

Model 1

Interactive test 1

Answer the following questions :

1 If $\overrightarrow{AB} = 3\,\vec{i} + 3\,\vec{j}$, $\overrightarrow{BC} = \vec{j}$, then $\|\overrightarrow{AC}\| =$ ....

(a) 6 (b) $3\sqrt{2}$ (c) 1 (d) 5

---

2 A cyclist covers 5 m. from a fixed point (O) due to the north , then 12 m. due to the east , then the total distance covered during the whole Journey = ... m.

(a) 13 (b) 12 (c) 7 (d) 17

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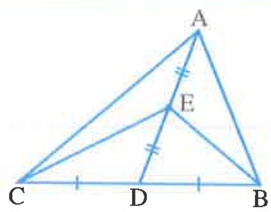
3 **In the opposite figure :**



D is the midpoint of $\overline{BC}$

, E is the midpoint $\overline{AD}$

Prove that : $\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{EB} + 2\,\overrightarrow{EC}$

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4 From the top of a light house 80 metres high , the measure of the angle of depression of a fixed target on the see equals 80° , then the distance between the fixed target and the top of the light house equals .... to nearest metre.

(a) 78 (b) 79 (c) 80 (d) 81

---

5 The equation of the straight line which passes through the intersection point of the two lines $x + 5 = 0$, $y - 3 = 0$ and the origin point is ....

(a) $5x - 3y = 0$ (b) $5x + 3y = 0$ (c) $3x - 5y = 0$ (d) $3x + 5y = 0$

---

6 If $B = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, $AB = I$, then the matrix $A =$ ....

(a) $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{10} & \frac{-2}{5} \\ \frac{-3}{10} & \frac{1}{5} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{-1}{10} \end{pmatrix}$

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7 The intercepted negative part of y-axis by the straight line : $2x - 3y - 6 = 0$ equals .... length unit.

(a) 2 (b) 3 (c) 6 (d) 5

8 If $\overrightarrow{A} = (k , k + 3)$, $\overrightarrow{B} = (2k , 5k - 3)$, then one of the values of k which makes $\overrightarrow{A} \parallel \overrightarrow{B}$ is ....

(a) 3 (b) – 3 (c) 2 (d) – 2

9 If $\overrightarrow{A} = (5 , 2)$, $\overrightarrow{B} = (2 , 5)$ and $\overrightarrow{C} = \overrightarrow{A} - \overrightarrow{B}$, then the polar form for $\overrightarrow{C}$ is ....

(a) $\left(3 , \frac{\pi}{4}\right)$ (b) $\left(3\sqrt{2} , \frac{3\pi}{4}\right)$ (c) $\left(3 , \frac{7\pi}{4}\right)$ (d) $\left(3\sqrt{2} , \frac{7\pi}{4}\right)$

10 The value of "a" which makes $\begin{vmatrix} 1 & 2 & -1 \\ 3 & a & 1 \\ -1 & 4 & -2 \end{vmatrix} = 0$ is ....

(a) 5 (b) – 2 (c) 1 (d) 3

11 If the straight line which passes through the point M (2 , 3) and $\overrightarrow{u}$ is a direction vector to it where $\overrightarrow{u} = (-1 , 2)$, then the parametric equations of the line L are ....

(a) $x = 2 - k$, $y = -3 + 2k$ (b) $x = 2 - k$, $y = 3 - 2k$

(c) $x = 2 + k$, $y = 3 + 2k$ (d) $x = 2 - k$, $y = 3 + 2k$

12 If $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & -7 & 6 \end{pmatrix}$, then $2A^t =$ ....

(a) $\begin{pmatrix} 4 & 6 & -2 \\ 8 & -14 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 4 \\ 6 & -7 \\ 2 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 8 \\ 6 & -14 \\ -2 & 12 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 8 \\ 3 & -14 \\ -1 & 11 \end{pmatrix}$

13 If $\overrightarrow{A} = \left(5 , \frac{5\pi}{6}\right)$, then the vector $\overrightarrow{A}$ in terms of the fundamental unit vectors equals ....

(a) $\frac{5}{2}\overrightarrow{i} + \frac{5\sqrt{3}}{2}\overrightarrow{j}$ (b) $\frac{5\sqrt{3}}{2}\overrightarrow{i} + \frac{5}{2}\overrightarrow{j}$ (c) $\frac{-5\sqrt{3}}{2}\overrightarrow{i} + \frac{5}{2}\overrightarrow{j}$ (d) $\frac{5}{2}\overrightarrow{i} - \frac{5\sqrt{3}}{2}\overrightarrow{j}$

14 The simplest form of the expression $\frac{\sin x \cos x \tan x + \sin x \cos x \cot x}{\sin x \sec x} =$ ....

(a) $\cot x$ (b) $\tan x$ (c) $\cos x$ (d) $\csc x$

15 The value of x which satisfies the equation : $5 \sin x = 12 \cos x$ where $x \in [0 , \pi]$ is .... (to the nearest second)

(a) $157° 22' 48''$ (b) $112° 37' 12''$ (c) $22° 37' 12''$ (d) $67° 22' 48''$

16 If $\overline{AE}$ is a median in ΔABC , M is the centroid of the triangle ABC , A (5 , 4) , M (7 , 8) , then $\overrightarrow{AE} =$ ....

(a) $\left(\frac{4}{3} , \frac{8}{3}\right)$ (b) $\left(\frac{2}{3} , \frac{4}{3}\right)$ (c) (3 , 6) (d) (1 , 2)

17 If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 5$, $d - c = 7$, then $\begin{vmatrix} a+2 & b+2 \\ c & d \end{vmatrix} =$ ....

(a) 5 (b) 14 (c) -9 (d) 19

18 Find the general solution for the equation : $\dfrac{\tan 5\,X}{\tan(90^\circ + 4\,X)} = -1$

19 If $\begin{vmatrix} a \sin X & 0 & 0 \\ 1 & a \cos X & 0 \\ \sec X & \cot X & a \tan X \end{vmatrix} = -a^3 \cos^2 X + 8$, then a = ...

(a) 8 (b) -2 (c) -8 (d) 2

20 The point which belongs to the solution set of the inequalities $X > 2$, $y < 4$, $X - y \leq 0$ is ....

(a) (2 , 3) (b) (0 , 4) (c) (3 , 3) (d) (3 , 1)

21 If $2\,X + \begin{pmatrix} -2 & -2 \\ 4 & 0 \end{pmatrix} = O$, then the matrix X = ...

(a) $\begin{pmatrix} -2 & -2 \\ 4 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 2 \\ -4 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$

22 The measure of the angle between the two lines

$L_1 : X + 2\,y + 5 = 0$, $L_2 : \vec{r} = (1 , 4) + k\,(1 , 2)$ equals ....

(a) zero (b) 45° (c) 90° (d) 135°

23 The length of the perpendicular drawn from the origin to the straight line $3\,X - 4\,y = 10$ equals .... length unit.

(a) 1 (b) 2 (c) 3 (d) 4

24 The area of the circular segment where the measure of its central angle is 30° and the length of the radius of its circle is $2\sqrt{3}$ cm. equals ....

(a) $\frac{\pi}{3} + 2$ (b) $\pi - 3$ (c) $\pi + 3$ (d) $\frac{\pi}{3} - 2$

25 The area of the convex quadrilateral whose diagonals lengths are 12 cm. , 13 cm. and cosine the included angle between them is $\frac{5}{13}$ equals ....

(a) 30 (b) 72 (c) 60 (d) 144

26 If B (0 , 3) , C (3 , 0) and A lies at third the distance from B to C , find the coordinates of the point A.

27 Find the maximum value of the objective function $P = 3x + 4y$ under constraints :

$x \geq 0$, $y \geq 0$, $x + y \leq 3$, $x - y \leq 1$

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28 The ratio in which the x-axis divides the directed line segment $\overrightarrow{BA}$ where

A (3 , 2) , B (5 , 6) equals ....

(a) 2 : 5 internally. (b) 5 : 2 externally. (c) 1 : 3 internally. (d) 3 : 1 externally.

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29 If $A = \begin{pmatrix} 4 & 0 \\ 3 & -4 \end{pmatrix}$, find A^{60}

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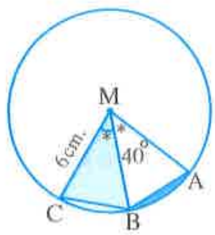
30 **In the opposite figure :**



Circle M , MC = 6 cm.

, m (∠ AMB) = m (∠ CMB) = 40°

, then the area of the shaded part = ...

(a) 4π (b) 5π (c) 6π (d) 7π

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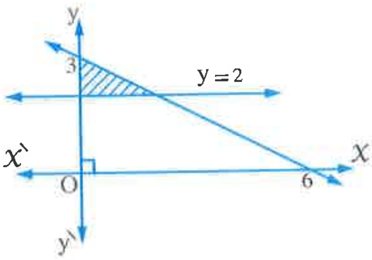
31 The shaded area in the opposite figure represents the solution set of the inequalities $y \geq 2$, $x \geq 0$ and ....



(a) $x + 2y - 6 \leq 0$

(b) $x + 2y + 6 \leq 0$

(c) $5x + 2y - 6 \geq 0$

(d) $x + 2y + 6 \geq 0$

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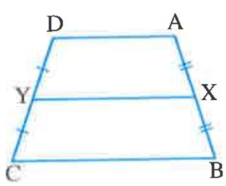
32 **In the opposite figure :**



ABCD is a trapezium , if $\overrightarrow{AD} + \overrightarrow{BC} = k\,\overrightarrow{yx}$

, then k = ... where $k \in \mathbb{R}$

(a) – 2 (b) – 1 (c) 1 (d) 2

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33 If $\vec{v}_A = 70\,\vec{e}$, $\vec{v}_B = -20\,\vec{e}$, then $\vec{v}_{AB} =$ .... $\vec{e}$

(a) – 90 (b) – 50 (c) 50 (d) 90

Model 2

Interactive test 2

Answer the following questions :

1 If the slope of a straight line $= \frac{-2}{3}$, then its direction vector is ....

(a) $(3 , -2)$ (b) $(-3 , 2)$

(c) $(6 , -4)$ (d) All the previous answers are correct.

2 If ℓ and m are the two roots of the equation : $x^2 - 3x + 1 = 0$

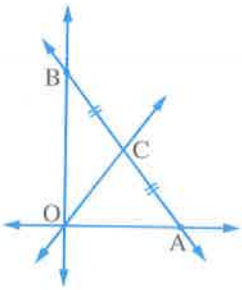
, then value of $\begin{vmatrix} \ell^2 m & -\ell^2 \\ m^2 & m \end{vmatrix} =$ ....

(a) zero (b) 1 (c) 2 (d) 3

3 **In the opposite figure :**

If the equation of the straight line $\overleftrightarrow{AB}$ is $\frac{x}{6} + \frac{y}{8} = 1$

, then parametric equation of the straight line $\overleftrightarrow{OC}$ is ....



(a) $x = 3 + 4k$, $y = 4 + 3k$

(b) $x = 4 + 3k$, $y = 4 + 4k$

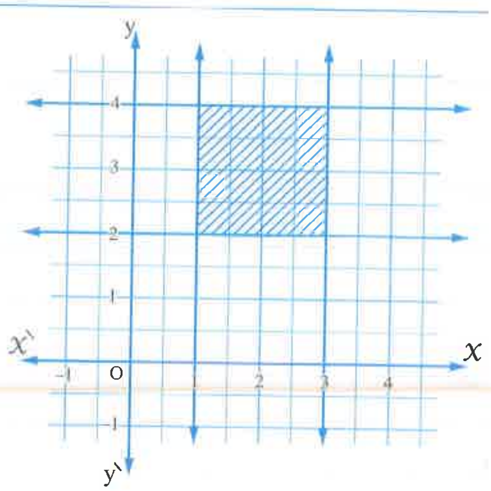
(c) $x = 3 + 3k$, $y = 4 + 4k$

(d) $x = 4 + 4k$, $y = 3 + 3k$

4 If the measure of the elevation angle of the parachute from a point at 60 m. height above a lake level is 30° and measure of the depression angle of the reflexed image of the parachute in the lake from the same point is 60° , then height of the parachute from the lake level = ... m.

(a) 120 (b) 60 (c) 90 (d) 150

5 The shaded region in the opposite graph represents the S.S. of the inequalities ....



(a) $x > 1$, $y > 2$

(b) $1 < x < 3$, $2 < y < 4$

(c) $1 \le x \le 3$, $2 \le y \le 4$

(d) $x + y \ge 3$, $x - y \le 7$

6 $(\sin^2 5° + \sin^2 10° + \sin^2 15° + \cdots + \sin^2 90°) = \cdots\cdots$

(a) $7\frac{1}{2}$ (b) $8\frac{1}{2}$ (c) $9\frac{1}{2}$ (d) $10\frac{1}{2}$

---

7 If $\overrightarrow{A} = \overrightarrow{i} + 3\overrightarrow{j}$, $\overrightarrow{B} = -10\overrightarrow{i} + \ell\overrightarrow{j}$ are two parallel vectors , then $\ell = \cdots\cdots$

(a) -30 (b) 6 (c) -6 (d) 3

---

8 Length of the drawn perpendicular from the point $(1 , 1)$ to the straight line : $x + y = 0$ equals $\cdots\cdots$ length unit.

(a) $\frac{\sqrt{2}}{2}$ (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 2

---

9 Measure of the acute angle between the straight line $\overrightarrow{r} = (2 , 2) + k(1 , 1)$ and the straight line $x = 0$ is $\cdots\cdots$

(a) $45°$ (b) $30°$ (c) $135°$ (d) $60°$

---

10 If $\overrightarrow{A} = 20\overrightarrow{i} - 15\overrightarrow{j}$, $\overrightarrow{B} = 7\overrightarrow{i} + 24\overrightarrow{j}$ and $\overrightarrow{M} = \overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{N} = \overrightarrow{A} - \overrightarrow{B}$, then $\cdots\cdots$

(a) $\overrightarrow{M} \parallel \overrightarrow{N}$ (b) $\overrightarrow{M} \perp \overrightarrow{N}$ (c) $\overrightarrow{M} = \overrightarrow{N}$ (d) $\|\overrightarrow{M}\| = \|\overrightarrow{N}\|$

---

11 General solution of the equation : $3\cot\left(\frac{\pi}{2} - \theta\right) = \sqrt{3}$ is $\cdots\cdots$

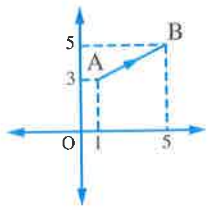
(a) $\frac{\pi}{6} + 2\pi n$ (b) $\frac{\pi}{6} + \pi n$ (c) $\frac{7\pi}{6} + 2\pi n$ (d) $\frac{\pi}{3} + \pi n$

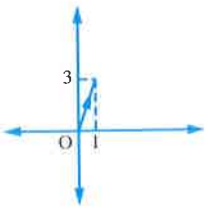
---

12 **In the opposite graph :**

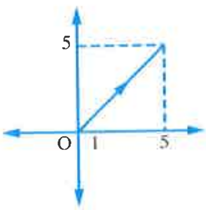
$A = (1 , 3)$, $B = (5 , 5)$

, then which of the following represent $\overrightarrow{AB}$

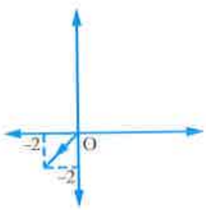




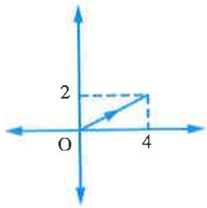
(a)



(b)



(c)



(d)

---

13 If A is a symmetric matrix , then which of the following can be a rule to deduce the element of matrix A ?

(a) $a_{ij} = 2i - j$ (b) $a_{ij} = i + j$ (c) $a_{ij} = i^j$ (d) $a_{ij} = 3i + 2j$

14 If a zero matrix O its order 3×3, then number of elements of the matrix = ...

(a) zero (b) $\varnothing$ (c) 3 (d) 9

15 Find the maximum value of the objective function $P = 3x + 2y$ under conditions :

$x \geq 0$, $y \geq 0$, $2x \leq 3y$, $2y + x \leq 7$

16 If $A = \begin{vmatrix} \sin 5\theta & -\cos 5\theta \\ \cos 5\theta & \sin 5\theta \end{vmatrix}$ = ...

(a) 1 (b) -1 (c) 5 (d) -5

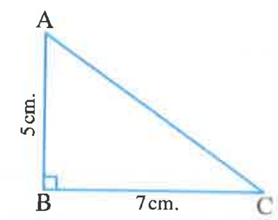
17 The area of a circular sector is 45 cm^2 and the length of the diameter of its circle is 20 cm. , then perimeter of this circular sector equals .... cm.

(a) 29 (b) 19 (c) 39 (d) 49

18 The area of the regular hexagon in which the length of its edge is 8 cm. equals .... cm^2

(a) $12\sqrt{3}$ (b) $24\sqrt{3}$ (c) $96\sqrt{3}$ (d) $144\sqrt{3}$

19 **In the opposite figure :**



m (∠ C) = ... to the nearest degree.

(a) 30 (b) 35

(c) 36 (d) 45

20 All of the following are unit vectors except ....

(a) (1 , 0) (b) (0 , – 1) (c) (1 , 1) (d) (0.6 , 0.8)

21 In Δ ABC : $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{AC}$ = ...

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{CA}$ (c) $2\,\overrightarrow{AC}$ (d) $2\,\overrightarrow{AB}$

22 The direction vector of the straight line whose parametric equations are $x + 3 = 2k$, $y = 5$ is ....

(a) (2 , 0) (b) (2 , – 3) (c) (2 , 3) (d) (2 , 5)

23 If $C \in \overline{AB}$, $3\,\overrightarrow{AB} = 5\,\overrightarrow{CB}$, then C divides $\overline{BA}$ by the ratio ....

(a) 2 : 3 (b) 3 : 2 (c) 3 : 5 (d) 5 : 3

24 The measure of the angle between the two straight lines $3x = 5$, $y = 3$ is ....

(a) 30° (b) 45° (c) 60° (d) 90°

25 If $\overrightarrow{AB} = 2\vec{i} + \vec{j}$, B (3 , – 1) , then the point of A is ....

(a) (1 , – 2) (b) (– 2 , 1) (c) (2 , 1) (d) (1 , 2)

26 If $\begin{vmatrix} x & 12 \\ 3 & x \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 10 & 5 \end{vmatrix}$, then $x =$ ....

(a) 2 (b) 5 (c) 6 (d) ± 6

27 If $A \times \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} = I$, then A = ...

(a) $\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

28 If $\vec{A} = -2\vec{i} - 2\vec{j}$, then the polar form of $\vec{A}$ is ....

(a) $\left(2\sqrt{2} , \frac{\pi}{4}\right)$ (b) $\left(2\sqrt{2} , \frac{3\pi}{4}\right)$ (c) $\left(2\sqrt{2} , \frac{5\pi}{4}\right)$ (d) $\left(2\sqrt{2} , \frac{7\pi}{4}\right)$

29 If $\vec{A} = 3\vec{i} - 4\vec{j}$, $\vec{B} = \vec{j}$, $\vec{C} = \left(5 , \frac{\pi}{18}\right)$, find the value of : $\|\vec{A}\| + \|\vec{B}\| + \|\vec{C}\|$

30 Find the area of circular segment whose chord length is 18 cm. and the radius length of its circle is 18 cm. to the nearest cm^2.

31 The solution set of the inequality : $x + 5 \leq 3x + 1 < 2x + 2$ in $\mathbb{R}$ is ....

(a) $\mathbb{R} - [1 , 2[$ (b) $]1 , 2]$ (c) $\varnothing$ (d) $\{1 , 2\}$

32 Find the different forms of the equation of the straight line which passes through the point (1 , 3) and is perpendicular to the straight line : $\vec{r} = (2 , 5) + k(-2 , 1)$

33 Find the area of the triangle whose vertices are A (2 , 4) , B (– 2 , 4) , C (0 , – 2)

Model 3

Interactive test 3

Answer the following questions :

1 ABCD is a parallelogram : $\overline{AC} \cap \overline{BD} = \{M\}$, then $\overrightarrow{AB} + \overrightarrow{AD} =$ ....

(a) $\overrightarrow{CA}$ (b) $\overrightarrow{BD}$ (c) $2\,\overrightarrow{MC}$ (d) $2\,\overrightarrow{DM}$

2 If (a , b) belongs to the solution set of the inequality $x + 2y \geq 5$ where a , b are integers , then the least value of the expression : $2a + 4b =$ ....

(a) 5 (b) -5 (c) 10 (d) 6

3 If A $(-2 , 1)$, B $(2 , 3)$, C $(-2 , -4)$ are three points , then the measure of the acute angle between the two straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ is ....

(d) $\tan^{-1}\left(\frac{-2}{3}\right)$ (b) $\tan^{-1}\left(\frac{2}{3}\right)$ (c) $\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\tan^{-1}\left(\frac{3}{2}\right)$

4 If ABCDEF is a regular hexagon whose geometrical centre (M) which of the following directed line segments are not equivalent ?

(a) $\overrightarrow{AB}$, $\overrightarrow{FM}$ (b) $\overrightarrow{AB}$, $\overrightarrow{ED}$ (c) $\overrightarrow{AB}$, $\overrightarrow{MC}$ (d) $\overrightarrow{AB}$, $\overrightarrow{MD}$

5 The matrix $\begin{pmatrix} a & 12 \\ 3 & a \end{pmatrix}$ has multiplicative inverse at ....

(a) $a = 6$ (b) $a = \pm 6$

(c) $a \in \mathbb{R} - \{6\}$ (d) $a \in \mathbb{R} - \{6 , -6\}$

6 Find the solution set of the equation $\begin{vmatrix} x-2 & 0 & 0 \\ 3 & x-3 & 0 \\ 4 & -1 & x \end{vmatrix} = \text{zero}$

7 The length of the perpendicular drawn from point $(-3 , 5)$ to the x-axis equals .... length unit.

(a) 8 (b) 5 (c) 3 (d) 2

8 If the area of the triangle whose vertices $(k , 0)$, $(4 , 0)$, $(0 , 2)$ is 4 square unit , then $k =$ ....

(a) zero or -8 (b) -4 or 4 (c) zero or 8 (d) 8 or -8

9 In Δ ABC , if $\sin^2 A + \cos^2 B = 1$, then Δ ABC is ....

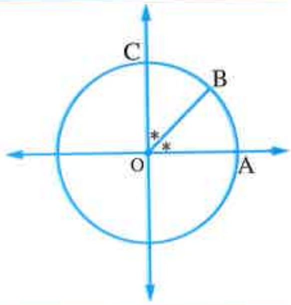
(a) an equilateral triangle. (b) an isosceles triangle.

(c) a scalene triangle. (d) a right-angled triangle.

10 If ABC is a right-angled triangle at B and AB > BC , the area of Δ ABC = 30 cm^2, AB + BC = 20 cm. , then m ($\angle$ A) $\approx$ ....

(a) $77^\circ\ 19'$ (b) $54^\circ\ 37'$ (c) $26^\circ\ 18'$ (d) $12^\circ\ 41'$

11 **In the opposite figure :**



If $\overrightarrow{OB}$ bisects $\angle$ AOC in a unit circle

, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} =$ ....

(a) $\sqrt{2}\ \overrightarrow{OB}$ (b) $2\ \overrightarrow{OB}$

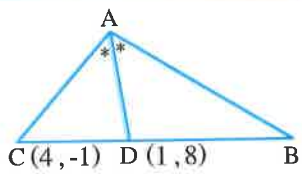
(c) $(\sqrt{2} + 1)\ \overrightarrow{OB}$ (d) $3\ \overrightarrow{OB}$

12 A small factory produces metal furniture 20 cupboard weekly at most of two different kinds A and B. If the profit from kind "A" is 80 pounds , and the profit from kind B is 100 pounds. The factory sells from kind A at least 3 times what it sells from the second kind. Find number of cupboard from each kind to satisfy the greatest possible profit to the factory.

13 If $\begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$ is a skew symmetric , then : $\dfrac{a+b+c+f}{d+x+y+z} =$ ....

(a) 1 (b) zero (c) -1 (d) e

14 **In the opposite figure :**



If 4 AC = 3 AB , find the coordinates of the point B.

15 Measure of the acute angle included between the two straight lines

$\sqrt{3}\,x - y = 4$, $y = 3$ equals ....

(a) 30° (b) 45° (c) 60° (d) 90°

16 If $\csc\theta - \cot\theta = \frac{1}{5}$, then $\csc\theta + \cot\theta =$ ....

(a) $\frac{-1}{10}$ (b) 5 (c) $\frac{1}{25}$ (d) 1

17 If $\overrightarrow{EM} = \left(4\sqrt{3} , 4\right)$, find the polar form of the vector $\overrightarrow{EM}$

18 If $\| 12 \overrightarrow{A} \| = 2 \| k \overrightarrow{A} \|$, then k = ...

(a) 6 (b) ± 6 (c) – 6 (d) 24

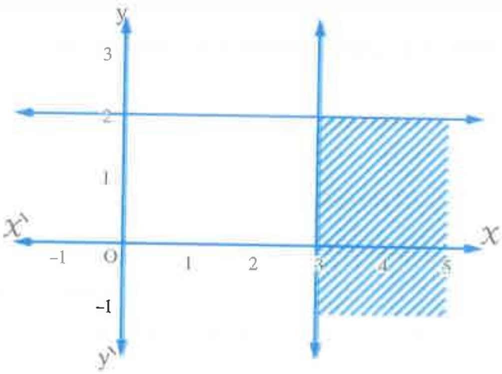
19 ABCD is a square in which A (2 , – 3) and the equation of $\overleftrightarrow{CD}$: $3 x - 4 y + 2 = 0$, then the area of the square ABCD = ... square unit.

(a) 4 (b) 9 (c) 16 (d) 25

20 $\dfrac{3}{1 + \tan^2 \theta} + 3 \sin^2 \theta =$ ....

(a) 3 (b) 1 (c) $3 \sin^2 \theta$ (d) $\sec^2 \theta$

21 The shaded part in the opposite figure represents the solution set of the inequalities : ....



(a) $x > 3$, $y < 2$

(b) $x \geq 3$, $y \geq 2$

(c) $x + 1 < 4$, $y + 1 < 3$

(d) $x + 1 \geq 4$, $2 y \leq 4$

22 If $\overrightarrow{xy} = (2 , 3)$, $\overrightarrow{yz} = (4 , 5)$, then $\overrightarrow{xz} =$ ....

(a) (6 , 8) (b) (2 , 2) (c) (8 , 8) (d) (4 , 3)

23 If the perimeter of a circular sector equals 10 cm. and the length of its arc equals 2 cm. , then its area = ... cm^2

(a) 4 (b) 8 (c) 10 (d) 20

24 The distance between the two parallel straight lines : $\vec{r} = (4 , 0) + k (4 , 3)$, $a x - 8 y + 4 = 0$ equals .... length unit.

(a) 20 (b) 0.2 (c) 28 (d) 2.8

25 The vector equation of the straight line which passes through the point (2 , – 3) and perpendicular to X-axis is ....

(a) $\vec{r} = (2 , -3) + k\,(1 , 0)$ (b) $\vec{r} = (3 , 2) + k\,(0 , 1)$

(c) $\vec{r} = (2 , -3) + k\,(0 , 7)$ (d) $\vec{r} = (-3 , 2) + k\,(0 , 1)$

26 If $\vec{A} = -10\,\vec{i} + k\,\vec{j}$, $\vec{B} = \vec{i} + 3\,\vec{j}$ and $\vec{A} \perp \vec{B}$, then k = ...

(a) – 30 (b) $\frac{10}{3}$ (c) $\frac{3}{10}$ (d) 30

27 The solution set of the equation $\sqrt{3}\tan\theta = 1$ where $90° < \theta < 270°$ is ....

(a) {30°} (b) {150°} (c) {210°} (d) {240°}

28 If $\begin{pmatrix} 4 & 2X-1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 5 & 3 \end{pmatrix}^t$, then $X =$ ....

(a) 1 (b) 2 (c) 3 (d) 5

29 If $X = \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$, then X^{-1} ....

(a) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{2} & -2 \\ 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & -2 \\ 0 & 1 \end{pmatrix}$

30 A plane 1000 metres high was observed by a person at an angle of elevation of measure 40° , find the distance between the plane and the observer to the neanst metre.

31 If $\vec{u} = (3 , -4)$ is a direction vector for a straight line , then all of the following are direction vectors to the same straight line except the vector ....

(a) (– 3 , 4) (b) (9 , – 12) (c) (3 , 4) (d) (1.5 , – 2)

32 If $A = \begin{pmatrix} X & 2 \\ y & -2 \end{pmatrix}$, $A \times A^{-1} = A^2$, then $X \times y =$ ....

(a) 3 (b) 2 (c) – 2 (d) – 3

33 If $v_A = 25\,\vec{e}$, $\vec{v}_B = -10\,\vec{e}$, then $v_{AB} =$ .... $\vec{e}$

(a) 35 (b) – 35 (c) 15 (d) – 15

Model 4

Interactive test 4

Answer the following questions :

1 If $\tan \theta = 3$, then $\sec^2 \theta =$ ....

(a) 9 (b) 10 (c) $\frac{1}{10}$ (d) $\frac{9}{10}$

2 Measure of the acute angle between the two straight lines : $x = 3y$, $x + 2y = 0$ is ....

(a) 15° (b) 30° (c) 45° (d) 60°

3 Which of the following points lies on the straight line $\vec{r} = (-2 , 1) + k(1 , -3)$ ....

(a) $\left(\frac{-5}{3} , -2\right)$ (b) $\left(\frac{-3}{2} , \frac{1}{2}\right)$ (c) $\left(\frac{3}{2} , -\frac{1}{2}\right)$ (d) $\left(\frac{-7}{3} , 2\right)$

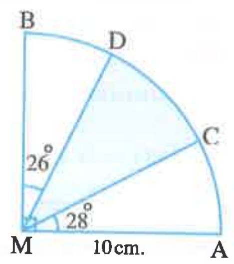
4 If $\vec{d} = (3 , 4)$ is a direction vector of a straight line , then its slope equals ....

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{-4}{3}$ (d) $\frac{-3}{4}$

5 If $\vec{A} = (k , 2)$, $\vec{B} = 2\vec{i} - \vec{j}$ and if $\vec{A} \perp \vec{B}$, then $k =$ ....

(a) 1 (b) -1 (c) ± 1 (d) zero

6 **In the opposite figure :**



The area of the shaded region = ... cm^2.

(a) 10π (b) 20π

(c) 30π (d) 40π

7 If the matrix A of order 3×2 and the matrix AB of order 3×1 , then matrix B of order ....

(a) 2×1 (b) 3×1 (c) 3×3 (d) 2×3

8 If $\vec{A} = (3 , -2)$, $\vec{B} = (7 , 1)$, then $\|\vec{A} + 2\vec{B}\| =$ .... length unit.

(a) $\sqrt{13}$ (b) $5\sqrt{2}$ (c) 11 (d) 17

9 Area of Δ ABC in which AB = 5 cm. , AC = 4 cm. , m ($\angle$ A) = 60°

equals .... cm^2.

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$ (c) 5 (d) 10

10 If position vector $\overrightarrow{A} = (\sqrt{3}, 1)$ rotates around origin with an angle of measure 45° anticlockwise , then the polar form of the vector $\overrightarrow{A}$ after rotation is ....

(a) (2 , 30°) (b) (2 , 45°) (c) (2 , 75°) (d) (4 , 75°)

11 If ABCD is a quadrilateral in which $\overrightarrow{BC} = 3\,\overrightarrow{AD}$, prove that : $\overrightarrow{AC} + \overrightarrow{BD} = 4\,\overrightarrow{AD}$

12 To make system of equations : $a_1 x + b_1 y = c_1$, $a_2 x + b_2 y = c_2$

has unique solution , it must be ....

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0$ (c) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ (d) $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \neq 0$

13 The point which belongs to the solution set of the inequalities :

$2x + y < 4$, $x + 3y < 6$ is ....

(a) (1 , – 4) (b) (2 , 1) (c) (1 , 2) (d) (3 , – 1)

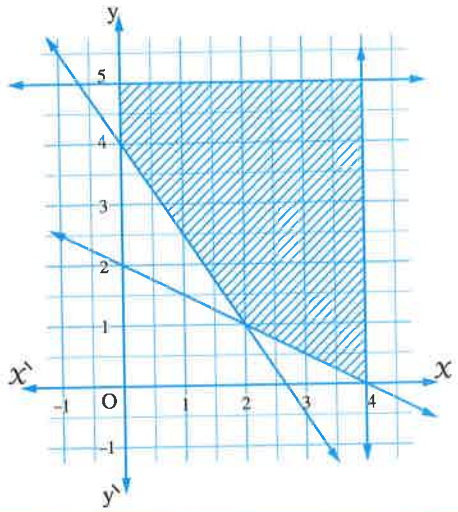
14 **In the opposite figure :**

The objective function

$P = \frac{1}{2} x + y$ is minimum at which of the following points ?

(a) (0 , 4) (b) (0 , 5)

(c) (2 , 1) (d) (4 , 5)

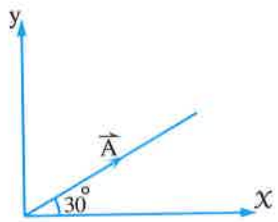


15 **In the opposite figure :**

$\|\overrightarrow{A}\| = 4$ cm. , then $\overrightarrow{A}$ = ...

(a) $(2, 2\sqrt{3})$ (b) $(2\sqrt{3}, 2)$

(c) $(4, \sqrt{3})$ (d) $(\sqrt{3}, 2)$



16 If $\overrightarrow{A}$, $\overrightarrow{B}$ are non-zero vectors and $\| \overrightarrow{A} + \overrightarrow{B} \| = \| \overrightarrow{A} - \overrightarrow{B} \|$, then ....

(a) $\overrightarrow{A} = -\overrightarrow{B}$ (b) $\overrightarrow{A}$ and $\overrightarrow{B}$ are equivalent.

(c) $\overrightarrow{A}$ and $\overrightarrow{B}$ are parallel. (d) $\overrightarrow{A}$ and $\overrightarrow{B}$ are perpendicular.

17 A light pole of height 8 metres gives a shade on the ground of length 5 metres , then the measure of the the elevation angle of the sun at that moment to the nearest degree equals ....

(a) 32° (b) 51° (c) 39° (d) 58°

18 The length of the perpendicular drawn from the origin to the straight line $\overrightarrow{r} = (5 , 0) + k (4 , 3)$ equals .... length unit.

(a) 15 (b) 5 (c) 3 (d) 4

19 $2 \sin \theta - \sqrt{3} = 0$ where $90° \leq \theta \leq 270°$, then $\theta =$ ....

(a) 60° (b) 120° (c) 240° (d) 300°

20 The perpendicular direction vector on the straight line $x = 3 + 2k$, $y = 4 - k$ equals ....

(a) (2 , 10) (b) (1 , 2) (c) (2 , 1) (d) (4 , – 2)

21 If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then $x =$ ....

(a) 1 (b) 2 (c) 3 (d) 4

22 Represent graphically the solution set of the following inequalities together :

$x \leq 4$, $y < x + 2$, $x + 2y \geq -2$ in $\mathbb{R} \times \mathbb{R}$

23 If $\overrightarrow{AB} = (2 , 3)$, $\overrightarrow{CB} = (-3 , 5)$, then $\overrightarrow{AC} =$ ....

(a) (5 , – 2) (b) (– 8 , 1) (c) (– 2 , 5) (d) (2 , 5)

24 If $\begin{vmatrix} x & -1 \\ 2 & x \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 2$, then $x =$ ....

(a) 3 or – 2 (b) – 3 or 2 (c) 3 or 2 (d) – 3 or – 2

25 $\sin^2 x + \cos^2 x + \cot^2 x =$ ....

(a) 1 (b) $\cot^2 x$ (c) $\csc^2 x$ (d) $\sec^2 x$

26 The area of circular sector equals 45 cm.^{2} and the length of the diameter of its circle equals 20 cm. , find the perimeter of this sector.

27 If A (0 , 0) is the image of the point B (4 , 2) by reflexion on the straight line L , then the equation of L is ....

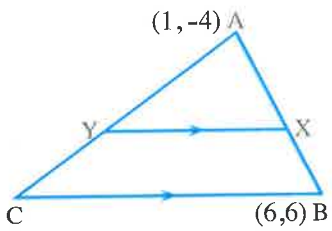
(a) $x = 2y$ (b) $2x + y - 5 = 0$ (c) $2x - y = 5$ (d) $x + y - 6 = 0$

28 If $\begin{pmatrix} 3^x & 2x + y \\ x + y & 3^y \end{pmatrix} = \begin{pmatrix} 25 & b \\ a & 5 \end{pmatrix}$, find the value of $\frac{b}{a}$

29 **In the opposite figure :**

$\overline{XY} \,//\, \overline{BC}$, $\frac{AY}{AC} = \frac{3}{5}$

, then $x =$ ....



(a) (2 , 4) (b) (4 , 2)

(c) (– 2 , 4) (d) (– 4 , 2)

30 If M is the midpoint of $\overline{XY}$, then $\overrightarrow{XM} + \overrightarrow{YM} =$ ....

(a) $2\,\overrightarrow{XM}$ (b) $2\,\overrightarrow{YM}$ (c) $\overrightarrow{XY}$ (d) $\vec{O}$

31 If $X = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} -3 & 5 \end{pmatrix}$, then $Y X =$ ....

(a) $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} -9 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} -9 & 15 \\ 6 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \end{pmatrix}$

32 If $X + \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}^t = O$, then $X =$ ....

(a) $\begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$ (d) I

33 Find the distance between the two straight lines : $3x - 4y + 20 = 0$, $3x - 4y + 10 = 0$

Model 5

Interactive test 5

Answer the following questions :

1 If $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 2 & -1 \\ x & 3 \end{pmatrix}$, then $x =$ ....

(a) 3 (b) – 3 (c) 5 (d) – 5

2 If $\| \overrightarrow{A} \| = \| \overrightarrow{B} \|$, then ....

(a) $\overrightarrow{A} = \overrightarrow{B}$

(b) $\overrightarrow{A} = -\overrightarrow{B}$

(c) $\overrightarrow{A} = \pm \overrightarrow{B}$

(d) It can't be find the relation between $\overrightarrow{A}$, $\overrightarrow{B}$

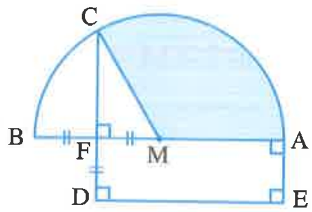
3 If A and B are the two images of the point (3 , 1) by reflection in x-axis and y-axis respectively , find the point that divides $\overrightarrow{AB}$ internally in the ratio 2 : 3

4 The point that belongs to the S.S. of the system of the following inequalitias :

$x > 2$, $y > 1$, $x + y \geq 3$ is ....

(a) (2 , 1) (b) (1 , 2) (c) (3 , 2) (d) (1 , 3)

5 **In the opposite figure :**



If the area of the rectangle AEDF = 27 cm^2.

, find area of the shaded part

6 If $0° \leq x \leq 360°$, then number of solutions of the equation $3 \sin x = \tan x$ is ....

(a) 2 (b) 3 (c) 4 (d) 5

7 Equation of the straight line passing through the point (2 , – 3) and parallel to x-axis is ....

(a) $x = -3$ (b) $y = -3$ (c) $x = 2$ (d) $y = 3$

8 If $\overrightarrow{A} + \overrightarrow{B} = (5, 12)$, $\overrightarrow{A} = 2\overrightarrow{i} + 9\overrightarrow{j}$, then $\overrightarrow{B} =$ ....

(a) $(3, 4)$ (b) $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$ (c) $(-3, -3)$ (d) $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

9 If $A = (3, 5)$, $B = (-1, m)$, $\|\overrightarrow{AB}\| = 4$ length unit , then $m =$ ....

(a) zero (b) 5 (c) -1 (d) -5

10 Measure of the included acute angle between the two straight lines :

$x - 3y + 5 = 0$, $x + 2y - 7 = 0$ equals ....

(a) 45° (b) 60° (c) 120° (d) 135°

11 A car moved 20 metres in direction of North , then moved the same distance in the direction of West , then the displacement of the car is ....

(a) 40 m. in the West direction. (b) 40 m. in the Western North direction.

(c) $20\sqrt{2}$ m. in the Western North direction. (d) $20\sqrt{2}$ m. in the Western South direction.

12 $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} =$ ....

(a) $\tan\theta$ (b) $\sin\theta + \cos\theta$ (c) $\sec\theta \csc\theta$ (d) 1

13 A regular hexagon of area $54\sqrt{3}$ cm^2 , then its side length .... cm.

(a) 5 (b) 6 (c) 7 (d) 8

14 If A is a diagonal matrix in the order 3×3 and $a_{ij} = 5$ for each $i = j$, then ....

(a) $A = I$ (b) $A = 5I$ (c) $A = 5O$ (d) $A = O$

15 The points $A(-1, 5)$, $B(2, 2)$ and $C(3, 1)$ are ....

(a) vertices of right-angled triangle of area 5 square unit.

(b) vertices of isosceles triangle of area 10 square unit.

(c) vertices of equilateral triangle of area 9 square unit.

(d) collinear.

16 Find the minimum value of the objective function $P = 2x + 3y$ under conditions

$x + y \leq 5$, $y \geq 1$, $x \geq 2$

17 If the point A $(0 , 0)$ is the image of the point B $(4 , 2)$ by reflection in the straight line L , then equation of L is ....

(a) $x = 2y$ (b) $2x + y = 5$

(c) $2x - y = 5$ (d) $x + y = 6$

18 From the top of a rock 100 metres high , the measure of the depression angle of the boat which is 200 m. away from the base of the rock equals (in radian) $\approx$ ....rad.

(a) 0.08 (b) 0.46 (c) 0.25 (d) 0.24

19 The area of the circular segment whose diameter length of its circle is 8 cm. and the measure of its central angle is 1.2^{rad} equals approximately .... cm^2

(a) 8.57 (b) 2.14 (c) 4.28 (d) 1.07

20 The length of the perpendicular drawn from the point $(3 , 1)$ to the straight line $4x + 3y - 5 = 0$ equals .... length unit.

(a) 2 (b) 3 (c) 4 (d) 5

21 If $3^{\sin\theta} = 1$ where $\theta \in]0 , 2\pi[$, then $\theta =$ ....°

(a) 45 (b) 90 (c) 180 (d) 270

22 If the polar from of $\overrightarrow{OA}$ is $\left(12 , \frac{2\pi}{3}\right)$, then the polar form of $\overrightarrow{AO}$ is ....

(a) $\left(12 , \frac{\pi}{6}\right)$ (b) $\left(12 , \frac{2\pi}{3}\right)$ (c) $\left(6 , \frac{4\pi}{3}\right)$ (d) $\left(12 , \frac{5\pi}{3}\right)$

23 If the slope of a straight line $= \frac{-2}{3}$, then its direction vector is ....

(a) $(3 , -2)$ (b) $(-3 , 2)$ (c) $(6 , -4)$ (d) all the previous.

24 The solution set for the equation $\begin{vmatrix} x+2 & 2 \\ x & x-3 \end{vmatrix} = 4$ is ....

(a) $\{3 , -2\}$ (b) $\{2 , -3\}$ (c) $\{5 , -2\}$ (d) $\{2 , -5\}$

25 If $A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}$, $A = A^{-1} \times B$, find the matrix B

26 The vector equation of X-axis is ....

(a) $\vec{r} = (1 , 1) + k (0 , 0)$ (b) $\vec{r} = (1 , 0) + k (1 , 1)$

(c) $\vec{r} = k (1 , 0)$ (d) $\vec{r} = k (0 , 1)$

27 If A is a square matrix in order 2×2 , $|2 A| = 8$, then $|3 A^t| =$ ....

(a) 9 (b) 12 (c) 18 (d) 24

28 If $\begin{pmatrix} 0 & 2 & 5 \\ X & 0 & 3 \\ 5 & 3 & 0 \end{pmatrix}$ is a symmetric matrix , then $X =$ ....

(a) 0 (b) 2 (c) 3 (d) -2

29 Find the norm of the resultant of the two forces : $\vec{F_1} = -3\vec{i} + \sqrt{3}\,\vec{j}$, $\vec{F_2} = \left(6 , \frac{\pi}{3}\right)$

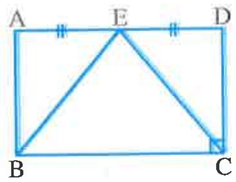
30 If $\vec{A} = (3 , k)$, $\vec{B} = (2 , -3)$ and $\vec{A} \perp \vec{B}$, then $k =$ ....

(a) 3 (b) 2 (c) -2 (d) -3

31 The solution set of the inequality $1 \le 2X - 1 < 5$ in $\mathbb{R}$ is ....

(a) $]1 , 3[$ (b) $]1 , 3]$ (c) $[1 , 3[$ (d) $[1 , 3]$

32 **In the opposite figure :**



ABCD is a rectangle

, E is the midpoint of $\overline{AD}$

, $\overrightarrow{EB} + \overrightarrow{BA} - \overrightarrow{DC} =$ ....

(a) $\overrightarrow{EB}$ (b) $\overrightarrow{BE}$ (c) $\overrightarrow{EC}$ (d) $\overrightarrow{CE}$

33 If $A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 7 \\ 8 & 9 \end{pmatrix}$, then $a_{21} + c_{12} =$ ....

(a) 5 (b) 4 (c) -5 (d) 3

Model 6

Interactive test 6

Answer the following questions :

1 S.S. of the two equations : $2x - 3y = 1$ and $3x + 2y = 8$ is ....

(a) $\{(1, 2)\}$ (b) $\{(2, 1)\}$ (c) $\{(2, 3)\}$ (d) $\{(3, 2)\}$

2 Find the area of circular sector , length of the diameter of its circle is 12 cm. and measure of its central angle 60°

3 Point of intersection of heights of the triangle in which its sides coincide on the straight lines : $x = 0$, $y = 0$ and $x + y = 1$ is ....

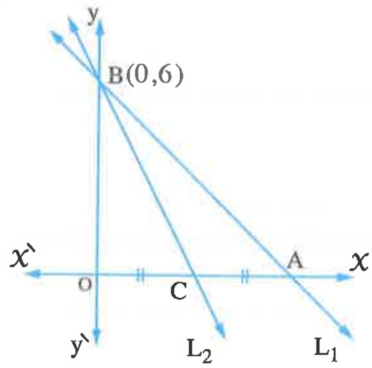
(a) $(1, 1)$ (b) $(0, 0)$ (c) $(1, 0)$ (d) $\left(\frac{1}{3}, \frac{1}{3}\right)$

4 If $\begin{vmatrix} k+\frac{1}{k} & 1 \\ 1 & k+\frac{1}{k} \end{vmatrix} = 15$, find the value of $k^2 + \frac{1}{k^2}$

5 **In the opposite figure :**

If area of $\Delta ABC = 15$ square unit

, $CA = CO$, find the equation of L_1



6 If $\tan\theta + \cot\theta = 2$, then $\tan^{2019}\theta + \cot^{2019}\theta =$ ....

(a) zero (b) 1 (c) 2 (d) 3

7 L_1 and L_2 are two straight lines , the tangent of the included angle between them equals $\frac{1}{3}$ and the slope of L_1 equals twice the slope of L_2 , then slope of the straight line $L_2 =$ ....

(a) $\pm \frac{1}{2}$ (b) ± 1

(c) $1 , \frac{1}{2}$ (d) All of the previous answers.

8 If the forces $\overrightarrow{F_1} = \left(8\sqrt{2}, \frac{3}{4}\pi\right)$, $\overrightarrow{F_2} = a\,\overrightarrow{i} + 3\,\overrightarrow{j}$, $\overrightarrow{F_3} = -5\,\overrightarrow{i} + (b+2)\,\overrightarrow{j}$ are acting in one point and equilibrium , then $\frac{a}{b} =$ ....

(a) 13 (b) – 13 (c) 1 (d) – 1

9 If A is a matrix in the order 2×3 , B^t is a matrix in the order 1×3 , then the matrix AB is in the order ....

(a) 3×3 (b) 3×1 (c) 2×1 (d) 1×2

10 If C is the midpoint of $\overline{AB}$ where C = (0 , 3) , B = (7 , 6) , then A = ...

(a) (7 , 6) (b) (3 , 6) (c) (– 7 , 0) (d) (0 , 6)

11 General solution of the equation : $\cos\theta = 1$ is ....

(a) $n\pi$ (b) $2n\pi$ (c) $\frac{\pi}{2} + n\pi$ (d) $\frac{\pi}{2} + 2n\pi$

12 If the point (4 , k) is lying on the axis of symmetry of the region of S.S. of the two inequalities : $x + y > a$, $x - y > a$, then k = ...

(a) 4 (b) – 4 (c) a (d) zero

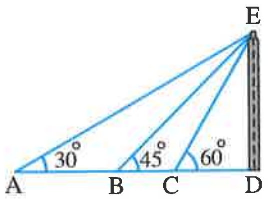
13 The straight line : $2x + 3y = 12$ makes with the two axes of coordinates a triangle of area .... square unit.

(a) 4 (b) 6 (c) 8 (d) 12

14 The point at which P has minimum value where $P = 35x + 10y$ from the following points is ....

(a) (0 , 10) (b) (0 , 20) (c) (0 , 40) (d) (20 , 10)

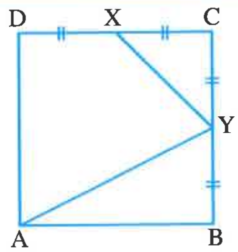
15 **In the opposite figure :**



If the elevation angles of the top of a tower from three different points on the horizontal line passing through the base of the tower are of measures 30° , 45° , 60° respectively , then AB : BC = ... : ....

(a) $1 : \sqrt{3}$ (b) 2 : 3 (c) $\sqrt{3} : \sqrt{2}$ (d) $\sqrt{3} : 1$

16 **In the opposite figure :**



ABCD is a square and $\overrightarrow{AY} + \overrightarrow{XY} = k\,\overrightarrow{XC}$, then k = ...

(a) 1 (b) 2

(c) 3 (d) 4

17 The measure of the central angle of a circular segment is 90° and its area is 56 cm², then the radius length of its circle approximately equals .... cm.

(a) 9.9 (b) 19.8 (c) 7 (d) 14

18 The measure of an angle in a rhombus is 50° and its side length is 12 cm. , then its area is .... cm².

(a) 36 sin 50° (b) 72 cos 50° (c) 144 sin 50° (d) 72 sin 50°

19 If $\overrightarrow{A} = (k , 2)$, $\overrightarrow{B} = (3 , k - 5)$ and $\overrightarrow{A} \perp \overrightarrow{B}$, then k = ...

(a) 1 (b) 2 (c) – 1 (d) – 2

20 If $\overrightarrow{A} = 3\overrightarrow{i} - 4\overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{j}$, $\overrightarrow{C} = \left(5 , \frac{\pi}{18}\right)$, then $\|\overrightarrow{A}\| + \|\overrightarrow{B}\| + \|\overrightarrow{C}\| =$ ....

(a) 9 (b) 10 (c) 11 (d) 12

21 The number of solutions of the equation : $\sin^2 x - 6 \sin x + 9 = 0$ is ....

(a) zero (b) 1 (c) 2 (d) 3

22 The value(s) of x which makes the matrix $\begin{pmatrix} x & 2 \\ 2 & x \end{pmatrix}$ has no multiplicative inverse is/are ....

(a) 2 or 4 (b) – 2 or – 4 (c) ± 2 (d) ± 4

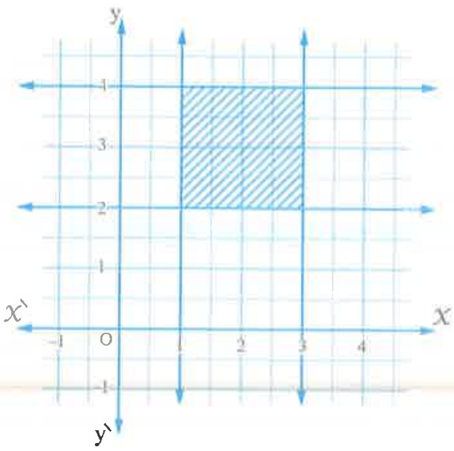
23 The shaded region in the given figure represents the solution set of the inequalities ....

(a) $x > 1$, $y > 2$

(b) $1 < x < 3$, $2 < y < 4$

(c) $1 \le x \le 3$, $2 \le y \le 4$

(d) $x + y \ge 3$, $x - y \le 7$



24 The length of the perpendicular drawn from the point $(1 , 1)$ to the straight line $x + y = 0$ equals .... length unit.

(a) $\frac{\sqrt{2}}{2}$ (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 2

25 If $\begin{vmatrix} x & y \\ z & \ell \end{vmatrix} = 4$, then $\begin{vmatrix} x - y & 4y \\ z - \ell & 4\ell \end{vmatrix} =$ ....

(a) 1 (b) 10 (c) 12 (d) 16

26 If $\vec{u} = (2 , -5)$ is a direction vector of a straight line , then all the following are direction vectors for the same straight line except ....

(a) $(-2 , 5)$ (b) $(6 , -15)$ (c) $(2 , 5)$ (d) $\left(-1 , 2\frac{1}{2}\right)$

27 If $\begin{pmatrix} 3 & a-2 \\ b & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 7 & 5 \end{pmatrix}^t$, then $\begin{pmatrix} a & b \end{pmatrix} =$ ....

(a) $\begin{pmatrix} 6 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 9 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 7 \end{pmatrix}$

28 Measure of the angle between the two straight lines $L_1 : x = 3$, $L_2 : \vec{r} = (2 , 5) + k (1 , 0)$ equals ....

(a) 90° (b) 180° (c) 45° (d) zero

29 If $A \times \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = I$, find the matrix 3 A

30 If $A = (k , 2)$, $B = (1 , -1)$ and $\| \overrightarrow{AB} \| = 5$, then $k =$ ....

(a) 5 (b) -3 (c) 5 or -3 (d) 15

31 If $A = (3 , -5)$, $B = (-1 , 5)$, $\vec{M} = (6 , k)$ and $\vec{M} \parallel \overrightarrow{AB}$, then $k =$ ....

(a) -15 (b) -10 (c) -5 (d) 5

32 The polar form of the vector $\vec{A} = -3\,\vec{j}$ is ....

(a) $\left(-3 , \frac{\pi}{2}\right)$ (b) $\left(3 , \frac{\pi}{2}\right)$ (c) $\left(-3 , \frac{3\pi}{2}\right)$ (d) $\left(3 , \frac{3\pi}{2}\right)$

33 If $\vec{A} = (-1 , 2)$, $\vec{B} = (3 , 7)$, $\vec{C} = (7 , 12)$, find $\vec{C}$ in terms of $\vec{A}$ and $\vec{B}$

Model 7

Interactive test 7

Answer the following questions :

1 If A , B are two matrices where AB $= \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$, then $B^t A^t =$ ....

(a) $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -1 \\ 4 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$

2 ABC is an equilateral triangle in which A (2 , – 1) and the equation of $\overleftrightarrow{BC}$ is $x + y = 2$, then the side length of the triangle ABC = ... length unit.

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{6}}{2}$ (c) $\frac{\sqrt{6}}{3}$ (d) $\sqrt{2}$

3 The S.S. of the equation : $\sin x + \cos x = 0$ where $180° < x < 360°$ equals ....

(a) {210°} (b) {225°} (c) {240°} (d) {315°}

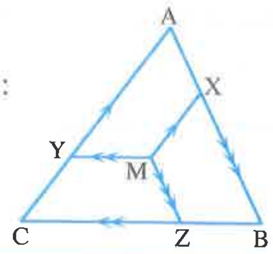
4 If $\begin{vmatrix} \ell & 2 & -1 \\ x-2 & m & 3 \\ 0 & 0 & n \end{vmatrix} = \ell\, m\, n$ where ℓ , m , n non-zero numbers , then $x =$ ....

(a) 2 (b) – 2 (c) 6 (d) mn

5 **In the opposite figure :**

If M is the point of intersection of the medians of Δ ABC , prove that :

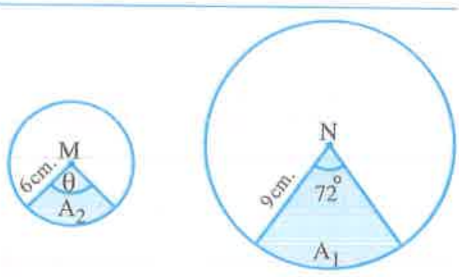
$\overrightarrow{MX} + \overrightarrow{MY} + \overrightarrow{MZ} = \overrightarrow{O}$



6 **In the opposite figure :**

M and N are two distinct circles and A_1 , A_2 are the areas of the two sectors.

If $\frac{A_1}{A_2} = \frac{9}{5}$, then $\theta =$ ....



(a) 72° (b) 80° (c) 90° (d) 100°

7 **The solution set of the inequality :** $-X \le y \le X$ is ....

(a)

(b)

(c)

(d)

---

8 If $\overrightarrow{M} = (2, 5)$, $\overrightarrow{N} = (1, 3)$, then $2\overrightarrow{M} - \overrightarrow{N} =$ ....

(a) $(3, 7)$ (b) $(1, 2)$ (c) $(3, 8)$ (d) $(0, 1)$

---

9 An equilateral triangle of side length 8 cm. , then its area = ... cm^2.

(a) $8\sqrt{3}$ (b) $16\sqrt{3}$ (c) $24\sqrt{3}$ (d) $32\sqrt{3}$

---

10 If $\overrightarrow{A} = \left(6, \frac{\pi}{2}\right)$, $\overrightarrow{B} = 4\overrightarrow{i} + 3\overrightarrow{j}$, then $\overrightarrow{AB} =$ ....

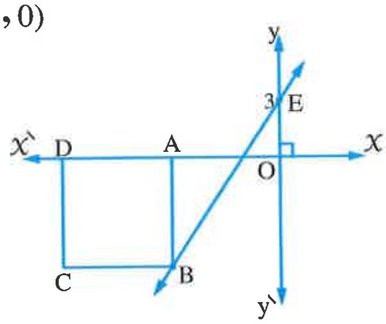
(a) $(-4, 3)$ (b) $(4, 3)$ (c) $(4, -3)$ (d) $(-4, -3)$

---

11 **In the opposite figure :**

If the area of the square ABCD = 36 square unit and D $(-12, 0)$

, then the vector equation of the straight line $\overleftrightarrow{EB}$ is ....



(a) $\overrightarrow{r} = (0, 3) + k(3, 2)$

(b) $\overrightarrow{r} = (0, 3) + k(-3, 2)$

(c) $\overrightarrow{r} = (-6, -6) + k(-2, 3)$

(d) $\overrightarrow{r} = (-6, -6) + k(2, 3)$

---

12 The point which aren't lying in the region of the solution of the inequality : $2X - y \le 7$ in $\mathbb{R} \times \mathbb{R}$ is ....

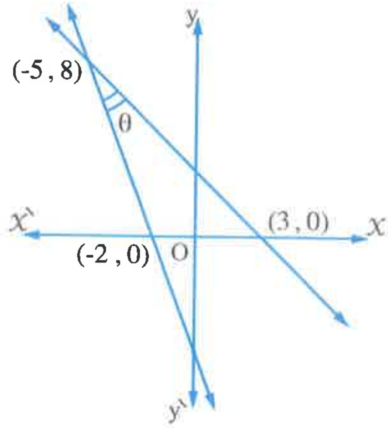
(a) $(0, 0)$ (b) $(2, 0)$ (c) $(3, -2)$ (d) $(5, 4)$

---

13 If $\overrightarrow{M} = (2, 1)$ is a direction vector of a straight line , then all of the following vectors are perpendicular to the straight line except the vector ....

(a) $(1, -2)$ (b) $(-2, 4)$ (c) $(-1, 2)$ (d) $(1, 2)$

14 **In the opposite figure :**



Tan θ = ...

(a) $\frac{8}{3}$

(b) $\frac{3}{11}$

(c) $\frac{5}{11}$

(d) $\frac{7}{11}$

---

15 If $\overrightarrow{F_1} = \overrightarrow{i} - 3\overrightarrow{j}$, $\overrightarrow{F_2} = 3\overrightarrow{i} + 6\overrightarrow{j}$, then the force $\overrightarrow{F_3}$ which makes the resultant of the three forces is a unit vector and acts in the direction of positive part of y-axis equals ....

(a) $-3\overrightarrow{i} - 3\overrightarrow{j}$ (b) $-4\overrightarrow{i} - 2\overrightarrow{j}$ (c) $-5\overrightarrow{i} - 3\overrightarrow{j}$ (d) $-4\overrightarrow{i} - 3\overrightarrow{j}$

---

16 If the length of the intercepted part of y-axis by a straight line is twice the intercepted part of the X-axis and the straight line passes through the point (1 , 2) , then the equation of the straight line is ....

(a) $2X + y = -4$ (b) $2X - y = 4$

(c) $2X - y + 4 = 0$ (d) $2X + y - 4 = 0$

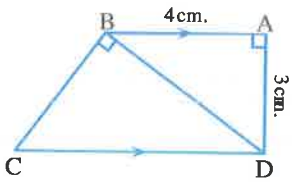
---

17 The area of the circular segment in a circle with radius length 10 cm. and its arc length 5 cm. approximately equals .... cm^2

(a) 0.13 (b) 0.51 (c) 2.05 (d) 1.03

---

18 **In the given figure :**



The length of $\overline{BC}$ = ...

(a) 5 cm. (b) $6\frac{2}{3}$ cm.

(c) $3\frac{3}{4}$ cm. (d) 3 cm.

---

19 $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$ = ...

(a) zero (b) sin θ (c) 1 (d) cos θ

20 $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \dots\dots$

(a) $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

---

21 If $\overrightarrow{A} = 2\overrightarrow{i} + 3\overrightarrow{j}$, $\overrightarrow{B} = (k-1)\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{A} \parallel \overrightarrow{B}$, then $k = \dots\dots$

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $-\frac{5}{3}$ (d) $-\frac{3}{5}$

---

22 If $\overrightarrow{AB} = (-3, 2)$, $\overrightarrow{BC} = (0, 2)$, then $\| \overrightarrow{AC} \| = \dots\dots$

(a) $\sqrt{13} + 2$ (b) $\sqrt{13} - 2$ (c) 4 (d) 5

---

23 Which of the following points belongs to the solution set of the system :

$x > 0$, $y > 0$, $2x + y > 6$?

(a) (1 , 3) (b) (0 , 0) (c) (2 , 3) (d) (4 , – 2)

---

24 Find the measure of the acute angle between the two straight lines :

$\ell_1 : \overrightarrow{r} = (4, 2) + k(-3, 1)$, $\ell_2 : 2x = 3 - y$

---

25 The matrix $\begin{pmatrix} x+3 & 2 \\ 2 & x-3 \end{pmatrix}$ has no multiplicative inverse at $x = \dots\dots$

(a) ± 3 (b) $\pm\sqrt{13}$ (c) 5 (d) ± 5

---

26 If ℓ , m are the roots of the equation : $x^2 - 4x - 10 = 0$

, find the value of the determinant $\begin{vmatrix} 2\ell & -1 \\ 3 & m \end{vmatrix}$

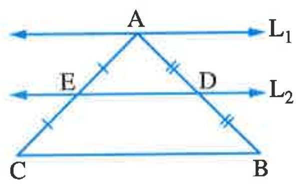
---

27 **In the given figure :**

If $L_1 : 3x - 4y + 10 = 0$, $L_2 : 3x = 4y$

, then the length of the perpendicular drawn

from A to $\overleftrightarrow{BC} = \dots\dots$ length unit.



(a) 1 (b) 2 (c) 4 (d) 6

28 The coordinates of the point C which divides $\overline{AB}$ internally in ratio 1 : 2 where A (5 , – 6) , B (– 1 , 3) is ....

(a) (0 , 0) (b) (3 , 3) (c) (– 3 , – 3) (d) (3 , – 3)

29 The general solution of the equation : $\cos \theta = -1$ is .... where $n \in \mathbb{Z}$

(a) $\frac{\pi}{2} + n\pi$ (b) $\frac{\pi}{3} + n\pi$ (c) $\pi + 2\pi n$ (d) $\frac{\pi}{6} + 2n\pi$

30 If $\overrightarrow{AB} = \overrightarrow{CD}$, $\overrightarrow{CD} = (6, 4)$, $\overrightarrow{A} = (-1, 3)$, then $\overrightarrow{B} =$ ....

(a) (5 , 7) (b) (– 5 , – 7) (c) (– 5 , 7) (d) (5 , – 7)

31 If $A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix}$, find the matrix B

32 The polar form of the vector $\overrightarrow{A} = \sqrt{3}\,\overrightarrow{i} + \overrightarrow{j}$ is ....

(a) $\left(2, \frac{\pi}{3}\right)$ (b) $\left(4, \frac{\pi}{3}\right)$ (c) $\left(2, \frac{\pi}{6}\right)$ (d) $\left(4, \frac{\pi}{6}\right)$

33 The area of the triangle with vertices (1 , 6) , (0 , 10) , (0 , 0) equals .... square unit.

(a) 5 (b) 10 (c) 15 (d) 20

Model 8

Interactive test 8

Answer the following questions :

1 If $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -3 & x \end{pmatrix} = I$, then $x =$ ....

(a) 4 (b) 5 (c) 1 (d) zero

2 If A is a square matrix , then $(A + A^t)$ is ....

(a) symmetric matrix. (b) skew symmetric matrix.

(c) zero matrix. (d) diagonal matrix.

3 Measure of the included angle between the two straight lines their slopes are $\frac{1}{2}$ and -2 equals ....

(a) 45° (b) 90° (c) 120° (d) 135°

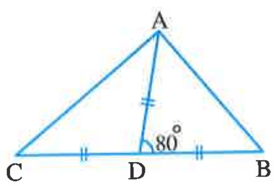
4 If x and y are two integers where $x > 0$, $y > 0$ and $x + y < 5$, then the number of ordered pairs (x , y) that satisfy the previous conditions are ....

(a) 4 (b) 5 (c) 6 (d) 7

5 The additive inverse of the vector $\overrightarrow{AB}$ is the vector ....

(a) $\overrightarrow{BA}$ (b) $\overrightarrow{O}$ (c) $-\overrightarrow{BA}$ (d) $\overrightarrow{AB}$

6 **In the opposite figure :**



If $D \in \overline{BC}$ where DA = DB = DC = 5 cm. , m (∠ ADB) = 80°

, then AC = ... cm.

(a) 10 sin 40° (b) 10 sin 50°

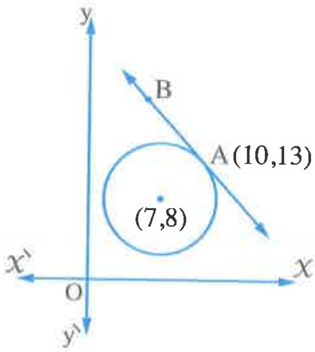
(c) 5 sin 80° (d) 5 sin 40°

7 $2 \sin^2 3 x^3 + 2 \cos^2 3 x^3 =$ ....

(a) 6 (b) 54 (c) 2 (d) $2 x^3$

8 **In the opposite figure :**

A circle whose centre (7 , 8) , the straight line $\overleftrightarrow{AB}$ is a tangent to it at A , then equation of the straight line $\overleftrightarrow{AB}$ is ....



(a) $5x + 3y = 95$ (b) $3x + 5y = 35$

(c) $3x + 5y + 95 = 0$ (d) $3x + 5y = 95$

9 If $\overrightarrow{AB} = \overrightarrow{CD}$ where $\overrightarrow{AB} = (5 , -11)$, $\vec{C} = (-1 , 8)$, then $\vec{D} =$ ....

(a) $(4 , -3)$ (b) $(-5 , -7)$ (c) $(4 , 3)$ (d) $(6 , -17)$

10 Area of the triangle subtended by the straight lines : $x = 0$, $y = 0$ and $\frac{x}{2} + \frac{y}{5} = 1$ equals .... square units.

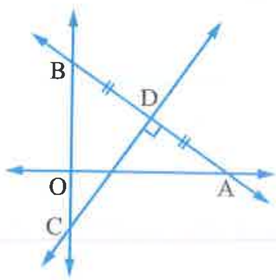
(a) 2 (b) 5 (c) 10 (d) 7

11 If $\vec{A}$ and $\vec{B}$ are two unit vectors , then ....

(a) $\|\vec{A} + \vec{B}\| = 2$ (b) $\|\vec{A} - \vec{B}\| = 2$ (c) $\|\vec{A} + \vec{B}\| \geq 2$ (d) $\|\vec{A} + \vec{B}\| \leq 2$

12 **In the opposite figure :**

If the equation of the straight line $\overleftrightarrow{AB}$ is : $2x + 3y = 12$, then the vector equation of the straight line $\overleftrightarrow{DC}$ is ....



(a) $\vec{r} = (2 , 3) + k (2 , 3)$

(b) $\vec{r} = (2 , 3) + k (3 , 2)$

(c) $\vec{r} = (3 , 2) + k (2 , 3)$

(d) $\vec{r} = (3 , 2) + k (3 , 2)$

13 Find the solution set of the equation $\begin{vmatrix} x^2 & 5 \\ 3 & x \end{vmatrix} = 12$ in $\mathbb{R}$

14 If $0° \leq \theta < 360°$ and $\cos \theta + 1 = 0$, then $\theta =$ ....

(a) 90° (b) 180° (c) 270° (d) 360°

15 The point that belongs to the S.S. of the following inequalities :

$x \geq 0$, $y \geq 0$, $2x + y < 4$ and $x + 3y < 6$ is ....

(a) $(1 , -3)$ (b) $(3 , 0)$ (c) $(2 , 3)$ (d) $(1 , 1)$

16 Which pair of the following vectors are perpendicular ?

(a) (3 , 0) , (2 , – 1)
(b) (– 2 , 5) , (4 , – 10)
(c) (2 , 0) , (0 , 2)
(d) (1 , – 4) , (2 , – 8)

17 From the top of a tower 80 m. high. , the measure of depression angle of a body lies on the horizontal plane that passing through the tower base equals 24° 12̀ , find the distance between the body and the base of the tower to the nearest metre.

18 If $\overrightarrow{A} = (6 , 8)$, $\overrightarrow{B} = (3 , m)$, $\overrightarrow{C} = (n , 9)$ and $\overrightarrow{A} // \overrightarrow{B}$, $\overrightarrow{B} \perp \overrightarrow{C}$, then m + n = ...

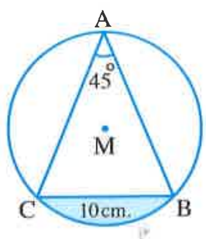
(a) 8
(b) – 8
(c) 12
(d) – 12

19 If $\sin \theta = \frac{a}{b}$, $\theta \in]0 , \frac{\pi}{2}[$, then $\sqrt{1 + \tan^2 \theta} =$ ....

(a) $\frac{1}{\sqrt{a^2 - b^2}}$
(b) $\frac{1}{\sqrt{a - b}}$
(c) $\frac{a}{\sqrt{1 + a^2}}$
(d) $\frac{b}{\sqrt{b^2 - a^2}}$

20 If $\overrightarrow{A} = \left(2 , \frac{\pi}{6}\right)$, find the Cartesian form of the vector $2\overrightarrow{A}$

21 **In the given figure :**



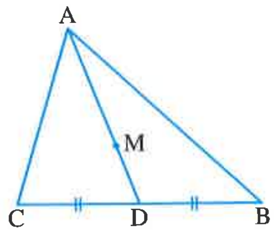
The area of the shaded part approximately equals .... cm.^{2}

(a) 7.1
(b) 28.5
(c) 14.3
(d) 2.02

22 The perimeter of a circular sector is (4 r) cm. where r is the radius length of its circle , then the radian measure of its central angle equals .... radian.

(a) $\frac{1}{2}$
(b) 8
(c) 2
(d) $\frac{1}{3}$

23 **In the opposite figure :**



M is the intersection point of the medians of Δ ABC

, $\overrightarrow{AB} + \overrightarrow{AC} = k\,\overrightarrow{AM}$, then k = ...

(a) $\frac{1}{3}$
(b) 3
(c) $\frac{1}{2}$
(d) 2

24 If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 6$ and $\begin{vmatrix} ak & kc \\ b & d \end{vmatrix} = -24$, then k = ...

(a) 4 (b) 3 (c) – 3 (d) – 4

25 If $\overrightarrow{A} = -7\overrightarrow{B}$, then ....

(a) $\overrightarrow{A} \perp \overrightarrow{B}$ (b) $\overrightarrow{A} \parallel \overrightarrow{B}$ (c) $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$ (d) $7\overrightarrow{A} = \overrightarrow{B}$

26 Measure of the acute angle between the two straight lines : $x - \sqrt{3}\,y = 5$, $y = 2$ equals ....

(a) 90° (b) 60° (c) 45° (d) 30°

27 If the slope of a straight line = 0.6 , then its direction vector could be ....

(a) (5 , 3) (b) (3 , 5) (c) (– 3 , 5) (d) (3 , – 5)

28 Find the coordinates of the point lies at $\frac{2}{5}$ the distance from A to B to the directed segment $\overrightarrow{AB}$ where A (3 , – 2) , B (– 1 , 5)

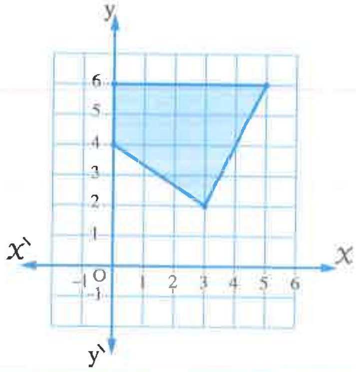
29 The value of the determinant $\begin{vmatrix} 4 & 0 & 0 \\ 7 & 2 & 0 \\ 1 & 5 & 1 \end{vmatrix} =$ ....

(a) 1 (b) 2 (c) 4 (d) 8

30 Find the length of the perpendicular drawn from the point (3 , 1) to the straight line $4x + 3y - 5 = 0$

31 By using the given figure , the point which makes the objective function : $P = 3x + 2y$ as small as possible ....

(a) (0 , 4) (b) (3 , 2)
(c) (5 , 6) (d) (0 , 6)



32 If $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$, then a + b = ...

(a) 3 (b) 4 (c) 5 (d) 6

33 If $\begin{pmatrix} x \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, then $x + y =$ ....

(a) 5 (b) 4 (c) 3 (d) 1

Interactive test 9

Model 9

Answer the following questions :

1 If A is a matrix of order 1×3 , B^t is a matrix of order 1×3 , then its possible to find ....

(a) $A + B$ (b) $B^t + A^t$ (c) $A\,B^t$ (d) $A\,B$

2 Projection of the point $(2 , 3)$ on the straight line $L : X + y = 11$ is ....

(a) $(-6 , 5)$ (b) $(6 , 5)$ (c) $(5 , 6)$ (d) $(-5 , 6)$

3 Measure of the obtuse angle included between the two straight lines :

$y = \left(2 - \sqrt{3}\right)(X + 5)$, $y = \left(2 + \sqrt{3}\right)(X - 7)$ is ....

(a) 150° (b) 60° (c) 135° (d) 120°

4 If A is a square matrix in order 2×2 and $|2A| = 8$, find $|3A^t|$

5 The productive rate of a factory is 120 units at most of two different kinds of articles the number of produced units from each kind equals X and y respectively.

If what is sold from the second kind doesn't less than half what is sold from the first kind.

Which of the following system of inequalities represent the previous data and conditions ?

(a) $X \geq 0$, $y \geq 0$, $X + y \leq 120$, $2y \leq X$

(b) $X \geq 0$, $y \geq 0$, $X + y \geq 120$, $y \leq 2X$

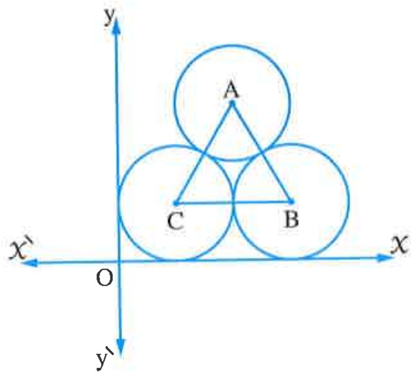
(c) $X \geq 0$, $y \geq 0$, $X + y \leq 120$, $2y \geq X$

(d) $X \geq 0$, $y \geq 0$, $X + y \leq 120$, $y \geq 2X$

6 **In the opposite figure :**

Three congruent circles are tangent to each other

If $C = (4 , 4)$, then equation of the straight line $\overleftrightarrow{AB}$ is ....



(a) $\vec{r} = (4 , 4) + k\left(1 , -\sqrt{3}\right)$

(b) $\vec{r} = (8 , 4) + k\left(1 , -\sqrt{3}\right)$

(c) $\vec{r} = (12 , 4) + k\left(1 , -\sqrt{3}\right)$

(d) $\vec{r} = (12 , 4) + k\left(-\sqrt{3} , 1\right)$

7 If $\overrightarrow{A} = (6, 4)$, $\overrightarrow{B} = (2, m)$ and $\overrightarrow{A} \perp \overrightarrow{B}$, then $m =$ ....

(a) -3 (b) 3 (c) -6 (d) 4

8 The direction vector of the straight line $a x + by + c = 0$ is ....

(a) (a, b) (b) $(a, -b)$ (c) (b, a) (d) $(b, -a)$

9 Number of solutions of the equation : $\sin x = 0$ where $x \in [0, 6\pi[$ is ....

(a) 2 (b) 4 (c) 6 (d) 8

10 If the measure of the included angle between the two straight lines :
$x = 7$, $y = a x + 2$ equals $90°$, then $a =$ ....

(a) zero (b) 1 (c) 90 (d) -1

11 Which of the following statement is incorrect ?

(a) If $\overrightarrow{A} = \overrightarrow{B}$, then $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$ (b) If $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$, then $\overrightarrow{A} = \overrightarrow{B}$

(c) If $\overrightarrow{A} \mathbin{/\mkern-6mu/} \overrightarrow{B}$, then $\overrightarrow{A} = k\overrightarrow{B}$ (d) If $\overrightarrow{A} = k\overrightarrow{B}$, then $\overrightarrow{A} \mathbin{/\mkern-6mu/} \overrightarrow{B}$

12 The quadrant that represent the S.S. of the system of the two inequalities : $x > 0$, $y > 0$ is the .... quadrant.

(a) first (b) second (c) third (d) fourth

13 If $\theta \in]0°, 360°[$, $2\cos\theta + \sqrt{3} = 0$, then θ may be equals ....

(a) $30°$ (b) $60°$ (c) $210°$ (d) $300°$

14 If $\sin\theta$, $\cos\theta$ are the two roots of the equation : $2x^2 + bx - 1 = 0$, then $b =$ ....

(a) zero (b) 2 (c) 3 (d) -4

15 If the area of a regular hexagon is $54\sqrt{3}$ cm^2 , then its side length equals .... cm.

(a) 6 (b) 12 (c) $6\sqrt{3}$ (d) $12\sqrt{3}$

16 Which of the following vectors represents the velocity of a car moves with speed of magnitude 100 km./hr. in the direction 60° West of North ?

(a) $50\overrightarrow{i} + 50\sqrt{3}\,\overrightarrow{j}$ (b) $50\sqrt{3}\,\overrightarrow{i} - 50\overrightarrow{j}$ (c) $-50\sqrt{3}\,\overrightarrow{i} + 50\overrightarrow{j}$ (d) $-50\overrightarrow{i} - 50\sqrt{3}\,\overrightarrow{i}$

17 If ABCD is a parallelogram , A (1 , 2) , B (X , – 3) , C (– 3 , 5) , D (– 7 , y)

, find $\overrightarrow{BD}$

18 If $\overrightarrow{A}$ = (7 , 8) , $\overrightarrow{B}$ = (4 , 4) , then all of the following is true except ....

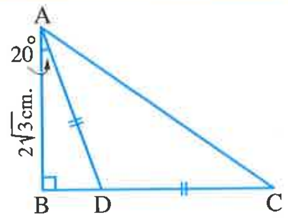
(a) $\overrightarrow{A} + \overrightarrow{B}$ = (11 , 12)

(b) $\overrightarrow{A} - \overrightarrow{B}$ = (3 , 4)

(c) 2 $\overrightarrow{A} - \overrightarrow{B}$ = (10 , 12)

(d) $\overrightarrow{AB} + \overrightarrow{A}$ = (4 , 6)

19 **In the opposite figure :**



The length of $\overline{AC}$ ≈ .... cm.

(a) 6

(b) 10

(c) 4

(d) 5

20 Find the solution set of the equation $\begin{vmatrix} X+2 & 2 \\ X & X-3 \end{vmatrix} = 4$

21 If X , y ∈ [0 , 2 π[, θ = X + y , then the values of θ which satisfy :

sin X sin y = 1 are ....

(a) {π , 2 π}

(b) {π , 3 π}

(c) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

(d) $\left\{\frac{\pi}{2}, \frac{\pi}{3}\right\}$

22 If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, then A^{2019} = ...

(a) A

(b) A^2

(c) 2019 A^3

(d) 2019 I

23 Equation of the straight line that is equidistant from the two straight lines :

y = – 2 , y = 10 is ....

(a) y = 8

(b) y = 4

(c) X = 4

(d) X = – 12

24 If the heigth of a circular segment is 5 cm. and the radius length of its circle is 10 cm.

, find the area of the circular segment to the nearest cm^2.

25 If 2 $\overrightarrow{AB}$ + 2 $\overrightarrow{BC}$ = k $\overrightarrow{CA}$, then k = ...

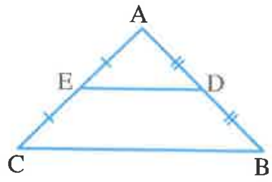
(a) 1

(b) 2

(c) – 1

(d) – 2

26 **In the opposite figure :**



D , E are the midpoints of $\overline{AB}$, $\overline{AC}$

, $\overrightarrow{AB} = \overrightarrow{M}$, $\overrightarrow{AC} = \overrightarrow{N}$

, then $\overrightarrow{DE} =$ ....

(a) $\overrightarrow{M} + \overrightarrow{N}$
(b) $\overrightarrow{M} - \overrightarrow{N}$
(c) $\frac{1}{2}\left(\overrightarrow{M} - \overrightarrow{N}\right)$
(d) $\frac{-1}{2}\left(\overrightarrow{M} - \overrightarrow{N}\right)$

---

27 If $\overrightarrow{A} = (6 , -8)$, $\overrightarrow{B} = (-3 , 4)$, find the ratio that the x-axis divides $\overline{AB}$

---

28 If A is a skew symmetric matrix , then $A + A^t =$ ....

(a) 2 A (b) $2 A^t$ (c) O (d) zero

---

29 If $X + \begin{pmatrix} 2 & 0 \\ 5 & 7 \end{pmatrix}^t = O$, then X = ...

(a) $\begin{pmatrix} 2 & -5 \\ 0 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & -5 \\ 0 & -7 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 5 \\ 0 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 0 \\ 5 & 7 \end{pmatrix}$

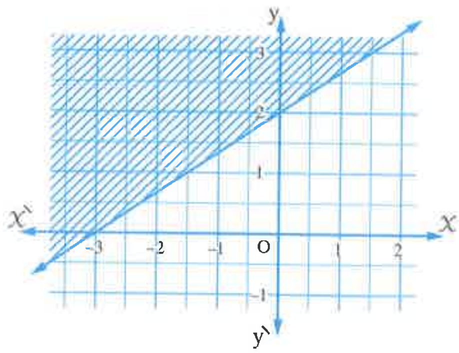
---

30 If $\overrightarrow{A} = (x , 4)$, $\overrightarrow{B} = (2 , y)$ and $\overrightarrow{A} \mathbin{//} \overrightarrow{B}$, then ....

(a) $x + 2y = 0$ (b) $x = 2y$ (c) $xy = 8$ (d) $\frac{x}{y} = 2$

---

31 Which of the following inequalities represent the opposite graph ?



(a) $x - 2y - 6 \leq 0$
(b) $2x - 3y + 6 \leq 0$
(c) $3x - 2y + 12 \leq 0$
(d) $3x + 2y + 12 \geq 0$

---

32 If $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 2 & -1 \\ x & 3 \end{pmatrix}$, then $x =$ ....

(a) 3 (b) – 3 (c) 5 (d) – 5

---

33 The area of the triangle bounded by the straight line $\frac{x}{4} + \frac{y}{3} = 1$ and the coordinates axes equals .... square unit.

(a) 6 (b) 12 (c) 4 (d) 3

Model 10

Interactive test 10

Answer the following questions :

1 If $X \times \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then X = ...

(a) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (c) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (d) $3\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

2 If AB = AC , then ....

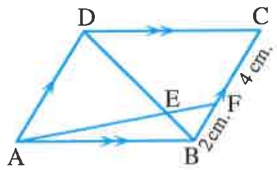
(a) B = C for all matrices A (b) B ≠ C for all matrices A

(c) B = C at | A | = zero (d) B = C at | A | ≠ zero

3 The normal straight line to the straight line $\vec{r} = (3 , 2) + k\left(1 , -\sqrt{3}\right)$ makes with the positive direction of X-axis positive angle of measure ....

(a) 120° (b) 30° (c) 60° (d) 150°

4 **In the opposite figure :**



If ABCD is a parallelogram in which BF = 2 cm. , FC = 4 cm. , then $\overrightarrow{AE}$ = ...

(a) $\frac{1}{2}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AD}$ (b) $\frac{1}{4}\overrightarrow{AB} + \frac{3}{4}\overrightarrow{AD}$

(c) $\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AD}$ (d) $\frac{3}{4}\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AD}$

5 If A , B are two square matrices of order 3 × 3 and | A | = – 1 , | B | = 3 , then | 3 AB | = ...

(a) – 9 (b) – 81 (c) – 27 (d) 81

6 If matrix A of order 2 × 3 and AB of order 2 × 4 , then B^t is a matrix of order ....

(a) 4 × 2 (b) 2 × 4 (c) 4 × 3 (d) 3 × 4

7 The expression $\frac{1-\cos^2\theta}{\sin^2\theta-1}$ in its simplest form equals ....

(a) $-\tan^2\theta$ (b) $-\cot^2\theta$ (c) $\tan^2\theta$ (d) $\cot^2\theta$

8 If $\overrightarrow{OC} = \left(12\sqrt{2} , \frac{3\pi}{4}\right)$ is a position vector of point C with respect to the origin , then point C is ....

(a) $(-6 , 6)$ (b) $(-12 , 12)$ (c) $(12 , -12)$ (d) $(6 , -6)$

9 If $\| 4k\overrightarrow{A} \| = \| -12\overrightarrow{A} \|$, then k = ...

(a) 3 (b) −3 (c) ±3 (d) ±4

10 Which of the following verbal expressions represents the inequality $y \geq 2x$

(a) Two numbers , one of them greater than twice the other.

(b) Two numbers , one of them not more than twice the other.

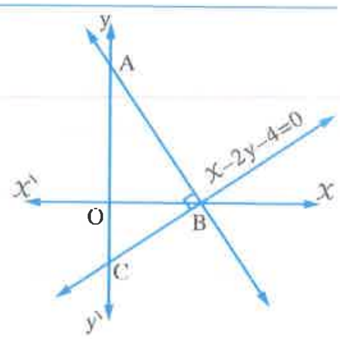
(c) Two numbers , one of them less than twice the other.

(d) Two numbers , one of them not less than twice the other.

11 The solution set of the equation $\csc\theta = -2$ where $\theta \in [0 , 2\pi[$ is ....

(a) $\{30° , 150°\}$ (b) $\{30° , 330°\}$ (c) $\{210° , 330°\}$ (d) $\{150° , 210°\}$

12 **In the opposite figure :**



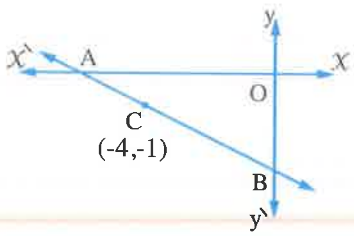
the area of Δ ABC

= .... square unit.

(a) 15 (b) 20

(c) 24 (d) 32

13 **In the opposite figure :**



If BC = 2 CA , then the equation of the straight line $\overleftrightarrow{AB}$ is ....

(a) $x + 2y + 6 = 0$ (b) $x - 2y - 6 = 0$

(c) $2x + y + 9 = 0$ (d) $x - 3y + 1 = 0$

14 The area of an equilateral triangle whose side length x cm. equals .... cm^2.

(a) x^2 (b) $\frac{\sqrt{3}}{2}x^2$ (c) $\frac{\sqrt{3}}{4}x^2$ (d) $\frac{1}{2}x^2$

---

15 If $\overrightarrow{A}$ is non-zero vector , then ....

(a) $-\overrightarrow{A} \perp \overrightarrow{A}$

(b) $-\overrightarrow{A}$ and $\overrightarrow{A}$ have same direction.

(c) $\|-\overrightarrow{A}\| < \|\overrightarrow{A}\|$

(d) $-\overrightarrow{A}$ and $\overrightarrow{A}$ have opposite directions.

---

16 The point .... belongs to the solution set of the inequality : $2x + y > 6$

(a) $(1 , 4)$ (b) $(0 , 0)$ (c) $(4 , -2)$ (d) $(2 , 3)$

---

17 Value (s) of x that makes the matrix $\begin{pmatrix} x & 4 \\ 9 & x \end{pmatrix}$ has no multiplicative inverse is ....

(a) $-6 , 4$ (b) $6 , 9$ (c) ± 6 (d) 36

---

18 A circular sector , its perimeter 10 cm. and length of its arc is 2 cm.

, then its area equals .... cm^2.

(a) 4 (b) 8 (c) 10 (d) 20

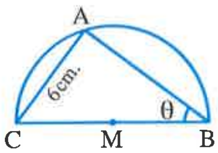
---

19 Length of the drawn perpendicular from the origin point to the straight line whose equation $3x - 4y - 15 = 0$ equals .... length unit.

(a) 3 (b) 4 (c) 5 (d) 15

---

20 **In the opposite figure :**



$\overline{BC}$ is a diameter in the circle M

, AC = 6 cm. , m ($\angle$ ABC) = θ , then area of Δ ABC = ... cm^2.

(a) $6 \sin \theta$ (b) $6 \tan \theta$ (c) $18 \tan \theta$ (d) $18 \cot \theta$

---

21 If $\sin A + \sin^2 A = 1$, find $\cos^2 A + \cos^4 A$

---

22 The vector equation of the x-axis is ....

(a) $\overrightarrow{r} = (1 , 1) + k (0 , 0)$

(b) $\overrightarrow{r} = (1 , 0) + k (1 , 1)$

(c) $\overrightarrow{r} = k (1 , 0)$

(d) $\overrightarrow{r} = k (0 , 1)$

23 If $\sin\theta + \cos\theta = 1.4$, then $\sin\theta\cos\theta =$ ....

(a) 0.96 (b) 0.24 (c) 0.48 (d) ± 0.24

---

24 If $\overrightarrow{A} = 2\overrightarrow{i} - \overrightarrow{j}$, $\overrightarrow{B} = \overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{C} = \overrightarrow{i} + 3\overrightarrow{j}$ and $\overrightarrow{A} \perp \left(k\overrightarrow{B} + \overrightarrow{C}\right)$, find the value of k

---

25 If (1 , y) lies on the region of the S.S. of the inequality : $x + 2y < 7$, then ....

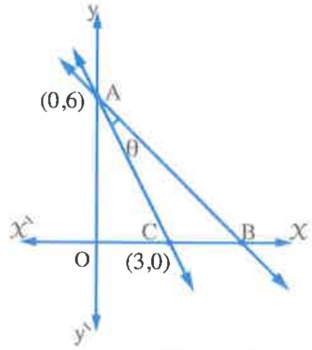
(a) $y < 3$ (b) $y > 3$ (c) $y = 3$ (d) $y > 7$

---

26 **In the opposite figure :**

If $\cos\theta = \frac{2}{\sqrt{5}}$

Find the coordinates of the point B



---

27 If $\overrightarrow{A} = 3\overrightarrow{B}$, then ....

(a) $\overrightarrow{A} \perp \overrightarrow{B}$ (b) $\overrightarrow{A} // \overrightarrow{B}$ (c) $\|\overrightarrow{A}\| = \|\overrightarrow{B}\|$ (d) $3\overrightarrow{A} = \overrightarrow{B}$

---

28 Triangle ABC in which D is the midpoint of $\overline{BC}$, then $\overrightarrow{AB} + \overrightarrow{AC} =$ ....

(a) $\overrightarrow{BC}$ (b) $2\overrightarrow{AD}$ (c) $\overrightarrow{O}$ (d) $\overrightarrow{AD}$

---

29 Find the value of x , if $\begin{vmatrix} 2x & 0 & 0 \\ 1 & 3x & 0 \\ 2 & 4 & -x \end{vmatrix} = 48$

---

30 If $\begin{vmatrix} 2k-1 & 2 \\ 3 & k+1 \end{vmatrix} = 2k^2 + 1$, then the value of k = ...

(a) 2 (b) 4 (c) 6 (d) 8

---

31 If $\overrightarrow{A} = k\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{B} = (k-2 , 2)$ and $\overrightarrow{A} // \overrightarrow{B}$, then k = ...

(a) 2 (b) -2 (c) ± 2 (d) 3

---

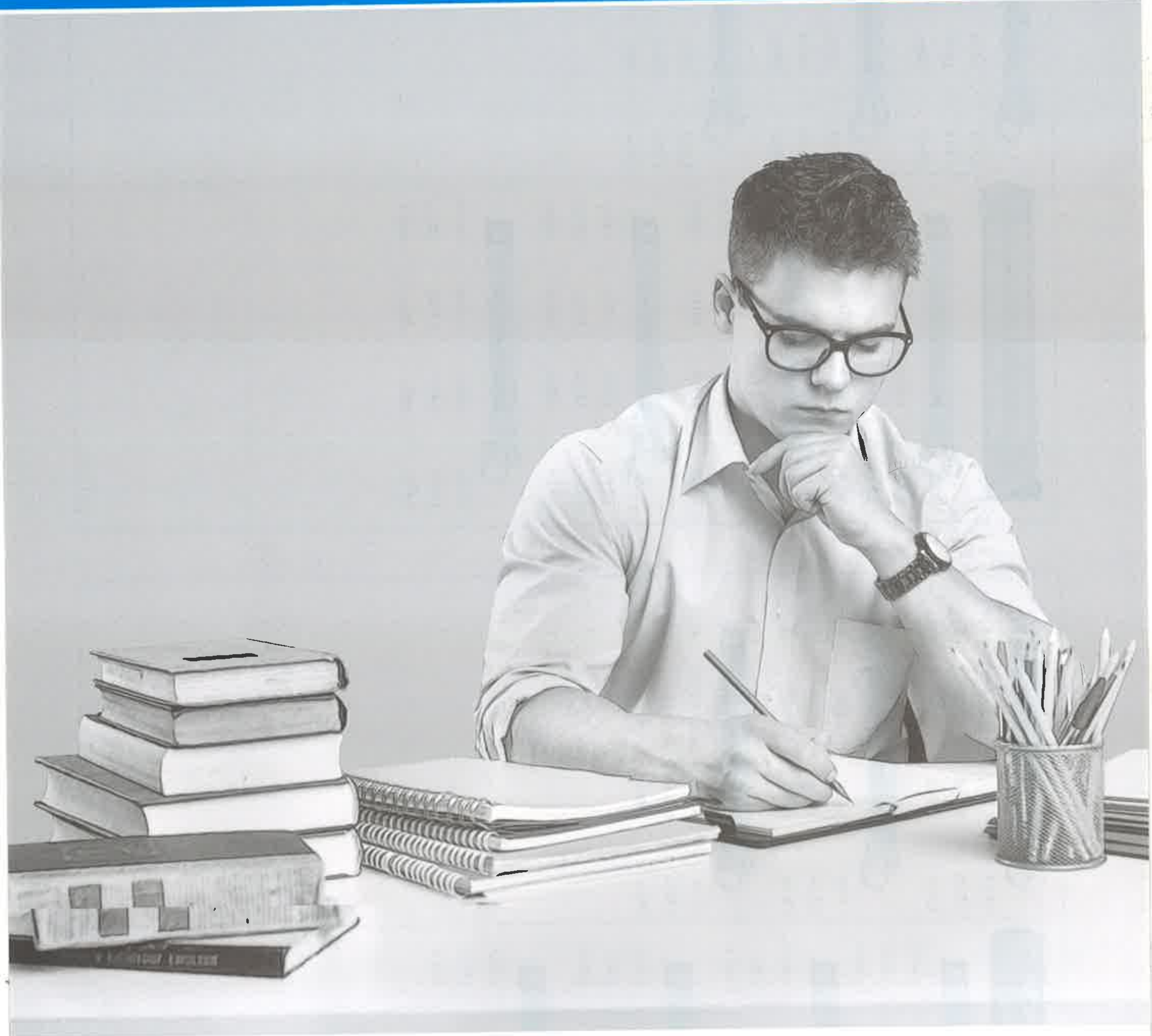
32 If $C \in \overline{AB}$, $3\overrightarrow{AB} = 5\overrightarrow{CB}$, then C divides $\overline{BA}$ by the ratio ....

(a) 2 : 3 (b) 3 : 2 (c) 3 : 5 (d) 5 : 3

---

33 Find the measure of the angle between the two straight lines $\overrightarrow{r} = (3 , 1) + k(2 , 1)$, $2x + y + 5 = 0$

ANSWERS



answers of accumulative quizzes on Algebra

Accumulative quiz 1

(1) b	(2) d	(3) b	(4) a
(5) b	(6) d	(7) b	(8) b
(9) b	(10) b	(11) c	(12) b

Accumulative quiz 2

(1) d	(2) d	(3) c	(4) d
(5) b	(6) a	(7) c	(8) c
(9) b	(10) c	(11) c	(12) a

Accumulative quiz 3

(1) a	(2) b	(3) d	(4) b
(5) d	(6) b	(7) c	(8) c
(9) a	(10) d	(11) a	(12) d

Accumulative quiz 4

(1) b	(2) d	(3) b	(4) c
(5) d	(6) c	(7) c	(8) b
(9) c	(10) b	(11) b	(12) b

Accumulative quiz 5

(1) d	(2) d	(3) a	(4) b
(5) b	(6) d	(7) c	(8) a
(9) a	(10) d	(11) d	(12) a

Accumulative quiz 6

(1) b	(2) c	(3) b	(4) d
(5) a	(6) a	(7) a	(8) b
(9) a	(10) a	(11) b	(12) c

Accumulative quiz 7

(1) c	(2) b	(3) d	(4) c
(5) c	(6) b	(7) b	(8) c
(9) c	(10) d	(11) d	(12) c

answers of accumulative quizzes on Trigonometry

Accumulative quiz 1

(1) b	(2) b	(3) c	(4) a
(5) a	(6) d	(7) c	(8) b
(9) d	(10) b	(11) d	(12) d

Accumulative quiz 2

(1) b	(2) c	(3) c	(4) a
(5) b	(6) c	(7) b	(8) d
(9) d	(10) c	(11) a	(12) d

Accumulative quiz 3

(1) b	(2) b	(3) c	(4) a
(5) c	(6) a	(7) c	(8) b
(9) a	(10) a	(11) d	(12) b

Accumulative quiz 4

(1) a	(2) c	(3) a	(4) d
(5) a	(6) c	(7) d	(8) a
(9) b	(10) c	(11) a	(12) b

Accumulative quiz 5

(1) a	(2) a	(3) b	(4) c
(5) b	(6) c	(7) c	(8) d
(9) b	(10) a	(11) c	(12) d

Accumulative quiz 6

(1) b	(2) b	(3) b	(4) b
(5) b	(6) d	(7) c	(8) a
(9) b	(10) a	(11) d	(12) b

Accumulative quiz 7

(1) a	(2) c	(3) d	(4) c
(5) a	(6) a	(7) c	(8) b
(9) a	(10) c	(11) a	(12) d

answers of accumulative quizzes on Analytic Geometry

Accumulative quiz 1

(1) a	(2) d	(3) c	(4) c
(5) d	(6) c	(7) d	(8) d

Accumulative quiz 2

(1) c	(2) c	(3) c	(4) d
(5) b	(6) a	(7) c	(8) c
(9) d	(10) c	(11) b	(12) d

Accumulative quiz 3

(1) a	(2) d	(3) a	(4) b
(5) d	(6) d	(7) b	(8) b
(9) b	(10) d	(11) c	(12) c

Accumulative quiz 4

(1) c	(2) d	(3) c	(4) c
(5) c	(6) b	(7) d	(8) b
(9) d	(10) c	(11) c	(12) d

Accumulative quiz 5

(1) c	(2) b	(3) a	(4) b
(5) b	(6) c	(7) d	(8) d
(9) a	(10) a	(11) c	(12) c

Accumulative quiz 6

(1) b	(2) c	(3) b	(4) b
(5) b	(6) d	(7) a	(8) a
(9) b	(10) b	(11) b	(12) c

Accumulative quiz 7

(1) a	(2) d	(3) b	(4) c
(5) a	(6) a	(7) c	(8) d
(9) b	(10) c	(11) a	(12) b

Accumulative quiz 8

(1) c	(2) d	(3) c	(4) b
(5) d	(6) c	(7) a	(8) c
(9) c	(10) d	(11) c	(12) c

Accumulative quiz 9

(1) c	(2) c	(3) c	(4) a
(5) d	(6) c	(7) a	(8) c
(9) c	(10) a	(11) b	(12) b

Answers of school book examinations on Algebra & Trigonometry

Model 1

1

(1) c (2) d (3) b (4) a (5) d

2

[a] Let the matrix equation be AX = C where

$A=\begin{pmatrix}2 & -3\\ 3 & 4\end{pmatrix}, X=\begin{pmatrix}x\\ y\end{pmatrix}, C=\begin{pmatrix}4\\ 23\end{pmatrix}$

$\therefore |A|=\begin{vmatrix}2 & -3\\ 3 & 4\end{vmatrix}=8+9=17$

$\therefore A^{-1}=\frac{1}{17}\begin{pmatrix}4 & 3\\ -3 & 2\end{pmatrix}=\begin{pmatrix}\frac{4}{17} & \frac{3}{17}\\ \frac{-3}{17} & \frac{2}{17}\end{pmatrix}$

$, X=A^{-1}C=\begin{pmatrix}\frac{4}{17} & \frac{3}{17}\\ \frac{-3}{17} & \frac{2}{17}\end{pmatrix}\begin{pmatrix}4\\ 23\end{pmatrix}$

$\therefore \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}5\\ 2\end{pmatrix}$

$\therefore x=5, y=2$

[b] L.H.S. $=\sin\theta\times\cos\theta\times\frac{\sin\theta}{\cos\theta}=\sin^2\theta$

$=1-\cos^2\theta=$ R.H.S.

3

[a] $A=\frac{1}{2}\begin{vmatrix}-4 & 2 & 1\\ 3 & 1 & 1\\ -2 & 5 & 1\end{vmatrix}$

$=\frac{1}{2}\left[-4\begin{vmatrix}1 & 1\\ 5 & 1\end{vmatrix}-2\begin{vmatrix}3 & 1\\ -2 & 1\end{vmatrix}+1\begin{vmatrix}3 & 1\\ -2 & 5\end{vmatrix}\right]$

$=\frac{1}{2}\left[-4\times(1-5)-2(3+2)+(15+2)\right]=11.5$

$\therefore$ The area of the triangle $=|A|=11.5$ square units.

[b] $\sin x=-\frac{1}{2}$ (negative).

$\therefore x$ lies in the 3rd or 4th quadrant

$, \because$ the acute angle whose $\sin=\frac{1}{2}$ is of measure 30°

$\therefore x=180°+30°=210°$ is equivalent to $\frac{7}{6}\pi$

or $x=360°-30°=330°$ is equivalent to $\frac{11}{6}\pi$

$\therefore$ The S.S. $=\left\{\frac{7}{6}\pi, \frac{11}{6}\pi\right\}$

4

[a] $\because \begin{vmatrix}x & 0 & 0\\ 1 & x & x\\ 5 & 2 & x\end{vmatrix}=3x$

$\therefore x\begin{vmatrix}x & x\\ 2 & x\end{vmatrix}=3x$

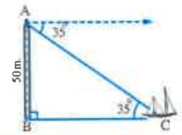
$\therefore x(x^2-2x)=3x$

$\therefore x(x^2-2x)-3x=0$

$\therefore x(x^2-2x-3)=0$

$\therefore x(x-3)(x+1)=0$

$\therefore x=0, 3$ or -1



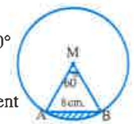
[b] $\because \sin C=\frac{AB}{AC}$

$\therefore \sin 35°=\frac{50}{AC}$

$\therefore AC\approx 87$ m.

5

[a] In Δ AMB :



$\because$ MA = MB = r , m ($\angle$ M) = 60°

$\therefore$ r = AB = 8 cm.

$\therefore$ The area of the circular segment

$=\frac{1}{2}(8)^2\left(\frac{\pi}{3}-\sin 60°\right)\approx 5.8$ cm².

[b] • $x\geq 0, y\geq 0$ are represented by $\overrightarrow{OX}\cup\overrightarrow{OY}\cup$ 1st quadrant

• Draw the boundary line

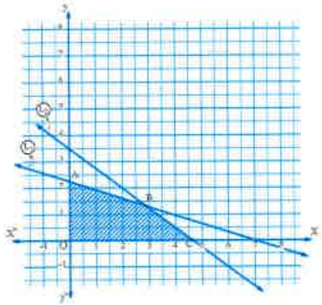
$L_1: x+3y=7$ (solid)

which passes through $\left(0, 2\frac{1}{3}\right)$, (7 , 0)

• Draw the boundary line

$L_2: 3x+4y=14$ (solid)

which passes through $\left(0, 3\frac{1}{2}\right)$, $\left(4\frac{2}{3}, 0\right)$



$\therefore$ The S.S. of the inequalities is the shaded region ABCO where A $\left(0, 2\frac{1}{3}\right)$, B $\left(2\frac{4}{5}, 1\frac{2}{5}\right)$, C $\left(4\frac{2}{3}, 0\right)$, O $(0, 0)$

, $\because$ the function is $P = 30\,x + 50\,y$

$\therefore [P]_A = 30\,(0) + 50\left(2\frac{1}{3}\right) = 116\frac{2}{3}$

, $[P]_B = 30\left(2\frac{4}{5}\right) + 50\left(1\frac{2}{5}\right) = 154$

, $[P]_C = 30\left(4\frac{2}{3}\right) + 50\,(0) = 140$, $[P]_O = 0$

$\therefore$ To get the greatest value of the function P

$x = 2\frac{4}{5}$, $y = 1\frac{2}{5}$

Model 2

1

(1) c (2) d (3) b (4) d

2

[a] $\because \Delta = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 7$

, $\Delta_x = \begin{vmatrix} 3 & -3 \\ 5 & 2 \end{vmatrix} = 6 + 15 = 21$

, $\Delta_y = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7$

$\therefore x = \frac{\Delta_x}{\Delta} = 3$, $y = \frac{\Delta_y}{\Delta} = 1$

[b] L.H.S. $= \cos x \times \frac{\sin x}{\cos x} \times \sin x$

$= \sin^2 x = 1 - \cos^2 x =$ R.H.S.

3

[a] $\begin{pmatrix} -2 & 2 \\ 4 & 3 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} -2 & 2 \\ 4 & 3 \end{vmatrix}} \begin{pmatrix} 3 & -2 \\ -4 & -2 \end{pmatrix}$

$= \frac{1}{-6-8}\begin{pmatrix} 3 & -2 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-3}{14} & \frac{1}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}$

$\therefore A = \begin{pmatrix} 12 & 23 \\ 8 & 13 \end{pmatrix}\begin{pmatrix} \frac{-3}{14} & \frac{1}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

[b] $\because \cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$ $\therefore \sin\theta = \frac{1}{2}$ (positive)

$\therefore \theta$ lies in the 1^{st} or 2^{nd} quad.

, $\because \sin 30° = \frac{1}{2}$ $\therefore \theta = 30°$ is equivalant to $\frac{\pi}{6}$

or $\theta = 180° - 30° = 150°$ is equivalant to $\frac{5}{6}\pi$

$\therefore$ The general solution is

$\theta = \frac{\pi}{6} + 2\,n\,\pi$ or $\theta = \frac{5\pi}{6} + 2\,n\,\pi$

4

[a] $\because A = \begin{pmatrix} 2 & 4 \\ -4 & 3 \end{pmatrix}$

$\therefore A^2 - 5A + 22I = \begin{pmatrix} 2 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -4 & 3 \end{pmatrix}$

$-5\begin{pmatrix} 2 & 4 \\ -4 & 3 \end{pmatrix} + \begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}$

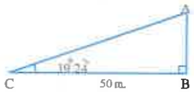
$= \begin{pmatrix} -12 & 20 \\ -20 & -7 \end{pmatrix} - \begin{pmatrix} 10 & 20 \\ -20 & 15 \end{pmatrix} + \begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$

[b] $\because \frac{1}{2} r^2 \left(\frac{\pi}{2} - \sin 90°\right) = 56$

$\therefore r^2 \approx 196$ $\therefore r \approx 14$ cm.

5



[a] $\tan 19° \, 24' = \frac{AB}{50}$

$\therefore AB = 50 \tan 19° \, 24'$

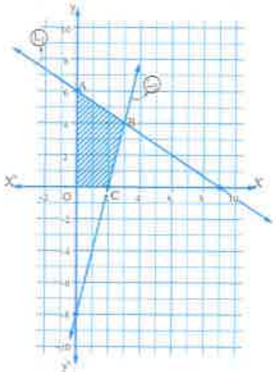
≈ 18 m.

$\therefore$ The height of the pole ≈ 18 m.

[b] • $x \geq 0$, $y \geq 0$ are represented by $\overrightarrow{OX} \cup \overrightarrow{OY} \cup 1^{st}$ quadrant.

• Draw the boundary line $L_1 : 2x + 3y = 18$ (solid) which passes through $(0, 6)$, $(9, 0)$

• Draw the boundary line $L_2 : -4x + y = -8$ (solid) which passes through $(0, -8)$, $(2, 0)$



$\therefore$ The S.S. of the inequalities is the shaded region ABCO

Where : A $(0, 6)$, B $(3, 4)$, C $(2, 0)$, O $(0, 0)$

, $\because$ the objective function is : $P = 2x + y$ has a maximum value.

$\therefore [P]_A = 2\,(0) + (6) = 6$, $[P]_B = 2\,(3) + (4) = 10$

, $[P]_C = 2\,(2) + 0 = 4$, $[P]_O = 0$

$\therefore$ The maximum value is $P = 10$

Answers of school book examinations on Analytic Geometry

Model (1)

1

(1) $\vec{i}+7\vec{j}$ (2) $\frac{3}{2}$ (3) $(2,-4)$

(4) 2 (5) $\vec{r}=(2,-3)+k(3,4)$

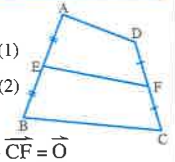
2

[a] $\because 8\|\vec{A}\|=5|k|\,\|\vec{A}\|$

$\therefore |k|=\frac{8}{5}$ $\therefore k=\pm\frac{8}{5}$

[b] $L=\frac{|5(1)-12(2)-7|}{\sqrt{5^2+12^2}}=2$ length units.

3



[a] $\because \overrightarrow{EB}+\overrightarrow{BC}+\overrightarrow{CF}=\overrightarrow{EF}$ (1)

$,\ \overrightarrow{EA}+\overrightarrow{AD}+\overrightarrow{DF}=\overrightarrow{EF}$ (2)

Adding (1), (2)

$\because \overrightarrow{EA}+\overrightarrow{EB}=\vec{O}\ ,\ \overrightarrow{DF}+\overrightarrow{CF}=\vec{O}$

$\therefore \overrightarrow{BC}+\overrightarrow{AD}=2\ \overrightarrow{EF}$

[b] $\because \vec{r}=(1,0)+t(1,1)$

$\therefore (x,y)=(1,0)+t(1,1)$

$\therefore x=1+t$ *i.e.* $t=x-1$

$,\ y=t$ $\therefore x-1=y$ $\therefore x-y-1=0$

$\therefore$ the general equation is :

$2x+y-5+k(x-y-1)=0$

$,\ \because$ the required straight line passes through $(5,3)$

$\therefore 2(5)+3-5+k(5-3,1)=0$

$\therefore 8+k=0$ $\therefore k=-8$

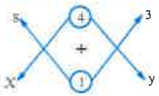
$\therefore$ The required equation is

$2x+y-5-8(x-y-1)=0$

$\therefore 2x+y-5-8x+8y+8=0$

$\therefore 9y-6x+3=0$ i.e. $3y-2x+1=0$

4



[a] **First :** Let $B=(x,y)$

, C divides $\overline{AB}$ internally

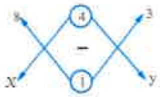
$\therefore \frac{4x+8}{5}=2$ $\therefore 4x+8=10$

$\therefore 4x=2$ $\therefore x=\frac{1}{2}$ $,\ \frac{4y+3}{5}=5$

$\therefore 4y+3=25$ $\therefore 4y=22$ $\therefore y=5\frac{1}{2}$

$\therefore B=\left(\frac{1}{2},5\frac{1}{2}\right)$

Second : Let $B=(x,y)$



, C divides $\overline{AB}$ externally

$\therefore \frac{4x-8}{3}=2$

$\therefore 4x-8=6$ $\therefore 4x=14$

$\therefore x=3\frac{1}{2}$ $,\ \frac{4y-3}{3}=5$

$\therefore 4y-3=15$ $\therefore 4y=18$

$\therefore y=4\frac{1}{2}$ $\therefore B=\left(3\frac{1}{2},4\frac{1}{2}\right)$

[b] $\overrightarrow{XY}=(4-3,2-5)=(1,-3)$

$,\ \overrightarrow{YZ}=(-5-4,-1-2)=(-9,-3)$

$,\ \because 1\times -9+(-3)(-3)=0$

$\therefore \overrightarrow{XY}\perp\overrightarrow{YZ}$

$\therefore \Delta XYZ$ is right-angled at Y

$\therefore \overline{XZ}$ is a diameter in the circumcircle

$\therefore XZ=\sqrt{(3+5)^2+(5+1)^2}=10$

$\therefore$ Radius of the circle = 5 length units.

, its area = 25π square units.

5

(1) $m_1=\frac{-3}{2}\ ,\ m_2=\frac{2}{3}$

$\therefore m_1\times m_2=\frac{-3}{2}\times\frac{2}{3}=-1$

$\therefore$ The angle between the two lines is right

(2) $L_1: 3x+2y-7=0$ (1)

$,\ L_2: 2x-3y+4=0$ (2)

Multiplying (1) by 3 , (2) by 2

$\therefore 9x+6y-21=0$ (3)

$,\ 4x-6y+8=0$ (4)

Adding (3) , (4)

$\therefore 13x-13=0$ $\therefore x=1$

, substituting in (1) :

$\therefore 3(1)+2y-7=0$ $\therefore y=2$

$\therefore$ The intersection point is $(1,2)$

$\therefore \vec{u}=(3,4)-(1,2)=(2,2)$

$\therefore \vec{r}=(3,4)+k(2,2)$

Model (2)

1

(1) $(-3,-1)$ (2) 5 length units.

(3) 1 : 2 internally (4) 90°

(5) $\sqrt{2}$ length units.

2

[a] $\because 4k\|\vec{A}\|=|-3|\,\|\vec{A}\|$ $\therefore k=\frac{3}{4}$

[b] $\because 2x-y+4=0$ $,\ x+y+5=0$

By adding : $\therefore 3x+9=0$

$\therefore x=-3$

$\therefore 2(-3)-y+4=0$ $\therefore y=-2$

$\therefore$ The point of intersection of the two lines is $(-3,-2)$

$\therefore \frac{y+2}{x+3}=\frac{0+2}{-1+3}$ $\therefore \frac{y+2}{x+3}=\frac{2}{2}=1$

$\therefore y+2=x+3$ $\therefore y-x-1=0$

3

[a] $\because$ ABCD is a parallelogram $\therefore \overrightarrow{AD}=\overrightarrow{BC}$

$\therefore \vec{D}-\vec{A}=\vec{C}-\vec{B}$

$\therefore \vec{D}=\vec{A}+\vec{C}-\vec{B}=(3,4)+(2,-2)-(5,-1)$

$=(0,3)$

$\therefore$ The point D is $(0,3)$

[b] $\because m_1=\frac{-2}{1}=-2\ ,\ m_2=\frac{-2}{1}=-2$

$\therefore m_1=m_2$ $\therefore$ The two straight lines are parallel

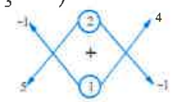
$,\ \because (0,4)\in$ the first straight line.

$\therefore$ The shortest distance between the two lines

$=\frac{|2(0)+(4)+2|}{\sqrt{2^2+1^2}}=\frac{6\sqrt{5}}{5}$ length units.

4

[a] $C=\left(\frac{2\times5+(-1)(1)}{3},\frac{2(-1)+(1)(4)}{3}\right)$



$=\left(3,\frac{2}{3}\right)$

[b] $\because \frac{|3(0)+4(0)+10|}{\sqrt{3^2+4^2}}$

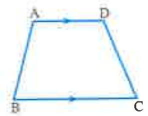
$=\frac{10}{5}=2$ length units.

$,\ \frac{|5(0)-12(0)+26|}{\sqrt{5^2+12^2}}=\frac{26}{13}=2$ length units.

$\therefore$ The two chords are equidistant from the centre of the circle.

$\therefore$ The two chords are equal in length.

5



(1) $\overrightarrow{AD}=(5-7,y+1)$

$=(-2,y+1)$

$,\ \overrightarrow{BC}=(2-3,1+1)$

$=(-1,2)$

$,\ \because \overrightarrow{AD}\,//\,\overrightarrow{BC}$

$\therefore -2\times2-(-1)(y+1)=0$

$\therefore -4+y+1=0$ $\therefore y=3$

(2) $\therefore \overrightarrow{AD}=(-2,4)$

$\therefore \|\overrightarrow{AD}\|=\sqrt{4+16}=2\sqrt{5}$ length units.

$,\ \|\overrightarrow{BC}\|=\sqrt{1+4}=\sqrt{5}$ length units.

$,\ \because$ the equation of $\overleftrightarrow{BC}$ is $\frac{y+1}{x-3}=-2$

$\therefore y+1=-2x+6$ $\therefore y+2x-5=0$

$\therefore$ The height of the trapezium $=\frac{|-1+2(7)-5|}{\sqrt{1+4}}$

$=\frac{8}{\sqrt{5}}$ length units.

$\therefore$ The area $=\frac{2\sqrt{5}+\sqrt{5}}{2}\times\frac{8}{\sqrt{5}}=3\sqrt{5}\times\frac{4}{\sqrt{5}}$

= 12 square units.

Answers of final Examination Models

Model 1

1 (d) 2 (d)

3

$\because \overline{AD}$ is a median in ΔABC

$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$ (1)

, $\because \overline{ED}$ is a median in ΔEBC

$\therefore \overrightarrow{EB} + \overrightarrow{EC} = 2\,\overrightarrow{ED} = \overrightarrow{AD}$ (2)

From (1) , (2) :

$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\left(\overrightarrow{EB} + \overrightarrow{EC}\right) = 2\,\overrightarrow{EB} + 2\,\overrightarrow{EC}$

4 (d) 5 (d) 6 (b) 7 (a) 8 (a)

9 (d) 10 (b) 11 (d) 12 (c) 13 (c)

14 (a) 15 (d) 16 (c) 17 (d)

18

$\because \frac{\tan 5X}{\tan(90^\circ + 4X)} = -1$ $\therefore \frac{\tan 5X}{-\cot 4X} = -1$

$\therefore \tan 5X = \cot 4X$

$\therefore 5X + 4X = 90^\circ + 180^\circ n$

$\therefore 9X = 90^\circ + 180^\circ n$ $\therefore X = 10^\circ + 20^\circ n$

$\therefore$ The general solution is $\boxed{10^\circ + 20^\circ n}$ where $n \in \mathbb{Z}$

19 (d) 20 (c) 21 (d) 22 (c)

23 (b) 24 (b) 25 (b)

26

$A = \left(\frac{1 \times 3 + 2 \times zero}{1+2}, \frac{1 \times zero + 2 \times 3}{1+2}\right) = (1, 2)$

27

$\because X \geq 0$, $y \geq 0$ represented by

$\overrightarrow{OX} \cup \overrightarrow{Oy} \cup$ the first quadrant

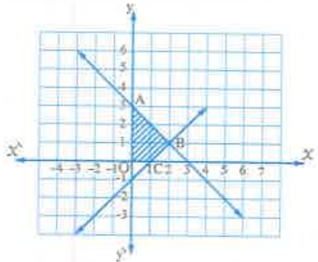
, the boundary line :

$X + y = 3$ passes through $(0, 3)$, $(3, 0)$

, the boundary line :

$X - y = 1$ passes through $(1, 0)$, $(0, -1)$

$\therefore$ The solution set is the shaded region ABCO



$\because [P]_A = 3(0) + 4(3) = 12$, $[P]_B = 3(2) + 4(1) = 10$

, $[P]_C = 3(1) + 4(0) = 3$, $[P]_O = 3(0) + 4(0) = 0$

The maximum value of the objective function = 12 at the point A (0 , 3)

28 (d)

29

$\because A^2 = \begin{pmatrix} 4 & 0 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

$= 16\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 16\,I = 2^4\,I$

$\therefore A^{60} = (A^2)^{30} = (2^4\,I)^{30} = 2^{120}\,I^{30} = 2^{120}\,I$

30 (a) 31 (a) 32 (a) 33 (d)

Model 2

1 (d) 2 (c) 3 (c) 4 (a)

5 (c) 6 (c) 7 (a) 8 (b)

9 (a) 10 (b) 11 (b) 12 (d)

13 (b) 14 (d)

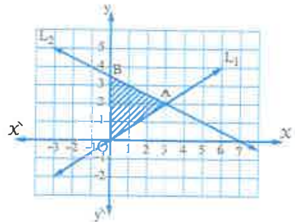
15

$\because X \geq 0$, $y \geq 0$ represented by $\overrightarrow{OX} \cup \overrightarrow{Oy} \cup$ the first quadrant

$L_1 : 2X = 3y$ passes through $(0, 0)$ and $(3, 2)$

, $L_2 : 2y + X = 7$ passes through $(0, 3.5)$ and $(7, 0)$

$\therefore$ The solution set is the shaded region AOB



$\therefore [P]_O = 3(0) + 2(0) = 0$

, $[P]_A = 3(3) + 2(2) = 13$

, $[P]_B = 3(0) + 2(3.5) = 7$

$\therefore$ The maximum value of the objective function = 13 at the point (3 , 2)

16 (a) 17 (a) 18 (c) 19 (c)

20 (c) 21 (c) 22 (a) 23 (b)

24 (d) 25 (a) 26 (d) 27 (d)

28 (c)

29

$\because \vec{A} = 3\,\vec{i} - 4\,\vec{j}$ $\therefore \|\vec{A}\| = \sqrt{(3)^2 + (4)^2} = 5$

, $\because \vec{E} = \vec{j}$ $\therefore \|\vec{B}\| = \sqrt{(0)^2 + (1)^2} = 1$

, $\because \vec{C} = \left(5, \frac{\pi}{18}\right)$ $\therefore \|\vec{C}\| = 5$

$\therefore \|\vec{A}\| + \|\vec{B}\| + \|\vec{C}\| = 11$

30

In ΔABM :

$\therefore m(\angle AMB) = 60^\circ$

$\therefore \theta^{rad} = \frac{\pi}{3}$

$\therefore$ The area of the circular segment

$= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta) = \frac{1}{2} \times (18)^2 \left(\frac{\pi}{3} - \sin 60^\circ\right)$

$\simeq 29\text{ cm}^2$

31 (c)

32

$\because$ The direction vector of the given line is $(-2, 1)$

$\therefore$ The direction vector of the required line is $(1, 2)$

$\therefore$ The vector equation : $\vec{r} = (1, 3) + k(1, 2)$

i.e. $(X, y) = (1, 3) + k(1, 2)$

The parametric equations are $X = 1 + k$, $y = 3 + 2k$

The cartesian equation $\frac{y-3}{X-1} = \frac{2}{1}$

$\therefore 2X - 2 = y - 3$

$\therefore$ The general form : $2X - y + 1 = 0$

33

$\because A = \frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ -2 & 4 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 12$

$\therefore$ The area of $\Delta ABC = |12| = 12$ square units.

Model 3

1 (c) 2 (c) 3 (b)

4 (d) 5 (d)

6

$\because A = \begin{vmatrix} X-2 & 0 & 0 \\ 3 & X-3 & 0 \\ 4 & -1 & X \end{vmatrix} = zero$

$\therefore (X-2)(X-3)(X) = zero$

$\therefore X = 2$ or $X = 3$ or $X = zero$

$\therefore$ The solution set $= \{2, 3, 0\}$

7 (b) 8 (c) 9 (b)

10 (d) 11 (c)

12

Let the number of cupboard of type A = X

, Let the number of cupboard of type B = y

$\therefore X \geq 0$, $y \geq 0$, $X + y \leq 20$, $X \geq 3y$

, The objective function : $P = 80X + 100y$

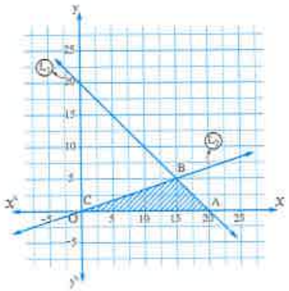
First : Determine the region that represents the S.S. of the inequalities as follows :

(1) The two inequalities $x \geq 0$, $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

(2) Draw the boundary line $L_1 : x + y = 20$ that passes through (0 , 20) , (20 , 0)

(3) Draw the boundary line $L_2 : x = 3y$ that passes through (0 , 0) , (15 , 5)

$\therefore$ The S.S. of the inequalities is the shaded region ABC where O (0 , 0) , A (20 , 0) , B (15 , 5)



Second : $\because$ The objective function : $P = 80x + 100y$

$\therefore [P]_O = 80 \times 0 + 100 \times 0 = 0$

, $[P]_A = 80 \times 20 + 100 \times 0 = 1600$

, $[P]_B = 80 \times 15 + 100 \times 5 = 1700$

$\therefore$ The maximum profit happens at producing 15 cupboards of type A and 5 cupboards of type B

13 (c)

14

$\because 4\,AC = 3\,AB$ $\qquad \therefore \frac{AB}{AC} = \frac{4}{3}$

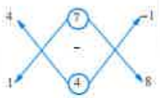
$\therefore \frac{BD}{DC} = \frac{AB}{AC} = \frac{4}{3}$

$\therefore$ B divides $\overline{CD}$ by the ratio 7 : 4 externally

$\therefore$ The coordinales of B

$= \left(\frac{7 \times 1 - 4 \times 4}{7 - 4} , \frac{7 \times 8 - 4 \times -1}{7 - 4}\right)$

$= (-3 , 20)$



15 (c) **16** (b)

17

$\because \|\overrightarrow{EM}\| = \sqrt{\left(4\sqrt{3}\right)^2 + (4)^2} = 8$

, $\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\qquad \therefore \theta = \frac{\pi}{6}$

$\therefore$ The polar form is $\left(8 , \frac{\pi}{6}\right)$

18 (b) **19** (c) **20** (a) **21** (d)

22 (a) **23** (a) **24** (d) **25** (c)

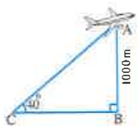
26 (b) **27** (c) **28** (c) **29** (d)

30

$\sin 40° = \frac{1000}{AC}$

$\therefore AC = \frac{1000}{\sin 40°}$

$\simeq 1556$ m.



31 (c) **32** (d) **33** (a)

Model 4

1 (b) **2** (c) **3** (d) **4** (b)

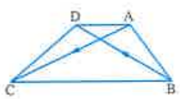
5 (a) **6** (a) **7** (a) **8** (d)

9 (b) **10** (c)

11

In Δ ABC :

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$



, in Δ ABD : $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$\therefore \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{AD}$

$= \overrightarrow{BC} + \overrightarrow{AD}$

, $\because \overrightarrow{BC} = 3\,\overrightarrow{AD}$

$\therefore \overrightarrow{AC} + \overrightarrow{BD} = 3\,\overrightarrow{AD} + \overrightarrow{AD} = 4\,\overrightarrow{AD}$

12 (c) **13** (a) **14** (c) **15** (b)

16 (d) **17** (d) **18** (c) **19** (b)

20 (b) **21** (d)

22

Draw the boundary lines $L_1 : x = 4$ (solid)

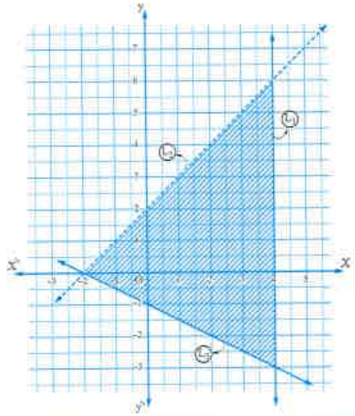
, $L_2 : y = x + 2$ (dashed)

that passes through (0 , 2) , (– 2 , 0)

, $L_3 : x + 2y = -2$ (solid)

that passes through (0 , – 1) , (– 2 , 0)

Then the S.S. of the inequalities is represented by the shaded region in the graph.



23 (a) **24** (b) **25** (c)

26

$\because$ The area of the sector = 45 cm^2.

$\therefore \frac{1}{2}\ell r = 45$ $\qquad \therefore \frac{1}{2}\ell \times 10 = 45$ $\qquad \therefore \ell = 9$ cm.

$\therefore$ The perimeter of the sector $= \ell + 2r = 9 + 20$
$= 29$ cm.

27 (b)

28

$\because 3^x = 25$, $3^y = 5$ $\qquad \therefore 3^x = 5^2 = 3^{2y}$

$\therefore x = 2y$

$\therefore a = x + y = 3y$, $b = 2x + y = 5y$

$\therefore \frac{b}{a} = \frac{5}{3}$

29 (b) **30** (d) **31** (d) **32** (c)

33

$\because m_1 = m_2 = \frac{3}{4}$

$\therefore$ The two lines are parallel , put $x = 4$ in the equation of the first line

$\therefore 3 \times 4 - 4y + 20 = 0$ $\qquad \therefore y = 8$

$\therefore (4 , 8) \in$ the first line

$\therefore$ The distance between the two straight lines

$= \frac{|3 \times 4 - 4 \times 8 + 10|}{\sqrt{(3)^2 + (-4)^2}} = 2$ length units.

Model 5

1 (d) **2** (d)

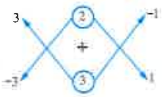
3

A = (3 , – 1) , B = (– 3 , 1)

point of division

$= \left(\frac{2 \times -3 + 3 \times 3}{2 + 3} , \frac{2 \times 1 + 3 \times -1}{2 + 3}\right)$

$= \left(\frac{3}{5} , -\frac{1}{5}\right)$



4 (c)

5

$\because$ The area of the rectangle AEDF = 27 cm^2.

$\therefore AF \times FD = 27$ $\qquad$, $\because BF = FM = \frac{1}{2}AM = \frac{1}{2}r$

$\therefore \frac{3}{2}r \times \frac{1}{2}r = 27$ $\qquad \therefore \frac{3}{4}r^2 = 27$

$\therefore r^2 = 36$ $\qquad \therefore r = 6$ cm.

, $\because \cos(\angle CMF) = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$

$\therefore m(\angle CMF) = 60°$

$\therefore m(\angle AMC) = 120°$

$\therefore$ The shaded area $= \frac{1}{2}\theta^{rad} r^2$

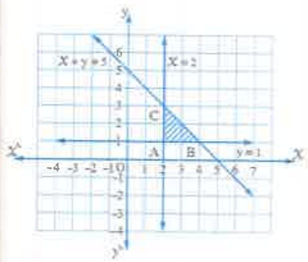
$= \frac{1}{2} \times \frac{120°\,\pi}{180°} \times (6)^2 = 12\pi$ cm^2.

6 (d) **7** (b) **8** (d) **9** (b)

10 (a) **11** (c) **12** (d) **13** (b)

14 (b) **15** (d)

16



The solution set of the inequalities is the shaded region ABC

, ∵ $[P]_A = 2(2) + 3(1) = 7$, $[P]_B = 2(4) + 3(1) = 11$

, $[P]_C = 2(2) + 3(3) = 13$

∴ The smallest value of the objective function $= 7$ at the point A (2 , 1)

17 (b) **18** (b) **19** (b) **20** (a)

21 (c) **22** (d) **23** (d) **24** (c)

25

$\because A = A^{-1} \times B$ $\therefore A \times A = A \times A^{-1} \times B$

$\therefore A^2 = B$

$\therefore B = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 8 \\ -4 & 1 \end{pmatrix}$

26 (c) **27** (c) **28** (b)

29

$\because \overrightarrow{F_2} = \left(6\cos\frac{\pi}{3}\right)\hat{i} + \left(6\sin\frac{\pi}{3}\right)\hat{j} = 3\hat{i} + 3\sqrt{3}\,\hat{j}$

$\therefore \overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} = (-3+3)\hat{i} + (\sqrt{3} + 3\sqrt{3})\hat{j} = 4\sqrt{3}\,\hat{j}$

$\therefore \|\overrightarrow{R}\| = 4\sqrt{3}$ force units.

30 (b) **31** (c) **32** (d) **33** (a)

Model 6

1 (b)

2

The area of the sector $= \frac{1}{2}\theta^{rad} r^2 = \frac{1}{2} \times \frac{60° \times \pi}{180°} \times (6)^2$

$= 6\pi$ cm^2

3 (b)

4

$\left(K + \frac{1}{K}\right)^2 - 1 = 15$

$\therefore K^2 + \frac{1}{K^2} + 2 - 1 = 15$

$\therefore K^2 + \frac{1}{K^2} = 14$

5

∵ The area of Δ ABC = 15

$\therefore \frac{1}{2} \times AC \times 6 = 15$ ∴ AC = 5 cm.

∴ OC = AC = 5 cm.

∴ A (10 , 0)

∴ The equation of the straight line L_1 is : $\frac{x}{10} + \frac{y}{6} = 1$

6 (c) **7** (d) **8** (d) **9** (c)

10 (c) **11** (b) **12** (d) **13** (d)

14 (a) **15** (d) **16** (c) **17** (d)

18 (c) **19** (b) **20** (c) **21** (a)

22 (c) **23** (c) **24** (b) **25** (d)

26 (c) **27** (c) **28** (a)

29

$\because A \times \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = I$

$\therefore A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

$\therefore 3A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

30 (c) **31** (a) **32** (d)

33

Let $\overrightarrow{C} = m\overrightarrow{A} + K\overrightarrow{B}$

∴ (7 , 12) = m (−1 , 2) + K (3 , 7)

∴ 3 K − m = 7 (1)

, 7 K + 2 m = 12 (2)

From (1) and (2) : ∴ K = 2 , m = −1

$\therefore \overrightarrow{C} = -\overrightarrow{A} + 2\overrightarrow{B}$

Model 7

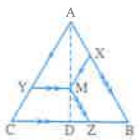
1 (b) **2** (c) **3** (d) **4** (a)

5

Construction :

Draw $\overline{AD}$ is a median in Δ ABC

, ∵ M is the point of intersection of the medians of Δ ABC



$\therefore \frac{MY}{DC} = \frac{MA}{AD} = \frac{2}{3}$

$\therefore \overrightarrow{MY} = \frac{2}{3}\overrightarrow{DC} = \frac{2}{3}\left(\frac{1}{2}\overrightarrow{BC}\right) = \frac{1}{3}\overrightarrow{BC}$

Similarly :

$\overrightarrow{MX} = \frac{1}{3}\overrightarrow{CA}$, $\overrightarrow{MZ} = \frac{1}{3}\overrightarrow{AB}$

$\therefore \overrightarrow{MX} + \overrightarrow{MY} + \overrightarrow{MZ} = \frac{1}{3}\overrightarrow{CA} + \frac{1}{3}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{AB}$

$= \frac{1}{3}(\overrightarrow{CA} + \overrightarrow{AB} + \overrightarrow{BC})$

$= \frac{1}{3} \times \overrightarrow{zero} = \overrightarrow{zero}$

6 (c) **7** (d) **8** (a) **9** (b) **10** (c)

11 (d) **12** (c) **13** (d) **14** (c) **15** (b)

16 (d) **17** (d) **18** (c) **19** (c) **20** (c)

21 (a) **22** (d) **23** (c)

24

$\because m_1 = -\frac{1}{3}$, $m_2 = -2$

$\therefore \tan\theta = \left|\frac{-\frac{1}{3} + 2}{1 + \left(-\frac{1}{3}\right)(-2)}\right| = 1$ $\therefore \theta = 45°$

25 (b)

26

$\because \ell m = \frac{-10}{1} = -10$

$\therefore \begin{vmatrix} 2 & \ell & -1 \\ 3 & & m \end{vmatrix} = 2\ell m + 3 = 2 \times -10 + 3 = -17$

27 (c) **28** (d) **29** (c) **30** (a)

31

$B = A^{-1}AB = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 4 & 1 \end{pmatrix}$

32 (c) **33** (a)

Model 8

1 (a) **2** (a) **3** (b) **4** (c)

5 (a) **6** (b) **7** (c) **8** (d)

9 (a) **10** (b) **11** (d) **12** (c)

13

$\because \begin{vmatrix} x^2 & 5 \\ 3 & x \end{vmatrix} = 12$

$\therefore x^3 - 15 = 12$ $\therefore x^3 = 27$

$\therefore x = 3$ ∴ S.S = {3}

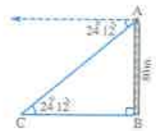
14 (b) **15** (d) **16** (c)

17

$\because \tan 24°\ 12' = \frac{80}{BC}$

$\therefore BC = \frac{80}{\tan 24°\ 12'}$

$\simeq 178$ m.



18 (b) **19** (d)

20

$\overrightarrow{A} = \left(2\cos\frac{\pi}{6}\right)\hat{i} + \left(2\sin\frac{\pi}{6}\right)\hat{j} = \sqrt{3}\,\hat{i} + \hat{j}$

$\therefore 2\overrightarrow{A} = 2\sqrt{3}\,\hat{i} + 2\hat{j}$

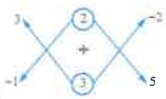
21 (c) **22** (c) **23** (b) **24** (d)

25 (b) **26** (d) **27** (a)

28

∵ The point C lies at $\frac{2}{5}$ of the distance AB

∴ C divides $\overrightarrow{AB}$ by the ratio 2 : 3



$\therefore C = \left(\frac{2 \times -1 + 3 \times 3}{2+3}, \frac{2 \times 5 + 3 \times -2}{2+3}\right) = \left(\frac{7}{5}, \frac{4}{5}\right)$

29 (d)

30

The length of the perpendicular $= \frac{|4(3) + 3(1) - 5|}{\sqrt{(4)^2 + (3)^2}}$

$= 2$ length units.

31 (a) **32** (d) **33** (a)

Model 9

1 (d) 2 (c) 3 (d)

4

$\because |2A| = 8$ $\therefore 2^2 |A| = 8$ $\therefore |A| = 2$

$\therefore |3A^t| = 3^2 |A^t| = 9|A| = 9 \times 2 = 18$

5 (c) 6 (c) 7 (a) 8 (d)

9 (c) 10 (a) 11 (b) 12 (a)

13 (c) 14 (a) 15 (a) 16 (c)

17

$\because$ ABCD is a parallelogram

$\therefore \vec{A} + \vec{C} = \vec{B} + \vec{D}$

$\therefore (1, 2) + (-3, 5) = (x, -3) + (-7 + y)$

$\therefore x - 7 = -2$ $\therefore x = 5$

and $y - 3 = 7$ $\therefore y = 10$

$\therefore \vec{B} = (5, -3)$, $\vec{D} = (-7, 10)$

$\therefore \overrightarrow{BD} = \vec{D} - \vec{B} = (-12, 13)$

18 (d) 19 (a)

20

$\because \begin{vmatrix} x+2 & 2 \\ x & x-3 \end{vmatrix} = 4$ $\therefore (x+2)(x-3) - 2x = 4$

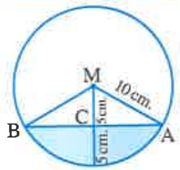
$\therefore x^2 - x - 6 - 2x - 4 = 0$ $\therefore x^2 - 3x - 10 = 0$

$\therefore (x-5)(x+2) = 0$ $\therefore x = 5$ or $x = -2$

$\therefore$ S.S $= \{5, -2\}$

21 (b) 22 (a) 23 (b)

24



$\because \cos(AMC) = \frac{5}{10}$

$\therefore m(\angle AMC) = 60°$

$\therefore m(\angle AMB) = 120°$

$\therefore$ The area of circular segment

$= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta) = \frac{1}{2}(10)^2 \left(\frac{2}{3}\pi - \sin 120°\right)$

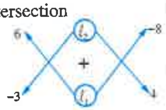
$\simeq 61 \text{ cm}^2$

25 (d) 26 (d)

27

Let $C = (x, 0)$ be the point of intersection of $\overline{AB}$ with the x-axis

, C divides $\overline{AB}$ by ratio $\ell_2 : \ell_1$



$\therefore \frac{4\ell_2 - 8\ell_1}{\ell_2 + \ell_1} = \text{zero}$ $\therefore 4\ell_2 = 8\ell_1$

$\therefore \frac{\ell_2}{\ell_1} = \frac{2}{1}$ $\therefore$ The ratio is 2 : 1 internally

28 (c) 29 (b) 30 (c)

31 (b) 32 (d) 33 (a)

Model 10

1 (b) 2 (d) 3 (b) 4 (d)

5 (b) 6 (c) 7 (a) 8 (b)

9 (c) 10 (d) 11 (c) 12 (b)

13 (a) 14 (c) 15 (d) 16 (d)

17 (c) 18 (a) 19 (a) 20 (d)

21

$\because \sin A + \sin^2 A = 1$

$\therefore \sin A = 1 - \sin^2 A = \cos^2 A$

, $\sin^2 A = \cos^4 A$

$\therefore \cos^2 A + \cos^4 A = \sin A + \sin^2 A = 1$

22 (c) 23 (c)

24

$\because \vec{A} = (2, -1)$

, $K\vec{B} + \vec{C} = (K+1, K+3)$

, $\because \vec{A} \perp (K\vec{B} + \vec{C})$

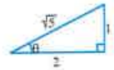
$\therefore 2(K+1) - (K+3) = 0$

$\therefore K - 1 = 0$ $\therefore K = 1$

25 (a)

26

Let $B = (K, 0)$



$\therefore$ The slope of $\overleftrightarrow{AB} = \frac{-6}{K}$

The slope of $\overleftrightarrow{AC} = \frac{6}{-3} = -2$

$\therefore \tan\theta = \left|\frac{-2 + \frac{6}{K}}{1 + (-2)\left(\frac{-6}{K}\right)}\right|$, $\because \cos\theta = \frac{2}{\sqrt{5}}$

$\therefore \tan\theta = \frac{1}{2}$ $\therefore \left|\frac{-2 + \frac{6}{K}}{1 + \frac{12}{K}}\right| = \frac{1}{2}$

$\therefore \frac{-2 + \frac{6}{K}}{1 + \frac{12}{K}} = \pm\frac{1}{2}$

$\therefore \frac{-2 + \frac{6}{K}}{1 + \frac{12}{K}} = \frac{1}{2}$ and hence

$-4 + \frac{12}{K} = 1 + \frac{12}{K}$

$\therefore -4 = 1$ (condradiction)

or $\frac{-2 + \frac{6}{K}}{1 + \frac{12}{K}} = -\frac{1}{2}$

$-4 + \frac{12}{K} = -1 - \frac{12}{K}$ $\therefore \frac{24}{K} = 3$ $\therefore K = 8$

$\therefore$ The point $B = (8, 0)$

27 (b) 28 (b)

29

$\because \begin{vmatrix} 2x & 0 & 0 \\ 1 & 3x & 0 \\ 2 & 4 & -x \end{vmatrix} = 48$

$\therefore 2x \times 3x \times -x = 48$

$\therefore -6x^3 = 48$ $\therefore x^3 = -8$

$\therefore x = -2$

30 (d) 31 (b) 32 (b)

33

$\because m_1 = \frac{1}{2}$, $m_2 = -\frac{2}{1}$

$\therefore m_1 \times m_2 = -1$

$\therefore$ The two straight lines are perpendicular

$\therefore$ The measure of the angle between the two straight lines = 90°